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ON THE ASYMPTOTIC EFFICIENCY OF THE MULTISAMPLE LOCATION–SCALE RANK TESTS AND THEIR ADJUSTMENT FOR TIES[1]

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Explicit formulas for the non-centrality parameters of the limiting chi-square distribution of proposed multisample rank based test statistics, aimed at testing the hypothesis of the simultaneous equality of location and scale parameters of underlying populations, are obtained by means of a general assertion concerning the location-scale test statistics. The finite sample behaviour of the proposed tests is discussed and illustrated by simulation estimates of the rejection probabilities. A modification for ties of a class of multisample location and scale test statistics, based on ranks and including the proposed test statistics, is presented. It is shown that under the validity of the null hypothesis these modified test statistics are asymptotically chi-square distributed provided that the score generating functions fulfill the imposed regularity conditions. An essential assumption is that the matrix, appearing in these conditions, is regular. Conditions sufficient for the validity of this assumption are also included.

Keywords: multisample rank test for location and scale, asymptotic non-centrality parameter, Pitman–Noether efficiency, adjustment for ties

AMS Subject Classification: 62G10, 62G20

1. INTRODUCTION

The topic of the paper is testing the multisample null hypothesis that the sampled populations have the same location parameters and the same scale parameters. It has been proposed in [12] to test this hypothesis by means of a test, which in the two sample case coincides with the Lepage test described in [8]. In the next section of this paper new test statistics for testing this null location-scale hypothesis are proposed, non-centrality parameters of their asymptotic chi-square distribution are computed for normal, logistic and Cauchy distribution and the resulting asymptotic efficiencies are discussed. The finite sample behaviour of the proposed tests is discussed in Section 2 and illustrated by simulation estimates of rejection probabilities both under the null and the alternative hypothesis, and the multiple comparisons

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procedure based on these new statistics are also briefly mentioned. The theoretical base for computation of the asymptotic non-centrality parameters can be found in Section 3, where a theorem on the Pitman–Noether asymptotic relative efficiency of the multisample location and scale rank tests, based on general regularity conditions, is stated and proved.

A general form of modification for ties of the proposed test statistics, guaranteeing their convergence in distribution to the chi-square distribution, is presented in Theorem 2.3 of Section 2, its proof is in Section 3; the application of this theorem to the mentioned statistics is at the end of Section 2.

2. MAIN RESULTS

It is supposed throughout the paper that $X_{j1}, \ldots, X_{jn_j}$ denotes for $j = 1, \ldots, k$ a random sample from the distribution of the random variable $\zeta_j = \sigma_j \varepsilon_j + \mu_j$, where $\sigma_j > 0$, μ_j are real numbers, these k random samples are independent and the distribution function

$$F(t) = \mathrm{P}(\varepsilon_j \le t) \tag{2.1}$$

does not depend on j. The topic of the paper is testing of the null hypothesis

$$H_0: \qquad \mu_1 = \mu_2 = \ldots = \mu_k\,, \qquad \sigma_1 = \sigma_2 = \ldots = \sigma_k \tag{2.2}$$

against the alternative, that there exist $i \neq j$ such that at least one of the non-equalities $\mu_i \neq \mu_j$, $\sigma_i \neq \sigma_j$ holds. Until stated otherwise, we assume that the function (2.1) is continuous.

Suppose that

$$(X_{1,1}, \ldots, X_{1,n_1}, \ldots, X_{k,1}, \ldots, X_{k,n_k}) \tag{2.3}$$

denotes the pooled random sample and $(R_{1,1}, \ldots, R_{1,n_1}, \ldots, R_{k,1}, \ldots, R_{k,n_k})$ are its ranks. The null hypothesis (2.2) is in [12] proposed to be tested by the statistic

$$T = T_K + T_B\,, \tag{2.4}$$

where T_K is the Kruskal–Wallis test statistic and T_B is the multisample Ansari–Bradley test statistic. We remark, the in the two-sample case $k = 2$ the statistic (2.4) coincides with the Lepage test statistic constructed in [8]. As a competitor of (2.4) for testing the null hypothesis (2.2) we propose the statistic

$$T_\Phi = Q_{\Phi^{-1}} + Q_{(\Phi^{-1})^2}\,, \tag{2.5}$$

where in the general notation

$$Q_\varphi = \frac{1}{\sigma_N^{2,\varphi}} \sum_{j=1}^{k} n_j \Big(\frac{S_j^{(\varphi)}}{n_j} - \tilde{\varphi} \Big)^2, \qquad S_j^{(\varphi)} = \sum_{i=1}^{n_j} \varphi\Big(\frac{R_{j,i}}{N+1} \Big)\,, \tag{2.6}$$

$$\begin{gathered} \sigma_N^{2,\varphi} = \sigma_N^{2,\varphi,\varphi}\,, \\ \sigma_N^{2,\varphi,\psi} = \tfrac{1}{N-1} \textstyle\sum_{i=1}^{N} \Big(\varphi\Big(\tfrac{i}{N+1}\Big) - \tilde{\varphi} \Big) \Big(\psi\Big(\tfrac{i}{N+1}\Big) - \tilde{\psi} \Big)\,, \quad \tilde{\varphi} = \tfrac{1}{N} \textstyle\sum_{i=1}^{N} \varphi\Big(\tfrac{i}{N+1}\Big)\,, \end{gathered} \tag{2.7}$$

$$N = n_1 + \ldots + n_k\,, \tag{2.8}$$

and Φ^{-1} denotes the quantile function of the $N(0,1)$ distribution function Φ, i. e., the equality $\Phi\Big(\Phi^{-1}(t)\Big) = t$ holds for every $t \in (0,1)$. Thus $Q_{\Phi^{-1}}$, $Q_{(\Phi^{-1})^2}$ is the multisample version of the van der Waerden and the Klotz test statistic, respectively. Finally, we propose to consider for use the test statistic

$$T_{SQ} = T_K + Q\,, \tag{2.9}$$

where T_K is the Kruskal–Wallis statistic and

$$\begin{aligned} Q &= \frac{1}{\sigma_N^2}\sum_{j=1}^{k} n_j\left(\frac{S_j^{(K)}}{n_j} - \frac{(N^2-1)}{12}\right)^2, \\ S_j^{(K)} &= \sum_{i=1}^{n_j}\left(R_{j,i} - \frac{N+1}{2}\right)^2, \qquad \sigma_N^2 = \frac{N(N+1)(N^2-4)}{180}\,. \end{aligned} \tag{2.10}$$

is the multisample version of the Mood test statistic. The next theorem can be proved by means of the assertion (II) of Theorem 3.1 of [12].

Theorem 2.1. Suppose that (2.2) holds. Then the weak convergences of distributions (cf. (2.4), (2.5), (2.9))

$$\mathcal{L}(T) \longrightarrow \chi^2_{2(k-1)}\,, \quad \mathcal{L}(T_\Phi) \longrightarrow \chi^2_{2(k-1)}\,, \quad \mathcal{L}(T_{SQ}) \longrightarrow \chi^2_{2(k-1)} \tag{2.11}$$

hold as $n_1 \to \infty, \ldots, n_k \to \infty$. Here $\chi^2_{2(k-1)}$ denotes the chi-square distribution with $2(k-1)$ degrees of freedom.

The test based on any of the statistics T, T_Φ and T_{SQ} rejects the null hypothesis (2.2) for large values of the employed test statistic. In accordance with the previous theorem the null hypothesis is rejected at the asymptotic significance level α whenever the observed value of the employed test statistic is larger than the $(1-\alpha)$th quantile $\chi^2_{2(k-1),1-\alpha}$ of the chi-square distribution with $2(k-1)$ degrees of freedom. In what follows we shall express the performance of these tests by means of asymptotic non-centrality parameters and illustrate the behaviour of their finite sample power by the simulation estimates. For this purpose we have chosen the normal, the logistic and the Cauchy distribution, because they have tails with different asymptotic behaviour.

To make the notation more precise for the sake of handling the Pitman alternatives, assume that the sample size from the jth population $n_j = n_j^{(u)}$, where $u = 1, 2, \ldots$ denotes the index of the experiment; thus also the total sample size $N = N_u$. We remark that the parts of the statements (2.11), (2.17) and (2.18) concerning $T = T_K + T_B$ have already been proved in [12], but for the sake of completeness they are included also into this paper.

(A 1) The sample sizes are such that

$$\lim_{u\to\infty} n_j^{(u)} = +\infty\,, \quad j = 1, \ldots, k\,, \tag{2.12}$$

and for the relative sample sizes

$$\hat{p}_j = \hat{p}_j^{(u)} = \frac{n_j^{(u)}}{N_u} \tag{2.13}$$

the relations

$$\lim_{u\to\infty} \hat{p}_j^{(u)} = p_j > 0\,, \qquad j = 1, \ldots, k \tag{2.14}$$

hold.

(A 2) The location and scale parameters of the jth population

$$\begin{gathered} \mu_j = \mu_j^{(u)} = \mu + \frac{\mu_j^{*(u)}}{\sqrt{N_u}}\,, \quad \sigma_j = \sigma_j^{(u)} = \sigma + \frac{\sigma_j^{*(u)}}{\sqrt{N_u}}\,, \\ \sigma > 0,\ \mu\,,\ \sigma_j^{*(u)}\,,\ \mu_j^{*(u)} \text{ are real numbers} \\ \text{for } j = 1, \ldots, k \text{ limits } \mu_j^* = \lim_{u\to\infty} \mu_j^{*(u)}\,,\ \ \sigma_j^* = \lim_{u\to\infty} \sigma_j^{*(u)} \text{ are real numbers.} \end{gathered} \tag{2.15}$$

The following theorem can be proved by means of Theorem 3.1 and Lemma 3.2 from the next section.

Theorem 2.2. Assume that (A 1), (A 2) hold and put

$$\overline{\mu} = \sum_{j=1}^{k} p_j \mu_j^*\,,\ \ \overline{\sigma} = \sum_{j=1}^{k} p_j \sigma_j^*\,,\ \ c_1 = \sum_{j=1}^{k} p_j \frac{(\mu_j^* - \overline{\mu})^2}{\sigma^2}\,,\ \ c_2 = \sum_{j=1}^{k} p_j \frac{(\sigma_j^* - \overline{\sigma})^2}{\sigma^2}\,. \tag{2.16}$$

(I) Suppose that F from (2.1) is the distribution function of the normal $N(0,1)$ distribution. Then the weak convergences of distributions

$$\mathcal{L}(T) \longrightarrow \chi^2_{2(k-1)}(\delta_T)\,,\quad \mathcal{L}(T_\Phi) \longrightarrow \chi^2_{2(k-1)}(\delta_\Phi)\,,\ \mathcal{L}(T_{SQ}) \longrightarrow \chi^2_{2(k-1)}(\delta_{SQ}) \tag{2.17}$$

hold as $u \to \infty$. Here the non-centrality parameters

$$\delta_T = \frac{3}{\pi} c_1 + \frac{12}{\pi^2} c_2\,,\quad \delta_\Phi = c_1 + 2c_2\,,\quad \delta_{SQ} = \frac{3}{\pi} c_1 + \frac{15}{\pi^2} c_2\,. \tag{2.18}$$

(II) Suppose that F from (2.1) is the distribution function of the logistic distribution with the density $f(x) = \exp(x)/(1+\exp(x))^2$. Then (2.17) holds with

$$\delta_T = \frac{1}{3} c_1 + \frac{(4\ln(2) - 1)^2}{3} c_2\,,\quad \delta_\Phi = \frac{1}{\pi} c_1 + 2\gamma_{lo}^2 c_2\,,\quad \delta_{SQ} = \frac{1}{3} c_1 + \frac{5}{4} c_2\,, \tag{2.19}$$

$$\gamma_{lo} = \int_0^1 \Phi^{-1}(t) \exp\Big((\Phi^{-1}(t))^2/2\Big) \sqrt{2\pi} t(1-t)\, \ln\Big(\frac{t}{1-t}\Big)\, \mathrm{d}t\,. \tag{2.20}$$

(III) Suppose that F from (2.1) is the distribution function of the Cauchy distribution with the density $f(x) = 1/(\pi(1+x^2))$. Then (2.17) holds with

$$\delta_T = \frac{3}{\pi^2} c_1 + \frac{48}{\pi^4} c_2\,,\quad \delta_\Phi = \frac{2}{\pi} \gamma_1^2 c_1 + \frac{1}{\pi} \gamma_2^2 c_2\,,\quad \delta_{SQ} = \frac{3}{\pi^2} c_1 + \frac{45}{\pi^4} c_2\,, \tag{2.21}$$

where

$$\begin{aligned}\gamma_1 &= \int_0^1 \exp\Big((\Phi^{-1}(t))^2/2\Big)\cos^2\Big((t-0.5)\pi\Big)\,\mathrm{d}t\,,\\ \gamma_2 &= \int_0^1 \Phi^{-1}(t)\exp\Big((\Phi^{-1}(t))^2/2\Big)\sin\Big((2t-1)\pi\Big)\,\mathrm{d}t\,.\end{aligned} \tag{2.22}$$

According to the previous theorem in the considered setting the asymptotic efficiency $e_{T^*,T^{**}} = \frac{\delta_{T^*}}{\delta_{T^{**}}}$ of the tests based on the statistics T^*, T^{**} depends on the value of the local alternative (2.15). By means of the built-in MATLAB function Φ^{-1} one obtains for the constants from (2.20) and (2.22) the values $\gamma_{lo} = 0.836$, $\gamma_1 = 0.581$ and $\gamma_2 = 0.930$, and since

$$\frac{\alpha^* c_1 + \beta^* c_2}{\alpha^{**} c_1 + \beta^{**} c_2} \in \Big\langle \min\Big(\frac{\alpha^*}{\alpha^{**}}, \frac{\beta^*}{\beta^{**}}\Big)\,,\, \max\Big(\frac{\alpha^*}{\alpha^{**}}, \frac{\beta^*}{\beta^{**}}\Big)\Big\rangle\,,$$

the results of the previous theorem can be organized into the following table.

Sampling from normal distribution		
$e_{T_\Phi,T} \in \langle 1.05\,,\, 1.65\rangle$	$e_{T_\Phi,T_{SQ}} \in \langle 1.05\,,\, 1.32\rangle$	$e_{T_{SQ},T} \in \langle 1\,,\, 1.25\rangle$
Sampling from logistic distribution		
$e_{T_\Phi,T} \in \langle 0.95\,,\, 1.33\rangle$	$e_{T_\Phi,T_{SQ}} \in \langle 0.95\,,\, 1.12\rangle$	$e_{T_{SQ},T} \in \langle 1\,,\, 1.19\rangle$
Sampling from Cauchy distribution		
$e_{T_\Phi,T} \in \langle 0.56\,,\, 0.71\rangle$	$e_{T_\Phi,T_{SQ}} \in \langle 0.60\,,\, 0.71\rangle$	$e_{T_{SQ},T} \in \langle 0.94\,,\, 1\rangle$

These results suggest that for the distributions with light tails (like the normal distribution) the best choice for testing (2.2) is the test based on the statistic (2.5). As has been explained in [12], the non-centrality parameter of the likelihood ratio statistic for testing (2.2) under the normality assumptions equals δ_Φ from (2.18). Hence if the sampling is drawn from normal distribution then the van der Waerden scores test statistic (2.5) has the non-centrality parameter the same as the LRT statistic, which is under the normality assumptions asymptotically optimal for testing (2.2). However, when one wants to have a test which performs well for distributions with heavy tails like the Cauchy distribution, then the test based on the statistic (2.4) is recommendable, the test statistic (2.9) is a compromise between T_Φ and T.

As has been already mentioned, these tests are based on the approximation of the critical constants by $(1-\alpha)$th quantile $\chi^2_{2(k-1),1-\alpha}$ of the chi-square distribution with $2(k-1)$ degrees of freedom. First we present some simulation estimates of the fit of the size of such a test with the chosen significance level α. In all the following tables we use the abbreviations $P(T,\alpha) = \mathrm{P}(T > \chi^2_{2(k-1),1-\alpha})$, $P(T_\Phi,\alpha) = \mathrm{P}(T_\Phi > \chi^2_{2(k-1),1-\alpha})$ and $P(T_{SQ},\alpha) = \mathrm{P}(T_{SQ} > \chi^2_{2(k-1),1-\alpha})$. Each particular simulation estimate is based on $N = 10000$ trials and is obtained by means of the MATLAB version 7.1.0.246 (R14) Service Pack 3.

The simulation results from the Tables 1 and 2 suggest that for the test based on T or on T_{SQ} and $k = 3$ or $k = 4$ the discrepancy between the actual size of the test and the nominal level $\alpha = 0.05$ or $\alpha = 0.1$ attains acceptable values if the minimal sample size is at least 10, for the test based on the normal scores statistic T_Φ this occurs when the minimal sample size is at least 18.

Table 1. Simulation estimates of the tail probabilities under validity of (2.2) for $k = 3$.

α	0.05	0.1	0.05	0.1	0.05	0.1
n_1, n_2, n_3	6 6 6		10 10 10		10 10 15	
$P(T, \alpha)$	0.032	0.078	0.040	0.093	0.042	0.091
$P(T_\Phi, \alpha)$	0.022	0.065	0.032	0.083	0.036	0.081
$P(T_{SQ}, \alpha)$	0.030	0.075	0.039	0.090	0.042	0.087

α	0.05	0.1	0.05	0.1	0.05	0.1	0.05	0.1
n_1, n_2, n_3	10 15 15		15 15 15		18 18 18		20 20 20	
$P(T, \alpha)$	0.042	0.101	0.042	0.094	0.048	0.096	0.048	0.099
$P(T_\Phi, \alpha)$	0.037	0.086	0.036	0.084	0.042	0.087	0.039	0.095
$P(T_{SQ}, \alpha)$	0.041	0.096	0.041	0.092	0.048	0.097	0.048	0.101

Table 2. Simulation estimates of the tail probabilities under validity of (2.2) for $k = 4$.

α	0.05	0.1	0.05	0.1	0.05	0.1	0.05	0.1
n_1, n_2, n_3, n_4	6 6 6 6		6 6 10 10		10 10 10 10		10 10 10 15	
$P(T, \alpha)$	0.034	0.081	0.037	0.085	0.040	0.090	0.040	0.090
$P(T_\Phi, \alpha)$	0.026	0.069	0.033	0.077	0.035	0.084	0.034	0.079
$P(T_{SQ}, \alpha)$	0.032	0.082	0.036	0.083	0.039	0.091	0.039	0.087

α	0.05	0.1	0.05	0.1	0.05	0.1	0.05	0.1
n_1, n_2, n_3 n_4	10 10 15 15		15 15 15 15		18 18 18 18		20 20 20 20	
$P(T, \alpha)$	0.039	0.089	0.041	0.090	0.047	0.099	0.047	0.098
$P(T_\Phi, \alpha)$	0.035	0.083	0.036	0.086	0.042	0.093	0.042	0.94
$P(T_{SQ}, \alpha)$	0.039	0.086	0.041	0.092	0.046	0.097	0.045	0.096

In the following tables of simulation estimates of the power under the alternative the best result for the given combination of sample sizes is printed in boldface letters. In accordance with the notation from the introduction of the paper μ_j denotes the location parameter and σ_j the scale parameter of the jth population. These simulation power estimates are in a fair agreement with the asymptotic theoretical results of Theorem 2.2, but as they show, it may happen in some cases that despite of the sampling from the normal populations (when the test based on T_Φ is asymptotically the best of the mentioned competitors), the power of the T_Φ based test does not exert its influence for medium sample sizes (the case $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 0.3$, $\sigma_2 = 1.5$, $\mu_3 = 0.8$, $\sigma_3 = 2$ and $n_1 = 15$, $n_2 = 25$, $n_3 = 35$). The superiority of the T based test when the sampling is drawn from Cauchy distribution appears to be in effect already for small sample sizes. However, as the knowledge of the type of the distribution need not be available and the T_{SQ} based test has in the simulations always got the rating either as the best or as the second best procedure, it is advisable to use the T_{SQ} based test in the case when the type of the distribution is unknown or when the maximum sample size is not larger than 15.

Table 3. Simulation estimates of the power for $k = 3$.

	$\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 0.3$, $\sigma_2 = 1.5$, $\mu_3 = 0.8$, $\sigma_3 = 2$					
α	0.05	0.1	0.05	0.1	0.05	0.1
	Sampling from normal distribution					
n_1, n_2, n_3	10 10 10		10 15 15		15 15 15	
$P(T, \alpha)$	0.223	0.366	0.262	0.413	0.389	0.540
$P(T_\Phi, \alpha)$	0.205	0.362	0.238	0.401	0.409	0.582
$P(T_{SQ}, \alpha)$	**0.241**	**0.393**	**0.280**	**0.444**	**0.432**	**0.588**
n_1, n_2, n_3	15 25 35		30 30 30		35 35 35	
$P(T, \alpha)$	0.539	0.691	0.770	0.862	0.851	0.918
$P(T_\Phi, \alpha)$	0.517	0.670	**0.845**	**0.925**	**0.920**	**0.965**
$P(T_{SQ}, \alpha)$	**0.579**	**0.735**	0.828	0.906	0.901	0.952
	Sampling from logistic distribution					
n_1, n_2, n_3	10 10 10		10 15 15		15 15 15	
$P(T, \alpha)$	0.167	0.289	0.196	0.319	0.288	0.423
$P(T_\Phi, \alpha)$	0.150	0.280	0.158	0.288	0.274	0.432
$P(T_{SQ}, \alpha)$	**0.178**	**0.310**	**0.198**	**0.335**	**0.311**	**0.457**
n_1, n_2, n_3	15 25 35		30 30 30		35 35 35	
$P(T, \alpha)$	0.398	0.544	0.612	0.738	0.695	0.800
$P(T_\Phi, \alpha)$	0.335	0.514	0.665	**0.799**	**0.759**	**0.865**
$P(T_{SQ}, \alpha)$	**0.422**	**0.584**	**0.677**	0.792	0.757	0.854
	Sampling from Cauchy distribution					
n_1, n_2, n_3	10 10 10		10 15 15		15 15 15	
$P(T, \alpha)$	**0.111**	**0.204**	**0.122**	**0.217**	**0.170**	**0.277**
$P(T_\Phi, \alpha)$	0.080	0.164	0.079	0.159	0.119	0.222
$P(T_{SQ}, \alpha)$	0.101	0.197	0.114	0.205	0.163	0.269
n_1, n_2, n_3	15 25 35		30 30 30		35 35 35	
$P(T, \alpha)$	**0.221**	**0.345**	**0.347**	**0.489**	**0.419**	**0.549**
$P(T_\Phi, \alpha)$	0.122	0.230	0.238	0.363	0.286	0.413
$P(T_{SQ}, \alpha)$	0.204	0.328	0.334	0.473	0.340	0.530
	$\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 0.3$, $\sigma_2 = 1.5$, $\mu_3 = 0.4$, $\sigma_3 = 0.8$					
α	0.05	0.1	0.05	0.1	0.05	0.1
	Sampling from normal distribution					
n_1, n_2, n_3	10 10 10		10 15 15		15 15 15	
$P(T, \alpha)$	0.184	0.314	0.280	0.422	0.309	0.450
$P(T_\Phi, \alpha)$	0.195	0.341	0.312	**0.483**	**0.372**	**0.527**
$P(T_{SQ}, \alpha)$	**0.208**	**0.342**	**0.325**	0.469	0.361	0.503
n_1, n_2, n_3	15 25 35		30 30 30		35 35 35	
$P(T, \alpha)$	0.587	0.711	0.647	0.769	0.725	0.826
$P(T_\Phi, \alpha)$	**0.740**	**0.844**	**0.787**	**0.881**	**0.861**	**0.927**
$P(T_{SQ}, \alpha)$	0.676	0.786	0.729	0.835	0.803	0.884
	Sampling from logistic distribution					
n_1, n_2, n_3	10 10 10		10 15 15		15 15 15	
$P(T, \alpha)$	0.142	0.254	0.213	0.345	0.236	0.363
$P(T_\Phi, \alpha)$	0.146	0.260	0.226	0.374	0.262	**0.407**
$P(T_{SQ}, \alpha)$	**0.158**	**0.273**	**0.245**	**0.380**	**0.268**	0.406
n_1, n_2, n_3	15 25 35		30 30 30		35 35 35	
$P(T, \alpha)$	0.470	0.602	0.512	0.641	0.578	0.707
$P(T_\Phi, \alpha)$	**0.583**	**0.717**	**0.609**	**0.740**	**0.687**	**0.801**
$P(T_{SQ}, \alpha)$	0.546	0.675	0.581	0.709	0.655	0.773
	Sampling from Cauchy distribution					
n_1, n_2, n_3	10 10 10		10 15 15		15 15 15	
$P(T, \alpha)$	**0.090**	**0.175**	**0.121**	**0.218**	**0.138**	**0.235**
$P(T_\Phi, \alpha)$	0.069	0.147	0.085	0.178	0.103	0.190
$P(T_{SQ}, \alpha)$	0.089	0.174	0.116	0.215	0.134	0.232
n_1, n_2, n_3	15 25 35		30 30 30		35 35 35	
$P(T, \alpha)$	**0.242**	**0.368**	**0.271**	**0.401**	**0.317**	**0.444**
$P(T_\Phi, \alpha)$	0.181	0.292	0.196	0.310	0.223	0.340
$P(T_{SQ}, \alpha)$	0.239	0.366	0.268	0.393	0.309	0.435

Here should be noted that the multiple comparisons procedure can be easily derived in the same way as Theorem 2.4 of [12] concerning the statistic (2.4). Thus ignoring for a while the index u of experiment and putting

$$D^{(\varphi)}_{j_1,j_2} = \frac{\dfrac{S^{(\varphi)}_{j_1}}{n_{j_1}} - \dfrac{S^{(\varphi)}_{j_2}}{n_{j_2}}}{\sqrt{\dfrac{1}{n_{j_1}} + \dfrac{1}{n_{j_2}}}} \sqrt{\frac{2}{\sigma^{2,\varphi}_N}}\,, \tag{2.23}$$

after the rejection of the null hypothesis by the test statistic declare the j_1th and the j_2th populations to be different, if at least one of the inequalities

$$|D^{(\varphi)}_{j_1,j_2}| > Q^{(\beta)}_k\,, \qquad |D^{(\psi)}_{j_1,j_2}| > Q^{(\beta)}_k\,, \tag{2.24}$$

holds; here $\beta = 1 - \sqrt{1-\alpha}$ and the rejection constant is defined by the equality $\mathrm{P}\Big(\max_{1\le i,j\le k} |y_i - y_j| > Q^{(\alpha)}_k \Big| \mathcal{L}(y) = N_k(\mathbf{0}, \mathbf{I}_k)\Big) = \alpha$, the approximation $\beta = 0.5\alpha$ is recommendable. If (2.2) is rejected by T_Φ, then $S^{(\varphi)}_j$, $\sigma^{2,\varphi}_N$ are defined by means of $\varphi(u) = \Phi^{-1}(u)$ and (2.6), (2.7), and $S^{(\psi)}_j$, $\sigma^{2,\psi}_N$ by $\psi(u) = (\Phi^{-1}(u))^2$. Analogously, if the null hypothesis is rejected by T_{SQ}, then $S^{(\varphi)}_j = S_j$, $\sigma^{2,\varphi}_N = w^2_N$, are the quantities $S_j = \sum_{i=1}^{n_j} R_{j,i}$, $w^2_N = \frac{N(N+1)}{12}$, and $S^{(\psi)}_j = S^{(K)}_j$, $\sigma^{2,\varphi}_N = \sigma^2_N$ are the quantities from (2.10).

Now we are going to pay attention to the modification of the previous test statistics for ties. In doing this we use first a general framework to achieve a more concise style. Throughout the rest of this section the distribution function (2.1) is not assumed to be continuous.

In accordance with [6], [1] and [7] the average scores are defined as follows. Suppose that $a_N(1), \ldots, a_N(N)$ are real numbers and $\tau = (\tau_1, \ldots, \tau_L)$ is a vector of positive integers such that $\tau_1 + \ldots + \tau_L = N$. Then by the average scores $\tilde{a}_N(i|\tau)$ modified for τ we understand the scores defined for $i = 1, \ldots, N$ by the formula

$$\tilde{a}_N(i|\tau) = \frac{1}{\tau_j} \sum_{t=1}^{\tau_j} a_N(\tau_1 + \ldots \tau_{j-1} + t) \quad \text{if } \tau_1 + \ldots + \tau_{j-1} < i \le \tau_1 + \ldots + \tau_{j-1} + \tau_j\,, \tag{2.25}$$

where for $j = 1$ we put $\tau_1 + \ldots + \tau_{j-1} = 0$. Let $Z^{(1)} \le \ldots \le Z^{(N)}$ denote the ordering of the pooled sample (2.3) according to the magnitude and

$$Z^{(1)} = \ldots = Z^{(\tau_1)} < Z^{(\tau_1+1)} = \ldots = Z^{(\tau_1+\tau_2)}\,,$$
$$Z^{(\tau_1+\ldots+\tau_{j-1})} < Z^{(\tau_1+\ldots+\tau_{j-1}+1)} = \ldots = Z^{(\tau_1+\ldots+\tau_{j-1}+\tau_j)}\,, \qquad j = 3, \ldots, L\,,$$
$$\tau_1 + \ldots + \tau_L = N\,.$$

Hence τ_j is the number of the elements of $Z^{(\cdot)}$ in the jth block of the ordering according to the magnitude and

$$\tau_N(Z) = (\tau_1, \ldots, \tau_L) \tag{2.26}$$

is called the vector of numbers of ties in Z.

The modification of the quadratic rank test statistic for ties is obtained by plugging the modified scores into the formula intended for setting where no ties occur. Assume for this purpose that the function $\varphi : (0,1) \to \mathbb{R}^1$ and define the scores and their arithmetic mean by the formulas

$$a_N^{(\varphi)}(i) = \varphi\Big(\frac{i}{N+1}\Big), \quad \overline{a}_N^{(\varphi)} = \frac{1}{N}\sum_{i=1}^{N} a_N^{(\varphi)}(i)\,. \tag{2.27}$$

In accordance with (2.25) by the scores $\tilde{a}_N^{(\varphi)}(i|\tau_N(Z))$ modified for ties we understand the numbers

$$\tilde{a}_N^{(\varphi)}(i|\tau_N(Z)) = \frac{1}{\tau_j}\sum_{t=1}^{\tau_j} a_N^{(\varphi)}(\tau_1 + \dots \tau_{j-1} + t) \quad \text{if } \sum_{s=1}^{j-1}\tau_s < i \le \sum_{s=1}^{j}\tau_s\,, \tag{2.28}$$

where $\tau_1, \dots, \tau_L$ are the integers from (2.26).

The k-sample quadratic rank statistic devised for setting where no ties occur is defined by the formula (2.6). Now let

$$\tilde{R}_{j,i} = \#\{(r,v);\, X_{r,v} \le X_{j,i}\,,\, r = 1,\dots,k\,,\, v = 1,\dots,n_r\,\}\,, \tag{2.29}$$

where as before, n_r denotes the size of sample from the rth population. Put (cf. (2.8), (2.28))

$$\begin{gathered}\tilde{S}_{j,N}^{(\varphi)} = \sum_{i=1}^{n_j} \tilde{a}_N^{(\varphi)}(\tilde{R}_{j,i}|\tau_N(Z))\,, \quad \tilde{\sigma}_N^{2,\varphi} = \tilde{\sigma}_N^{2,\varphi,\varphi}\,, \\ \tilde{\sigma}_N^{2,\varphi,\psi} = \frac{1}{N-1}\sum_{i=1}^{N}\Big(\tilde{a}_N^{(\varphi)}(i|\tau_N(Z)) - \overline{a}_N^{(\varphi)}\Big)\Big(\tilde{a}_N^{(\psi)}(i|\tau_N(Z)) - \overline{a}_N^{(\psi)}\Big)\,.\end{gathered} \tag{2.30}$$

Then

$$\tilde{Q}^{(\varphi)} = \tilde{Q}_{n_1,\dots,n_k}^{(\varphi)} = \frac{1}{\tilde{\sigma}_N^{2,\varphi}}\sum_{j=1}^{k} n_j\left(\frac{\tilde{S}_{j,N}^{(\varphi)}}{n_j} - \overline{a}_N^{(\varphi)}\right)^2 \tag{2.31}$$

is the statistic (2.6) modified for ties. We shall carry out the modification of location-scale rank tests by means of these quantities. To achieve convergence in distribution we impose this regularity assumption.

(A 3) The null hypothesis (2.2) hold. Let the functions $\varphi : (0,1) \to \mathbb{R}^1$, $\psi : (0,1) \to \mathbb{R}^1$ be expressible as a finite sum of monotone square integrable functions, F is the function (2.1),

$$\mathcal{C} = \sigma(F) \tag{2.32}$$

denote the σ-ring of subsets of $(0,1)$ generated by the intervals $\{(0, F(t));\, t \in \mathbb{R}^1\}$, and $\mathrm{E}[\varphi\,|\,\mathcal{C}]$ denote the function of the argument $t \in (0,1)$ fulfilling the equality

$$\int_A \mathrm{E}[\varphi\,|\,\mathcal{C}](t)\,\mathrm{d}t = \int_A \varphi(t)\,\mathrm{d}t \tag{2.33}$$

for every set $A \in \sigma(F)$. Put

$$\overline{\varphi} = \int_0^1 \varphi(t)\,\mathrm{d}t\,, \qquad \overline{\psi} = \int_0^1 \psi(t)\,\mathrm{d}t\,, \tag{2.34}$$

$$V_{\tilde{\varphi}} = \int_0^1 \Big(\mathrm{E}[\varphi\,|\,\mathcal{C}](t) - \overline{\varphi}\Big)^2\,\mathrm{d}t\,, \quad V_{\tilde{\psi}} = \int_0^1 \Big(\mathrm{E}[\psi\,|\,\mathcal{C}](t) - \overline{\psi}\Big)^2\,\mathrm{d}t\,, \tag{2.35}$$

$$V_{\tilde{\varphi},\tilde{\psi}} = \int_0^1 \Big(\mathrm{E}[\varphi\,|\,\mathcal{C}](t) - \overline{\varphi}\Big)\Big(\mathrm{E}[\psi\,|\,\mathcal{C}](t) - \overline{\psi}\Big)\,\mathrm{d}t\,. \tag{2.36}$$

The matrix

$$\widetilde{\mathbf{V}} = \begin{pmatrix} V_{\tilde{\varphi}} & V_{\tilde{\varphi},\tilde{\psi}} \\ V_{\tilde{\varphi},\tilde{\psi}} & V_{\tilde{\psi}} \end{pmatrix} \tag{2.37}$$

is regular.

Theorem 2.3. Let the functions $\varphi : (0,1) \to \mathbb{R}^1$, $\psi : (0,1) \to \mathbb{R}^1$ and in accordance with (2.27)–(2.31) put

$$\tilde{T}_{n_1,\dots,n_k} = \frac{\tilde{\sigma}_N^{2,\varphi}\,\tilde{\sigma}_N^{2,\psi}}{\tilde{\sigma}_N^{2,\varphi}\tilde{\sigma}_N^{2,\psi} - (\tilde{\sigma}_N^{2,\varphi,\psi})^2}[\tilde{Q}^{(\varphi)} + \tilde{Q}^{(\psi)} - \tilde{Q}^{(\varphi,\psi)}]\,, \tag{2.38}$$

$$\tilde{Q}^{(\varphi,\psi)} = 2\frac{\tilde{\sigma}_N^{2,\varphi,\psi}}{\tilde{\sigma}_N^{2,\varphi}\tilde{\sigma}_N^{2,\psi}}\sum_{j=1}^k n_j\left(\frac{\tilde{S}_{j,N}^{(\varphi)}}{n_j} - \overline{a}_N^{(\varphi)}\right)\left(\frac{\tilde{S}_{j,N}^{(\psi)}}{n_j} - \overline{a}_N^{(\psi)}\right). \tag{2.39}$$

If both (2.12) and (A 3) are fulfilled then the convergence in distribution

$$\tilde{T}_{n_1,\dots,n_k} \longrightarrow \chi^2_{2(k-1)} \tag{2.40}$$

to the chi-square distribution with $2(k-1)$ degrees of freedom holds as $n_1 \to \infty, \dots, n_k \to \infty$.

An essential condition of this theorem is the assumption (A 3). Its validity can be verified by means of the following sufficient conditions.

Lemma 2.1. Let there exist real numbers z_1, z_2, z_3 such that

$$0 < F(z_1) < F(z_2) < F(z_3) < 1\,. \tag{2.41}$$

Suppose that $\varphi : (0,1) \to \mathbb{R}^1$, $\psi : (0,1) \to \mathbb{R}^1$ are square integrable functions and φ is strictly increasing. Put

$$\varphi_j^* = \frac{1}{F_j - F_{j-1}}\int_{F_{j-1}}^{F_j} \varphi(t)\,\mathrm{d}t\,, \quad \psi_j^* = \frac{1}{F_j - F_{j-1}}\int_{F_{j-1}}^{F_j} \psi(t)\,\mathrm{d}t\,,$$

where $F_0 = 0$, $F_j = F(z_j)$, $F_4 = 1$.

(I) Let for each $\delta \in \langle 0, \frac{1}{2})$

$$\varphi(\frac{1}{2} - \delta) + \varphi(\frac{1}{2} + \delta) = 1. \tag{2.42}$$

If $\psi(t) = \min\{\,\varphi(t)\,,\,1-\varphi(t)\,\}$, then the matrix (2.37) is regular.

(II) Let $\varphi(\frac{1}{2}+\delta) = -\varphi(\frac{1}{2}-\delta)$ for each $\delta \in \langle 0, \frac{1}{2})$ and $\psi(t) = \varphi(t)^2$. If either

$$\frac{\varphi_3^* - \varphi_2^*}{\varphi_1^* - \varphi_2^*} \neq \frac{\psi_3^* - \psi_2^*}{\psi_1^* - \psi_2^*}\,, \qquad \frac{\varphi_2^* - \varphi_3^*}{\varphi_4^* - \varphi_3^*} \neq \frac{\psi_2^* - \psi_3^*}{\psi_4^* - \psi_3^*} \tag{2.43}$$

or

$$\frac{\psi_1^* - \overline{\psi}}{\varphi_1^*} \neq \frac{\psi_2^* - \overline{\psi}}{\varphi_2^*}\,, \qquad \frac{\psi_4^* - \overline{\psi}}{\varphi_4^*} \neq \frac{\psi_3^* - \overline{\psi}}{\varphi_3^*} \tag{2.44}$$

then the matrix (2.37) is regular.

First we apply these results to the statistic (2.4). Let $\tilde{T}_K$ denote the Kruskal–Wallis test statistic modified for ties, i. e.,

$$\tilde{T}_K = \frac{1}{\tilde{\sigma}_N^2} \sum_{j=1}^{k} n_j \Big(\frac{\tilde{S}_j}{n_j} - \frac{N+1}{2} \Big)\,, \tag{2.45}$$

$$\tilde{S}_j = \sum_{i=1}^{n_j} R_{j,i}\,, \qquad \tilde{\sigma}_N^2 = \frac{1}{N-1} \sum_{j=1}^{k} \sum_{i=1}^{n_j} \Big(R_{j,i} - \frac{N+1}{2} \Big)^2 \tag{2.46}$$

and $R_{j,i}$ is the usual midrank of X_{ji} computed from the pooled sample (2.3). Further, let $b_N = (1, 2, 3, \ldots, m, m, \ldots, 3, 2, 1)$ and $b_N = (1, 2, 3, \ldots, m, \frac{N+1}{2}, m, \ldots, 3, 2, 1)$ for $N = 2m$ and $N = 2m+1$, respectively, i. e., if $b_N(i)$ stands for the ith coordinate of b_N, then $b_N(1), \ldots, b_N(N)$ are the score of the Ansari–Bradley test statistic. In accordance with (2.25), (2.26) let

$$\tilde{b}_N(i) = \frac{1}{\tau_j} \sum_{t=1}^{\tau_j} b_N(\tau_1 + \ldots \tau_{j-1} + t) \quad \text{ if } \tau_1 + \ldots + \tau_{j-1} < i \leq \tau_1 + \ldots + \tau_{j-1} + \tau_j\,,$$

denote their modification for ties. Put (cf. (2.29))

$$\tilde{S}_j^{(b)} = \sum_{i=1}^{n_j} \tilde{b}_N(\tilde{R}_{j,i})\,, \quad \tilde{v}_N^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\tilde{b}_N(i) - \lambda_N)^2\,, \quad \tilde{T}_B = \frac{1}{\tilde{v}_N^2} \sum_{j=1}^{k} n_j \Big(\frac{\tilde{S}_j^{(b)}}{n_j} - \lambda_N \Big)^2,$$

where $\lambda_N = \frac{(N+2)}{4}$ if N is even and $\lambda_N = \frac{(N+1)^2}{4N}$ if N is odd, i. e., $\tilde{T}_B$ denotes the Ansari–Bradley test statistics modified for ties. Use the notation from these two steps and put

$$\tilde{r}_N^{(1)} = \sum_{j=1}^{k} \sum_{i=1}^{n_j} \Big(R_{j,i} - \frac{N+1}{2} \Big)^2, \qquad \tilde{r}_N^{(2)} = \sum_{i=1}^{N} (\tilde{b}_N(i) - \lambda_N)^2\,, \tag{2.47}$$

$$\tilde{r}_N^{(1,2)} = \sum_{j=1}^{k} \sum_{i=1}^{n_j} \Big(R_{j,i} - \frac{N+1}{2} \Big) \Big(\tilde{b}_N(\tilde{R}_{j,i}) - \lambda_N \Big)\,,$$

$$\tilde{T}_M = 2(N-1) \frac{\tilde{r}_N^{(1,2)}}{\tilde{r}_N^{(1)} \tilde{r}_N^{(2)}} \sum_{j=1}^{k} n_j \left(\frac{\tilde{S}_j}{n_j} - \frac{N+1}{2} \right) \left(\frac{\tilde{S}_j^{(b)}}{n_j} - \lambda_N \right).$$

Then

$$\tilde{T} = \frac{\tilde{r}_N^{(1)}\tilde{r}_N^{(2)}}{\tilde{r}_N^{(1)}\tilde{r}_N^{(2)} - (\tilde{r}_N^{(1,2)})^2}\Big[\tilde{T}_K + \tilde{T}_B - \tilde{T}_M\Big] \tag{2.48}$$

is the modification of the statistic (2.4) for ties. If (2.41) holds for some z_1, z_2, z_3, then an application of Theorem 2.3 and Lemma 2.1 (I) yields that the statistic (2.48) converges in distribution to chi-square distribution with $2(k-1)$ degrees of freedom as $n_1 \to \infty, \ldots, n_k \to \infty$ provided that (2.2) holds.

To handle the statistic (2.9) put $c_N(i) = (i - (N+1)/2)^2$, $i = 1, \ldots, N$. Let $\tilde{c}_N(i|\tau_N(Z))$ denote in accordance with (2.25) and (2.26) the modification of these scores for ties,

$$\tilde{S}_j^{(K)} = \sum_{i=1}^{n_j} \tilde{c}_N(\tilde{R}_{j,i}|\tau_N(Z))\,, \qquad \tilde{\sigma}_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}\Big(\tilde{c}_N(i|\tau_N(Z)) - \frac{N^2-1}{12}\Big)^2,$$

$$\tilde{Q} = \frac{1}{\tilde{\sigma}_N^2}\sum_{j=1}^{k} n_j\Big(\frac{\tilde{S}_j^{(K)}}{n_j} - \frac{(N^2-1)}{12}\Big)^2,$$

and (cf. (2.47), (2.46))

$$r_N^{*(1)} = \tilde{r}_N^{(1)}\,, \;\; r_N^{*(2)} = \sum_{i=1}^{N}(\tilde{c}_N(i) - \frac{N^2-1}{12})^2\,,$$

$$r_N^{*(1,2)} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\Big(R_{j,i} - \frac{N+1}{2}\Big)\Big(\tilde{c}_N(\tilde{R}_{j,i}) - \frac{N^2-1}{12}\Big)\,,$$

$$T_M^* = 2(N-1)\frac{r_N^{*(1,2)}}{r_N^{*(1)}r_N^{*(2)}}\sum_{j=1}^{k} n_j\left(\frac{\tilde{S}_j}{n_j} - \frac{N+1}{2}\right)\left(\frac{\tilde{S}_j^{(K)}}{n_j} - \frac{N^2-1}{12}\right).$$

Then

$$\tilde{T}_{SQ} = \frac{r_N^{*(1)}r_N^{*(2)}}{r_N^{*(1)}r_N^{*(2)} - (r_N^{*(1,2)})^2}\Big[\tilde{T}_K + \tilde{Q} - T_M^*\Big] \tag{2.49}$$

is the modification of the statistic T_{SQ} for ties. An application of Theorem 2.3 and (2.43) yields that under the validity of (2.2) and (2.41) for some z_1, z_2, z_3, the statistic $\tilde{T}_{SQ}$ has asymptotically chi-square distribution with $2(k-1)$ degrees of freedom.

The modification $\tilde{T}_\Phi$ of the statistic T_Φ from (2.5) can be computed directly by means of Theorem 2.3 and its convergence (2.40) in distribution can be established by means of (2.44).

3. PROOFS

The aim of the first two assumptions is to ensure the validity of the Chernoff–Savage theorem, the assumption (AS 4) will be used for functions φ, ψ satisfying (AS 1) and

(AS 2). In this part of the paper until the end of the proof of Lemma 3.2 the distribution function (2.1) is assumed to have the form $F(x) = \int_{-\infty}^{x} f(z)\,\mathrm{d}z$.

(AS 1) $\psi : (0,1) \to \mathbb{R}^1$ and there exist functions $g_\psi^{(i)} : (0,1) \to \mathbb{R}^1$, $i = 1,2$ and finitely many real numbers $a_0 = 0 < \ldots < a_v = 1$ such that for all $t \in (0,1) - \{a_0,\ldots,a_v\}$ the first two derivatives of ψ exist and

$$\psi'(t) = g_\psi^{(1)}(t)\,, \qquad \psi''(t) = g_\psi^{(2)}(t)\,,$$

$g_\psi^{(1)}$ is right-continuous and

$$\psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2} g_\psi^{(1)}(t)\,\mathrm{d}t\,, \qquad g_\psi^{(1)}(z_2) - g_\psi^{(1)}(z_1) = \int_{z_1}^{z_2} g_\psi^{(2)}(t)\,\mathrm{d}t$$

for all $0 < t_1 < t_2 < 1$, the second equality holds whenever $z_1 < z_2$ belong to (a_i, a_{i+1}) and $i = 0,\ldots,v-1$.

(AS 2) There exist positive real numbers K, δ such that for all $t \in (0,1)$

$$|\psi(t)| \le K(t(1-t))^{\delta-1/2}\,, \quad \left|g_\psi^{(1)}(t)\right| \le K(t(1-t))^{\delta-3/2}\,, \quad \left|g_\psi^{(2)}(t)\right| \le K(t(1-t))^{\delta-5/2}\,.$$

(AS 3) Suppose that the assumptions (A 1), (A 2), (AS 1) and (AS 2) hold, the numbers $p_j^* = p_j^{*(u)}$, $j = 1,\ldots,k$ are such that $p_1^{*(u)} + \ldots + p_k^{*(u)} = 1$ and $\lim_{u\to\infty} p_j^{*(u)} = p_j$, $j = 1,\ldots,k$ are the limits from (2.14). Put (cf. (2.15), (2.16))

$$F_{u,j}(x) = \mathrm{P}(\,\sigma_j^{(u)}\varepsilon_j + \mu_j^{(u)} < x\,)\,, \qquad H_u^*(x) = \sum_{j=1}^{k} p_j^{*(u)} F_{u,j}(x)\,, \tag{3.1}$$

$$\begin{aligned} \mu_{u,j}^{(\psi)} &= \int_{-\infty}^{+\infty} \psi(H_u^*(x))\,\mathrm{d}F_{u,j}(x)\,, \\ \nu_j^{(\psi)} &= \int_{-\infty}^{+\infty} \left(\frac{\sigma_j^* - \overline{\sigma}}{\sigma} x + \frac{\mu_j^* - \overline{\mu}}{\sigma}\right) g_\psi^{(1)}(F(x)) f^2(x)\,\mathrm{d}x\,, \end{aligned} \tag{3.2}$$

where $\int_{-\infty}^{+\infty} |x|^d\,|g_\psi^{(1)}(F(x))| f^2(x)\,\mathrm{d}x < +\infty$ for $d = 0$ and $d = 1$. Then (cf. (2.34))

$$\lim_{u\to\infty} \sqrt{N_u}\left(\mu_{u,j}^{(\psi)} - \overline{\psi}\right) = \nu_j^{(\psi)}\,. \tag{3.3}$$

(AS 4) If the vector $(\mu_1^*, \sigma_1^*, \ldots, \mu_k^*, \sigma_k^*)' \in \mathbb{R}^{2k}$ is such that for some $t \ne t^*$ at least one of the non-equalities $\mu_t^* \ne \mu_{t^*}^*$ or $\sigma_t^* \ne \sigma_{t^*}^*$ holds, then there exists an index j such that at least one of the numbers (cf. (2.16))

$$\int_{-\infty}^{+\infty} \Big((\mu_j^* - \overline{\mu}) + (\sigma_j^* - \overline{\sigma})x\Big) g_\varphi^{(1)}(F(x)) f^2(x)\,\mathrm{d}x\,,$$

$$\int_{-\infty}^{+\infty} \Big((\mu_j^* - \overline{\mu}) + (\sigma_j^* - \overline{\sigma})x\Big) g_\psi^{(1)}(F(x)) f^2(x)\,\mathrm{d}x$$

is different from zero.

Lemma 3.1. If ψ fulfills (AS 1) and (AS 2), then as $N \to \infty$ (cf. (2.7), (2.34))

$$\sqrt{N}(\tilde{\psi} - \overline{\psi}) \longrightarrow 0\,. \tag{3.4}$$

P r o o f. Making use of (AS 1), (AS 2) one obtains that

$$\left|\frac{1}{N}\sum_{i=2}^{N-1}\psi\Big(\frac{i}{N+1}\Big) - \int_{\frac{1}{N}}^{\frac{N-1}{N}}\psi(t)\,\mathrm{d}t\right| \le \sum_{i=2}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\Big|\psi\Big(\frac{i}{N+1}\Big) - \psi(t)\Big|\,\mathrm{d}t \le$$
$$\le \sum_{i=2}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left[\int_{\frac{i-1}{N}}^{\frac{i}{N}}\Big|g_{\psi}^{(1)}(z)\Big|\,\mathrm{d}z\right]\mathrm{d}t \le \frac{K}{N}\int_{\frac{1}{N}}^{\frac{N-1}{N}}\Big(z(1-z)\Big)^{\delta-3/2}\,\mathrm{d}z = \frac{1}{\sqrt{N}}\,o(1)\,,$$

$$\left|\tilde{\psi} - \overline{\psi}\right| \le \frac{1}{N}\left|\psi(\frac{1}{N+1})\right| + \int_0^{\frac{1}{N}}|\psi(t)|\,\mathrm{d}t + \frac{1}{\sqrt{N}}\,o(1) + \int_{\frac{N-1}{N}}^1|\psi(t)|\,\mathrm{d}t + \frac{1}{N}\left|\psi\left(\frac{N}{N+1}\right)\right|$$

which together with (AS 2) implies (3.4). □

Before stating the next theorem we remark that some conditions sufficient for validity of (AS 3) can be found in Lemma 3.2.

Theorem 3.1. Suppose that the functions $\psi : (0,1) \to \mathbb{R}^1$, $\varphi : (0,1) \to \mathbb{R}^1$ fulfill the assumptions (AS 1) – (AS 3) and the matrix

$$\mathbf{V}_{\varphi,\psi} = \begin{pmatrix} V_{\varphi} & V_{\varphi,\psi} \\ V_{\varphi,\psi} & V_{\psi} \end{pmatrix} \tag{3.5}$$

is regular. Here $V_{\varphi} = V_{\varphi,\varphi}$, $V_{\varphi,\psi} = \int_0^1(\varphi(t) - \overline{\varphi})(\psi(t) - \overline{\psi})\,\mathrm{d}t$.

(I) Assume that (A 1), (A 2) hold and put (cf. (2.6))

$$Q_{n_1,\dots,n_k} = \frac{1}{1-\hat{\rho}^2}(Q_{\varphi} + Q_{\psi} - Q_{\varphi,\psi})\,, \tag{3.6}$$

where (cf. (2.7),(2.6))

$$\hat{\rho} = \frac{\sigma_N^{2,\varphi,\psi}}{\sqrt{\sigma_N^{2,\varphi}\sigma_N^{2,\psi}}}\,, \qquad Q_{\varphi,\psi} = 2\frac{\sigma_N^{2,\varphi,\psi}}{\sigma_N^{2,\varphi}\sigma_N^{2,\psi}}\sum_{j=1}^{k} n_j\Big(\frac{S_j^{(\varphi)}}{n_j} - \tilde{\varphi}\Big)\Big(\frac{S_j^{(\psi)}}{n_j} - \tilde{\psi}\Big)\,. \tag{3.7}$$

In accordance with (A 2) let

$$\theta_u = \Big(\mu + \frac{\mu_1^{*(u)}}{\sqrt{N_u}}, \sigma + \frac{\sigma_1^{*(u)}}{\sqrt{N_u}}, \dots, \mu + \frac{\mu_k^{*(u)}}{\sqrt{N_u}}, \sigma + \frac{\sigma_k^{*(u)}}{\sqrt{N_u}}\Big) \tag{3.8}$$

denote these Pitman alternatives. Then the weak convergence of distributions

$$\mathcal{L}\Big(Q_{n_1^{(u)},\dots,n_k^{(u)}}|\mathrm{P}_{\theta_u}\Big) \longrightarrow \chi^2_{2(k-1)}(\delta_{\varphi,\psi}) \tag{3.9}$$

holds as $u \to \infty$, and the non-centrality parameter of this chi-square distribution with $2(k-1)$ degrees of freedom is

$$\delta_{\varphi,\psi} = \boldsymbol{\nu}_{\varphi,\psi}{}' \Big[\mathbf{V}_{\varphi,\psi}^{-1} \otimes \mathbf{M}(\mathbf{p}) \Big] \boldsymbol{\nu}_{\varphi,\psi} \,. \tag{3.10}$$

Here

$$\mathbf{p} = (p_1, \dots, p_k)' \,, \quad \mathbf{M}(\hat{\mathbf{p}}) = \operatorname{diag}(\hat{\mathbf{p}}) - \hat{\mathbf{p}}\hat{\mathbf{p}}' \,, \quad \hat{\mathbf{p}} = (\hat{p}_1, \dots, \hat{p}_k)' \,. \tag{3.11}$$

$\boldsymbol{\nu}_{\varphi,\psi} = (\nu_1^{(\varphi)}, \dots, \nu_k^{(\varphi)}, \nu_1^{(\psi)}, \dots, \nu_k^{(\psi)})'$ is the vector with the coordinates (3.2) and if (cf. (2.16)) at least one of the non-equalities $\mu_t^* \neq \overline{\mu}$ or $\sigma_t^* \neq \overline{\sigma}$ holds for some t and (AS 4) is fulfilled, then (3.10) is a positive real number.

(II) Suppose that the critical constants $\{l_u\}_{u=1}^{\infty}$ are such that under the validity of the null hypothesis (2.2) and (A 1)

$$\lim_{u\to\infty} \mathrm{P}(Q_{n_1^{(u)},\dots,n_k^{(u)}} \geq l_u) = \alpha \in (0,1) \,. \tag{3.12}$$

Further, let (A 2) be fulfilled and the Pitman alternatives (3.8) be such that for the limits in (2.15) and the overlined terms from (2.16) at least one of the non-equalities $\mu_t^* \neq \overline{\mu}$ or $\sigma_t^* \neq \overline{\sigma}$ holds for some t, and

$$\lim_{u\to\infty} \mathrm{P}_{\theta_u}(Q_{n_1^{(u)},\dots,n_k^{(u)}} \geq l_u) = \beta \in (\alpha,1) \,. \tag{3.13}$$

Assume that the functions $\varphi^* : (0,1) \to \mathbb{R}^1$, $\psi^* : (0,1) \to \mathbb{R}^1$ fulfill (AS 1) – (AS 4), the matrix $\mathbf{V}_{\varphi^*,\psi^*}$ defined by (3.5) is regular and the quadratic test statistic

$$Q^*_{n_1,\dots,n_k} = \frac{1}{1 - \hat{\rho^*}^2} (Q_{\varphi^*} + Q_{\psi^*} - Q_{\varphi^*,\psi^*})$$

is defined by means of φ^*, ψ^* in the same way as the statistic (3.6). Then there exist sample sizes $\{n_j^{*(u)}\}_{u=1}^{\infty}$, $j = 1, \dots, k$, such that for $N_u^* = n_1^{*(u)} + \ldots + n_k^{*(u)}$

$$\lim_{u\to\infty} n_j^{*(u)} = +\infty \,, \quad \lim_{u\to\infty} \frac{n_j^{*(u)}}{N_u^*} = p_j \,, \quad j = 1, \dots, k \tag{3.14}$$

are the numbers from (2.14), and if under the validity of (2.2)

$$\lim_{u\to\infty} \mathrm{P}(Q^*_{n_1^{*(u)},\dots,n_k^{*(u)}} \geq l_u^*) = \alpha \in (0,1) \tag{3.15}$$

is the number from (3.12), then for the alternatives (3.8)

$$\lim_{u\to\infty} \mathrm{P}_{\theta_u}(Q^*_{n_1^{*(u)},\dots,n_k^{*(u)}} \geq l_u^*) = \beta \in (\alpha,1) \tag{3.16}$$

is the number from (3.13). For any sample sizes satisfying (3.14) – (3.16) the asymptotic relative efficiency

$$e_{Q,Q^*} = \lim_{u\to\infty} \frac{N_u^*}{N_u} = \frac{\delta_{\varphi,\psi}}{\delta_{\varphi^*,\psi^*}} \,, \tag{3.17}$$

where the non-centrality parameter $\delta_{\varphi^*,\psi^*}$ of the statistic Q^* is obtained by plugging φ^* instead of φ and ψ^* instead of ψ into (3.10).

Proof. Put (cf. (2.7))

$$Z_j^{(\varphi)} = \frac{S_j^{(\varphi)} - n_j\tilde{\varphi}}{\sqrt{\sigma_N^{2,\varphi}}}\,, \qquad \mathbf{Z}^{(\varphi)} = \begin{pmatrix} Z_1^{(\varphi)} \\ \vdots \\ Z_k^{(\varphi)} \end{pmatrix}, \qquad \mathbf{Z} = \begin{pmatrix} \mathbf{Z}^{(\varphi)} \\ \mathbf{Z}^{(\psi)} \end{pmatrix}. \tag{3.18}$$

The covariance matrix of $\mathbf{Z}$ under the hypothesis (2.2) is (cf. (3.11))

$$\mathbf{\Sigma} = \mathrm{var}(\mathbf{Z}) = N\hat{\mathbf{K}} \otimes \mathbf{M}(\hat{\mathbf{p}})\,, \qquad \hat{\mathbf{K}} = \begin{pmatrix} 1 & \hat{\rho} \\ \hat{\rho} & 1 \end{pmatrix}, \tag{3.19}$$

and $\hat{\rho}$ is defined in (3.7). Since $\hat{\rho} \to \rho$, assume that $|\hat{\rho}| < 1$. Since the sum of rows of the matrix $(\mathbf{M}(\hat{p}), \mathbf{Z}^{(\varphi)})$ is zero vector and $\mathrm{rank}(\mathbf{M}(\hat{\mathbf{p}})) = k-1$, obviously $\mathrm{rank}(\mathbf{M}(\hat{p}), \mathbf{Z}^{(\varphi)}) = k-1$. Thus denoting $\mathcal{M}(\mathbf{C})$ the column space of the matrix $\mathbf{C}$ we see that

$$\mathbf{Z}^{(\varphi)} \in \mathcal{M}(\mathbf{M}(\hat{p}))\,, \qquad \mathcal{M}\begin{pmatrix} \mathbf{M}(\hat{p}) \\ \mathbf{0} \end{pmatrix} \subset \mathcal{M}(\mathbf{\Sigma})\,.$$

In this way $\mathbf{Z} \in \mathcal{M}(\mathbf{\Sigma})$ and therefore $\mathbf{Z}'\overline{\mathbf{\Sigma}}\mathbf{Z}$ does not depend on the choice of the g-inverse $\overline{\mathbf{\Sigma}}$ of the matrix $\mathbf{\Sigma}$. Put

$$\hat{\mathbf{B}} = \mathrm{diag}(\hat{\mathbf{c}}) - \hat{\mathbf{c}}\hat{\mathbf{c}}'\,, \qquad \hat{\mathbf{c}} = (\hat{p}_1, \dots, \hat{p}_{k-1})'\,. \tag{3.20}$$

Since the matrices

$$\overline{\mathbf{\Sigma}} = \tfrac{1}{N}\hat{\mathbf{K}}^{-1} \otimes \begin{pmatrix} \hat{\mathbf{B}}^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix},$$

$$\mathbf{\Sigma}^* = \tfrac{1}{N}\hat{\mathbf{K}}^{-1} \otimes \left[\mathrm{diag}\left(\frac{1}{\sqrt{\hat{p}_1}}, \dots, \frac{1}{\sqrt{\hat{p}_k}}\right)\left(\mathbf{I_k} - \sqrt{\hat{\mathbf{p}}}(\sqrt{\hat{\mathbf{p}}})'\right)\mathrm{diag}\left(\tfrac{1}{\sqrt{\hat{p}_1}}, \dots, \tfrac{1}{\sqrt{\hat{p}_k}}\right)\right]$$

are generalized inverses of the matrix $\mathbf{\Sigma}$, we obtain that $Q_{n_1,\dots,n_k} = \mathbf{Z}'\mathbf{\Sigma}^*\mathbf{Z} = \mathbf{Z}'\overline{\mathbf{\Sigma}}\mathbf{Z}$ and therefore

$$Q_{n_1,\dots,n_k} = \frac{1}{N}\tilde{\mathbf{Z}}'\hat{\mathbf{\Lambda}}^{-1}\tilde{\mathbf{Z}}\,, \qquad \hat{\mathbf{\Lambda}}^{-1} = \hat{\mathbf{K}}^{-1} \otimes \hat{\mathbf{B}}^{-1}\,, \tag{3.21}$$

$$\tilde{\mathbf{Z}} = \begin{pmatrix} \tilde{\mathbf{Z}}^{(\varphi)} \\ \tilde{\mathbf{Z}}^{(\psi)} \end{pmatrix}, \qquad \tilde{\mathbf{Z}}^{(\varphi)} = \begin{pmatrix} Z_1^{(\varphi)} \\ \vdots \\ Z_{k-1}^{(\varphi)} \end{pmatrix}, \qquad \tilde{\mathbf{Z}}^{(\psi)} = \begin{pmatrix} Z_1^{(\psi)} \\ \vdots \\ Z_{k-1}^{(\psi)} \end{pmatrix}. \tag{3.22}$$

Since the transformation $x \longrightarrow (x-\mu)/\sigma$ does not change the ranks of observations, we may assume that (3.8) holds with $\mu = 0$, $\sigma = 1$ and we shall prove the theorem under this assumption with the perturbations $\mu_j^{*(u)}/(\sigma\sqrt{N_u})$ and $\sigma_j^{*(u)}/(\sigma\sqrt{N_u})$.

(I) Since the matrix $\hat{\mathbf{\Lambda}}^{-1}$ is regular, we may proceed similarly as on pp. 119–121 of [10]. To utilize the Chernoff–Savage theorem put $\theta_0 = (0, 1, \ldots, 0, 1)'$ and

$$\mathbf{T}_{n_1,\ldots,n_k} = \left(\frac{S_1^{(\varphi)}}{n_1}, \ldots, \frac{S_{k-1}^{(\varphi)}}{n_{k-1}}, \frac{S_1^{(\psi)}}{n_1}, \ldots, \frac{S_{k-1}^{(\psi)}}{n_{k-1}} \right)' , \tag{3.23}$$

$$\tilde{\sigma}^{[i,\varphi]}_{n_1,\ldots,n_k} = \frac{\sqrt{\sigma_N^{2,\varphi}}}{\hat{p}_i \sqrt{N}} , \qquad \mathbf{D}_{n_1,\ldots,n_k} = \operatorname{diag}\Big(\tilde{\sigma}^{[1,\varphi]}_{n_1,\ldots,n_k}, \ldots, \tilde{\sigma}^{[k-1,\varphi]}_{n_1,\ldots,n_k}, \tilde{\sigma}^{[1,\psi]}_{n_1,\ldots,n_k}, \ldots, \tilde{\sigma}^{[k-1,\psi]}_{n_1,\ldots,n_k} \Big). \tag{3.24}$$

Now, let $\Theta = \{(a_1, b_1, \ldots, a_k, b_k)'; \, b_j > 0, \, a_j \in \mathbb{R}^1, j = 1, \ldots, k\}$. For $\vartheta = (a, b)$, $\theta = (\vartheta_1, \ldots, \vartheta_k)' \in \Theta$ use the notation

$$F_\vartheta(x) = F((x-a)/b) , \qquad \hat{H}_\theta(x) = \sum_{j=1}^{k} \hat{p}_j F_{\vartheta_j}(x) , \tag{3.25}$$

and put

$$\boldsymbol{\mu}^{(\varphi)}_{n_1,\ldots,n_k}(\theta) = \left(\int_{-\infty}^{+\infty} \varphi\Big(\hat{H}_\theta(x)\Big) \, \mathrm{d}F_{\vartheta_1}(x), \ldots, \int_{-\infty}^{+\infty} \varphi\Big(\hat{H}_\theta(x)\Big) \, \mathrm{d}F_{\vartheta_{k-1}}(x) \right)' . \tag{3.26}$$

To express that the concerned quantities vary with the index of experiment, put

$$\mathbf{D}_u = \mathbf{D}_{n_1^{(u)},\ldots,n_k^{(u)}} , \qquad \boldsymbol{\mu}^{(\varphi)}_u(\theta) = \boldsymbol{\mu}^{(\varphi)}_{n_1^{(u)},\ldots,n_k^{(u)}}(\theta) , \qquad \boldsymbol{\mu}_u(\theta) = \begin{pmatrix} \boldsymbol{\mu}^{(\varphi)}_u(\theta) \\ \boldsymbol{\mu}^{(\psi)}_u(\theta) \end{pmatrix} , \tag{3.27}$$

$$\mathbf{T}_u = \mathbf{T}_{n_1^{(u)},\ldots,n_k^{(u)}} . \tag{3.28}$$

Since $\theta_u \to \theta_0$ and (AS 1), (AS 2) hold, an application of the Chernoff–Savage theorem yields that

$$\mathcal{L}\Big[\mathbf{D}_u^{-1}(\mathbf{T}_u - \boldsymbol{\mu}_u(\theta_u)) | \mathrm{P}_{\theta_u} \Big] \to N_{2(k-1)}\Big(\mathbf{0}, \mathbf{\Lambda}\Big) , \qquad \mathbf{\Lambda} = \mathbf{K} \otimes \mathbf{B} . \tag{3.29}$$

Here (cf. (2.25))

$$\mathbf{K} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} , \qquad \rho = \frac{V_{\varphi,\psi}}{\sqrt{V_\varphi V_\psi}} , \qquad \mathbf{B} = \operatorname{diag}(\mathbf{c}) - \mathbf{c}\mathbf{c}' , \qquad \mathbf{c} = (p_1, \ldots, p_{k-1})'$$

are the limiting values of their counterparts from (3.19) and (3.20). But the assumption (AS 3) and (3.24) imply that

$$\lim_{u\to\infty} \mathbf{D}_u^{-1}\Big(\boldsymbol{\mu}_u(\theta_u)) - \boldsymbol{\mu}_u(\theta_0)\Big) = \mathbf{C}_{p_1,\ldots,p_k} \boldsymbol{\kappa} , \tag{3.30}$$

where $\boldsymbol{\kappa} = (\mu_1^*, \sigma_1^*, \ldots, \mu_k^*, \sigma_k^*)'$ and for $i = 1, \ldots, k-1$

$$\begin{aligned} \mathbf{C}_{p_1,\ldots,p_k}(i, 2j-1) &= \frac{p_i(\delta_{ij} - p_j)}{\sigma\sqrt{V_\varphi}} \int_{-\infty}^{+\infty} g^{(1)}_\varphi(F(x)) f^2(x) \, \mathrm{d}x \qquad j = 1, \ldots, k , \\ \mathbf{C}_{p_1,\ldots,p_k}(i, 2j) &= \frac{p_i(\delta_{ij} - p_j)}{\sigma\sqrt{V_\varphi}} \int_{-\infty}^{+\infty} g^{(1)}_\varphi(F(x)) x f^2(x) \, \mathrm{d}x \qquad j = 1, \ldots, k , \end{aligned} \tag{3.31}$$

with δ_{ij} denoting the Kronecker delta, the case $k-1+i$ is the same except for φ being replaced with ψ. Put $\tilde{\boldsymbol{\mu}}_u = (\tilde{\varphi}, \ldots, \tilde{\varphi}, \tilde{\psi}, \ldots, \tilde{\psi})' \in \mathbb{R}^{2(k-1)}$. Then

$$\mathbf{D}_u^{-1}(\mathbf{T}_u - \tilde{\boldsymbol{\mu}}_u) = \tilde{\mathbf{Z}}/\sqrt{N_u} \tag{3.32}$$

and employing Lemma 3.1 we see that

$$\begin{aligned}\mathbf{D}_u^{-1}(\mathbf{T}_u - \boldsymbol{\mu}_u(\theta_u)) &= \mathbf{D}_u^{-1}(\mathbf{T}_u - \tilde{\boldsymbol{\mu}}_u) + \mathbf{D}_u^{-1}(\boldsymbol{\mu}_u(\theta_0) - \boldsymbol{\mu}_u(\theta_u)) + o(1) \\ &= \tilde{\mathbf{Z}}/\sqrt{N_u} - \mathbf{C}_{p_1,\ldots,p_k}\boldsymbol{\kappa} + o(1)\,, \end{aligned} \tag{3.33}$$

which together with (3.29) means that

$$\mathcal{L}\Big[\frac{\tilde{\mathbf{Z}}}{\sqrt{N_u}}|\mathrm{P}_{\theta_u}\Big] \to N_{2(k-1)}\Big(\mathbf{C}_{p_1,\ldots,p_k}\boldsymbol{\kappa}, \boldsymbol{\Lambda}\Big)\,.$$

This convergence together with (3.21) implies that (3.9) holds with

$$\delta_{\varphi,\psi} = \boldsymbol{\kappa}'\mathbf{C}'_{p_1,\ldots,p_k}\boldsymbol{\Lambda}^{-1}\mathbf{C}_{p_1,\ldots,p_k}\boldsymbol{\kappa}\,. \tag{3.34}$$

But with the notation from (3.2) the ith coordinate by (3.31)

$$(\mathbf{C}_{p_1,\ldots,p_k}\boldsymbol{\kappa})_i = \begin{cases} p_i\nu_i^{(\varphi)}/\sqrt{V_\varphi} & i = 1, \ldots, k-1\,, \\ p_i\nu_i^{(\psi)}/\sqrt{V_\psi} & i = k, \ldots, 2(k-1)\,, \end{cases} \tag{3.35}$$

and as $\mathbf{B}^{-1} = \mathrm{diag}(\frac{1}{p_1}, \ldots, \frac{1}{p_{k-1}}) + \frac{1}{p_k}\mathbf{1}\mathbf{1}'$, after some computation one obtains that

$$\boldsymbol{\kappa}'\mathbf{C}'_{p_1,\ldots,p_k}\boldsymbol{\Lambda}^{-1}\mathbf{C}_{p_1,\ldots,p_k}\boldsymbol{\kappa} = \boldsymbol{\nu}_{\varphi,\psi}{}'\Big(\mathbf{V}_{\varphi,\psi}^{-1} \otimes (\mathrm{diag}(p_1, \ldots, p_k) - \mathbf{p}(\mathbf{p})')\Big)\boldsymbol{\nu}_{\varphi,\psi} \tag{3.36}$$

which proves (3.10). Finally, suppose that for some t at least one of the non-equalities $\mu_t^* \neq \mu_{t*}^*$ or $\sigma_t^* \neq \sigma_{t*}^*$ holds and (AS 4) is fulfilled. Since it is only the matter of identification of the sampled distributions, one may assume that $t = 1$, but then using (3.35) and (AS 4) one obtains that (3.34) is a positive real number.

(II) In this part of the proof the notation from the proof of (I) will be used, and to distinguish the concepts corresponding to φ^*, ψ^*, we shall mark them with the superscript *.

First assume that (3.14)–(3.16) hold. Express the parameters (3.8) in the form of Pitman alternatives in the terms of the starred sample sizes, i. e.,

$$\theta_u = \Big(\frac{a_1^{*(u)}}{\sqrt{N_u^*}}, 1 + \frac{b_1^{*(u)}}{\sqrt{N_u^*}}, \ldots, \frac{a_k^{*(u)}}{\sqrt{N_u^*}}, 1 + \frac{b_k^{*(u)}}{\sqrt{N_u^*}}\Big)'\,. \tag{3.37}$$

Suppose that $\limsup_{u\to\infty} |a_{i_0}^{*(u)}| = +\infty$ for some i_0. Then there exists a subsequence such that $\lim_{v\to\infty} |a_{i_0}^{*(u_v)}| = +\infty$, which together with (3.37) and (2.15) means that

$$\lim_{v\to\infty} \frac{N_{u_v}^*}{N_{u_v}} = +\infty\,. \tag{3.38}$$

By means of (3.21) and (3.32)

$$\beta^*_u = \mathrm{P}_{\theta_u}\Big(Q^*_{n_1^{*(u)},\dots,n_k^{*(u)}} \ge l^*_u\Big) = \mathrm{P}_{\theta_u}\Big((\mathbf{T}^*_u - \tilde{\boldsymbol{\mu}}^*_u)'\mathbf{D}^{*-1}_u \hat{\boldsymbol{\Lambda}}^{*-1}\mathbf{D}^{*-1}_u(\mathbf{T}^*_u - \tilde{\boldsymbol{\mu}}^*_u) \ge l^*_u\Big), \tag{3.39}$$

where $\tilde{\boldsymbol{\mu}}^*_u = (\tilde{\varphi^*},\dots,\tilde{\varphi^*},\tilde{\psi^*},\dots,\tilde{\psi^*})'$, $\tilde{\psi^*} = \tilde{\psi^*}_{N^*_u}$. Now let $\hat{p^*}_j = n_j^{*(u)}/N^*_u$ and (cf. (3.27), (3.24))

$$\Delta_u^{[i,\varphi^*]} = \frac{\pi_i\Big(\boldsymbol{\mu}_u^{*(\varphi^*)}(\theta_u) - \boldsymbol{\mu}_u^{*(\varphi^*)}(\theta_0)\Big)}{\tilde{\sigma}^{[i,\varphi^*]}_{n_1^{*(u)},\dots,n_k^{*(u)}}}$$

where π_i denotes the ith coordinate. Then taking into account the property $\sqrt{N^*}(\overline{\varphi^*} - \tilde{\varphi^*}_{N^*}) = o(1)$ one obtains that

$$\mathbf{D}^{*-1}_u(\mathbf{T}^*_u - \tilde{\boldsymbol{\mu}}^*_u) = \mathbf{D}^{*-1}_u(\mathbf{T}^*_u - \boldsymbol{\mu}^*_u(\theta_u)) + \boldsymbol{\Delta}^*_u + o(1)\,, \tag{3.40}$$

and since similarly as in (3.30)

$$\Delta_u^{[i,\varphi^*]} = \sqrt{\frac{N^*_u}{N_u}}(M_i + o(1))\Big[\Big(\mathbf{C}^*_{p_1,\dots,p_k}\boldsymbol{\kappa}\Big)_i + o(1)\Big]\,,$$

where M_i is a positive real number, by means of (3.38), (3.35) and (AS 4)

$$\lim_{v\to\infty} \|\boldsymbol{\Delta}^*_{u_v}\| = +\infty\,. \tag{3.41}$$

As $\hat{\boldsymbol{\Lambda}}^{*-1}$ converges to a positive definite matrix and by the Chernoff–Savage theorem $\mathbf{D}^{*-1}_u(\mathbf{T}^*_u - \boldsymbol{\mu}^*_u(\theta_u)) = \mathcal{O}_P(1)$, the validity of (3.40), (3.41) together with (3.39) and the convergence of $\{l^*_u\}_{u=1}^\infty$ to the $(1-\alpha)$th quantile of the chi-square distribution yield that $\lim_{v\to\infty}\beta^*_{u_v} = 1$, which is a contradiction with (3.16).

Hence all the sequences $\{a_i^{*(u)}\}_{u=1}^\infty$, $\{b_i^{*(u)}\}_{u=1}^\infty$ are bounded and without the loss of the generality in proving (3.17) we assume that

$$\lim_{u\to\infty} a_i^{*(u)} = a_i\,, \quad \lim_{u\to\infty} b_i^{*(u)} = b_i \quad \text{are real numbers for } \; i = 1,\dots,k\,. \tag{3.42}$$

Thus the assumptions of the assertion (I) are fulfilled and therefore the distributions $\mathcal{L}(Q^*_{n_1^{*(u)},\dots,n_q^{*(u)}}|\mathrm{P}_{\theta_u}) \to \chi^2_{2(k-1)}(\delta^*_{\varphi^*,\psi^*})$ as $u\to\infty$. Since the critical constants l_u from (3.12) and l^*_u from (3.15) converge to the $(1-\alpha)$th quantile of the $\chi^2_{2(k-1)}$ distribution as $u\to\infty$, from (3.13) and (3.16) we see that $\delta^*_{\varphi^*,\psi^*} = \delta_{\varphi,\psi}$. Since for $\boldsymbol{\kappa}^*_u = (a_1^{*(u)}, b_1^{*(u)},\dots,a_k^{*(u)}, b_k^{*(u)})'$ and $\boldsymbol{\kappa}_u = (\mu_1^{*(u)},\sigma_1^{*(u)},\dots,\mu_k^{*(u)},\sigma_k^{*(u)})'$ by (3.8) and (3.37) the equality $\boldsymbol{\kappa}^*_u = (N^*_u/N_u)^{1/2}\boldsymbol{\kappa}_u$ holds, by means of (3.34)

$$1 = \frac{\delta^*_{\varphi^*,\psi^*}}{\delta_{\varphi,\psi}} = \lim_{u\to\infty}\frac{\boldsymbol{\kappa}^{*'}_u\mathbf{C}^{*'}_{p_1,\dots,p_k}\boldsymbol{\Lambda}^{*-1}\mathbf{C}^*_{p_1\dots p_k}\boldsymbol{\kappa}^*_u}{\boldsymbol{\kappa}'\mathbf{C}'_{p_1,\dots,p_k}\boldsymbol{\Lambda}^{-1}\mathbf{C}_{p_1\dots p_k}\boldsymbol{\kappa}} = \lim_{u\to\infty}\frac{N^*_u}{N_u}\frac{\boldsymbol{\kappa}'_u\mathbf{C}^{*'}_{p_1,\dots,p_k}\boldsymbol{\Lambda}^{*-1}\mathbf{C}^*_{p_1\dots p_k}\boldsymbol{\kappa}_u}{\boldsymbol{\kappa}'\mathbf{C}'_{p_1,\dots,p_k}\boldsymbol{\Lambda}^{-1}\mathbf{C}_{p_1\dots p_k}\boldsymbol{\kappa}}$$

which together with (3.36) yields (3.17).

Since by means of (I), (3.34) and $n_j^{*(u)} = [n_j^{(u)}\delta_{\varphi,\psi}/\delta_{\varphi^*,\psi^*}]$ one can show that (3.14) – (3.16) hold, the assertion (II) is true. □

An essential assumption of the previous theorem is the condition (AS 3). One way how to establish its validity is the topic of the next lemma.

Lemma 3.2. Let us consider the following conditions.

(C 1) There exist numbers $\alpha_1 < \alpha_2$ from $(0,1)$ such that the sign of $g_\psi^{(2)}$ is constant on $(0,\alpha_1)$ and it is constant also on $(\alpha_2,1)$.

(C 2) The distribution function (2.1) has a density f with respect to the Lebesgue measure on line, which is positive, bounded and (with the possible exception of finitely many numbers) continuous on $\mathbb{R}^1$. There exist real numbers $M_1 < M_2$ such that f is non-decreasing on $(-\infty, M1\rangle$ and non-increasing on $\langle M_2, +\infty)$.

(C 3) There exist a number $\alpha \in (0,1\rangle$ with the following property. The inequality

$$\int_{-\infty}^{+\infty} |x| \left| g_\psi^{(1)}(F(x)) \right| f^{2\alpha}(x)\,\mathrm{d}x < +\infty \tag{3.43}$$

holds and for every real numbers $\beta > 0$, $\gamma > 0$, $M \geq M_0 = M_0(\alpha)$ there exists a number $H > 0$ such that for all $x \geq M$ and $N \geq N_0(\gamma,\beta,\alpha)$

$$f\Big[x(1-\frac{\gamma}{\sqrt{N}}) - \frac{\beta}{\sqrt{N}}\Big] \leq H\left(f\Big[x(1+\frac{\gamma}{\sqrt{N}}) + \frac{\beta}{\sqrt{N}}\Big]\right)^\alpha, \tag{3.44}$$

and for all $x \leq -M$ and $N \geq N_0(\gamma,\beta,\alpha)$

$$f\Big[x(1-\frac{\gamma}{\sqrt{N}}) + \frac{\beta}{\sqrt{N}}\Big] \leq H\left(f\Big[x(1+\frac{\gamma}{\sqrt{N}}) - \frac{\beta}{\sqrt{N}}\Big]\right)^\alpha. \tag{3.45}$$

If the conditions (A 1), (A 2), (AS 1), (AS 2) and (C 1) – (C 3) hold, then the assumption (AS 3) is fulfilled.

P r o o f. The proof is similar to the proof of (3.39) in [12] but with the important difference that the interchange of the limit and the integration sign will be now substantiated not by the Lebesgue theorem but by the Pratt theorem from [9]. Similarly as on p. 730 of [12] the equality $\mu_{u,j}^{(\psi)} - \overline{\psi} = \int_{-\infty}^{+\infty} G_u(x)\,\mathrm{d}x$ holds, where

$$\sqrt{N_u}G_u(x) = \frac{\psi(y_{N_u})-\psi(y)}{y_{N_u}-y}\sum_{i=1}^k p_i^*\frac{F(x_{i,N_u})-F(x)}{x_{i,N_u}-x}\sqrt{N_u}(x_{i,N_u}-x)f(x), \tag{3.46}$$

$$\begin{aligned} y(x) = F(x), \quad & y_{N_u}(x) = \sum_{i=1}^k p_i^* F(x_{i,N_u}), \\ x_{i,N_u} = \frac{\sqrt{N_u}\sigma + \sigma_j^{*(u)}}{\sqrt{N_u}\sigma + \sigma_i^{*(u)}}\, x & + \frac{\mu_j^{*(u)} - \mu_i^{*(u)}}{\sqrt{N_u}\sigma + \sigma_i^{*(u)}}. \end{aligned} \tag{3.47}$$

Thus, as $u \to \infty$, the limit of (3.46) equals a.e. the function under the integration sign in $\nu_j^{(\psi)}$ from (3.2). Hence it is sufficient to find integrable functions $\{\xi_{N_u}\}$, ξ such that for all $u \geq u^*$

$$\left|\sqrt{N_u}G_u(x)\right| \leq \xi_{N_u}(x), \tag{3.48}$$

$$\lim_{u\to\infty} \xi_{N_u}(x) = \xi(x), \quad \lim_{u\to\infty}\int_{-\infty}^{+\infty} \xi_{N_u}(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \xi(x)\,\mathrm{d}x, \tag{3.49}$$

because then (3.3) follows from the Pratt theorem proved in [9]. We shall utilize that

$$x_{i,N_u} = x\left(1 + \frac{\gamma_{i,N_u}}{\sqrt{N_u}}\right) + \frac{\beta_{i,N_u}}{\sqrt{N_u}}\,, \qquad \lim_{u\to\infty} \gamma_{i,N_u} = \gamma_i\,,\ \lim_{u\to\infty} \beta_{i,N_u} = \beta_i \tag{3.50}$$

are real numbers. Throughout the rest of this proof assume that M is a fixed positive constant. Making use of the assumptions and (3.46) one obtains that there exists $C > 0$ such that for all $u \geq u_0$ and $|x| \leq M$

$$\left|\sqrt{N_u} G_u(x)\right| \leq C\,. \tag{3.51}$$

Without the loss of generality we may assume that (cf. (AS 1), (C 1) and (C 2))

$$\begin{gathered} -M + 1 < M_1\,,\ M_2 < M - 1\,, \\ F(-M+1) < \min\{\alpha_1\,, a_1\}\,,\ \max\{a_{v-1}\,, \alpha_2\} < F(M-1)\,, \\ f(M_1) < 1\,,\ f(M_2) < 1\,,\ f \text{ is continuous on } (-\infty, M_1) \cup (M_2, +\infty). \end{gathered} \tag{3.52}$$

Since (3.50) holds there exist positive real numbers γ, β such that for $u \geq u_1$ and $x \geq M$

$$M - 1 < x\Big(1 - \frac{\gamma}{\sqrt{N_u}}\Big) - \frac{\beta}{\sqrt{N_u}} < x_{i,N_u} < x\Big(1 + \frac{\gamma}{\sqrt{N_u}}\Big) + \frac{\beta}{\sqrt{N_u}}\,, \tag{3.53}$$

$$y(x)\,,\, y_{N_u}(x) \in \left\langle F\Big[x\Big(1 - \frac{\gamma}{\sqrt{N_u}}\Big) - \frac{\beta}{\sqrt{N_u}}\Big]\,,\, F\Big[x\Big(1 + \frac{\gamma}{\sqrt{N_u}}\Big) + \frac{\beta}{\sqrt{N_u}}\Big]\right\rangle\,. \tag{3.54}$$

This together with (AS 1) and the monotonicity of $g_\psi^{(1)}(F(t))$ on $\langle M-1, +\infty)$ means that

$$\begin{aligned} &\frac{\Big|\psi(y_{N_u}(x)) - \psi(y(x))\Big|}{\Big|y_{N_u}(x) - \psi(y(x))\Big|} \\ &\leq \left|g_\psi^{(1)}\Big(F\Big[x(1 - \frac{\gamma}{\sqrt{N_u}}) - \frac{\beta}{\sqrt{N_u}}\Big]\Big)\right| + \left|g_\psi^{(1)}\Big(F\Big[x(1 + \frac{\gamma}{\sqrt{N_u}}) + \frac{\beta}{\sqrt{N_u}}\Big]\Big)\right|. \end{aligned} \tag{3.55}$$

Similarly, by means of (3.53) and (3.52)

$$\frac{|F(x_{i,N_u}) - F(x)|}{|x_{i,N_u} - x|} \leq f\Big(x(1 - \frac{\gamma}{\sqrt{N_u}}) - \frac{\beta}{\sqrt{N_u}}\Big) \tag{3.56}$$

and by (3.53) for $x \geq M$

$$\sqrt{N_u}\,|x_{i,N_u} - x| \leq x\gamma + \beta\,. \tag{3.57}$$

Combining (3.46) and (3.55)–(3.57) and employing (C 3) one obtains that there exist constants K_1, K_2 such that for $u \geq u_2$ and $x \geq M$

$$\left|\sqrt{N_u} G_u(x)\right| \leq \xi_{N_u}^{(1)}(x) + \xi_{N_u}^{(2)}(x)\,, \tag{3.58}$$

and these functions are for $x \geq M$ defined by the formulas

$$\xi_{N_u}^{(1)}(x) = \xi^{*(1)}\Big(x(1-\tfrac{\gamma}{\sqrt{N_u}}) - \tfrac{\beta}{\sqrt{N_u}}\Big),\ \xi^{*(1)}(x) = K_1\Big|g_\psi^{(1)}\Big(F(x)\Big)\Big|f^2(x)|x|\,,$$
$$\xi_{N_u}^{(2)}(x) = \xi^{*(2)}\Big(x(1+\tfrac{\gamma}{\sqrt{N_u}}) + \tfrac{\beta}{\sqrt{N_u}}\Big),\ \xi^{*(2)}(x) = K_2H^2\Big|g_\psi^{(1)}\Big(F(x)\Big)\Big|f^{2\alpha}(x)|x|\,. \tag{3.59}$$

But (3.52) and the assumptions of the lemma imply that $\lim_{u\to\infty}\xi_{N_u}^{(j)}(x) = \xi^{*(j)}(x)$ for $x \geq M$ and $j = 1, 2$, and since $\int_{M-1}^{\infty}\xi^{*(j)}(x)\,\mathrm{d}x$ is a real number, for $j = 1, 2$

$$\int_M^{\infty}\xi_{N_u}^{(j)}(x)\,\mathrm{d}x \longrightarrow \int_M^{\infty}\xi^{*(j)}(x)\,\mathrm{d}x \tag{3.60}$$

as $u \to \infty$. Analogously, for $x \leq -M$ the inequality (3.58) holds with

$$\xi_{N_u}^{(1)}(x) = \xi^{*(1)}\Big(x(1-\frac{\gamma}{\sqrt{N_u}}) + \frac{\beta}{\sqrt{N_u}}\Big),\quad \xi_{N_u}^{(2)}(x) = \xi^{*(2)}\Big(x(1+\frac{\gamma}{\sqrt{N_u}}) - \frac{\beta}{\sqrt{N_u}}\Big). \tag{3.61}$$

Proceeding in this way and putting (cf. (3.51), (3.59) and (3.61))

$$\xi(x) = \begin{cases} \xi^{*(1)}(x) + \xi^{*(2)}(x) & |x| \geq M, \\ C & |x| < M, \end{cases} \quad \xi_{N_u}(x) = \begin{cases} \xi_{N_u}^{(1)}(x) + \xi_{N_u}^{(2)}(x) & |x| \geq M, \\ C & |x| < M, \end{cases}$$

one obtains that (3.48) and (3.49) hold. □

In the rest of this section we shall use the notation

$$\|\eta\| = \Big(\int_0^1 \eta^2(t)\,\mathrm{d}t\Big)^{1/2}. \tag{3.62}$$

The following lemma is a reformulation of Theorem 4.2 of [1] but for the sake of completeness we prefer to include it into the paper.

Lemma 3.3. Suppose that the function $\varphi : (0,1) \to \mathbb{R}^1$ is expressible as a finite sum of monotone square integrable functions, $F : \mathbb{R}^1 \to \langle 0, 1\rangle$ is a right-continuous distribution function and $\mathcal{C}$ is the σ-algebra (2.32). If (2.2) holds, then for the modified scores (2.28) and the function from (2.33)

$$\lim_{N\to\infty}\int_0^1 \bigg(\tilde{a}_N^{(\varphi)}(1+[tN]\,\Big|\,\tau_N(Z)) - \mathrm{E}[\varphi\,|\,\mathcal{C}](t)\bigg)^2\,\mathrm{d}t = 0 \tag{3.63}$$

almost surely. Here $[x]$ denotes the largest integer not exceeding x.

P r o o f. Let $\{Z_n\}_{n=1}^{\infty}$ be i.i.d random variables and $\mathrm{P}(Z_1 \leq t) = F(t)$. Suppose that $F_N(t) = \#\Big\{j \in \{1,\dots,N\};\, Z_j \leq t\Big\}/N$ denotes the e.d.f. generated by $Z_1,\dots,Z_N$, $\mathcal{C}_N = \sigma(F_N)$ is the σ-ring generated by the intervals $\{(0, F_N(t));\, t \in \mathbb{R}^1\}$ and similarly as in (2.33), $\mathrm{E}[\varphi\,|\,\mathcal{C}_N]$ denotes this conditional expectation related to the Lebesgue measure on $(0,1)$. If $t \in (0,1)$ and with the notation from (2.26) the inequalities $\frac{\tau_1+\ldots+\tau_{j-1}}{N} \le t < \frac{\tau_1+\ldots+\tau_{j-1}+\tau_j}{N}$ hold, then

$$\mathrm{E}\Big[\varphi(\tfrac{1+[zN]}{N+1})\,|\mathcal{C}_N\Big](t) = \tilde{a}_N^{(\varphi)}\Big(1+[tN]|\tau_N(Z)\Big)\,. \tag{3.64}$$

But by the Jensen inequality

$$\int_0^1 \Big(\mathrm{E}\Big[\varphi(\tfrac{1+[zN]}{N+1})\,|\mathcal{C}_N\Big](t) - \mathrm{E}[\varphi\,|\mathcal{C}_N](t)\Big)^2\,\mathrm{d}t \le \int_0^1 \Big(\varphi(\tfrac{1+[zN]}{N+1}) - \varphi(z)\Big)^2\,\mathrm{d}z \to 0$$

as $N\to\infty$, where the last convergence follows from Lemma 1 on p. 195 of [5]. Thus employing the subadditivity of the norm one obtains that it is sufficient to prove that

$$\lim_{N\to\infty}\int_0^1 \Big(\mathrm{E}\Big[\varphi\,|\mathcal{C}_N\Big](t) - \mathrm{E}\Big[\varphi\,|\mathcal{C}\Big](t)\Big)^2\,\mathrm{d}t = 0 \tag{3.65}$$

a. s. But as the norm is subadditive, we may assume without the loss of generality that the function φ is non-decreasing. Then according to the proof of Lemma 1 on p. 195 of [5] given $\varepsilon > 0$ there exists an integer m such that for the function of $t \in (0,1)$ defined by the formula

$$\varphi_m(t) = \sum_{i=2}^{m-1}\varphi(\frac{i}{m+1})\chi_{\langle\frac{i-1}{m},\frac{i}{m})}(t)$$

(with χ denoting now the indicator function of the set) the inequality $\|\varphi-\varphi_m\| < \varepsilon$ holds. Hence by means of the Jensen inequality and (3.62)

$$\begin{aligned}\|\mathrm{E}[\,\varphi\,|\mathcal{C}_N\,] - \mathrm{E}[\,\varphi\,|\mathcal{C}\,]\| &\le 2\varepsilon + \Big\|\mathrm{E}[\,\varphi_m\,|\mathcal{C}_N\,\Big] - \mathrm{E}[\,\varphi_m\,|\mathcal{C}\,]\Big\| \\ &\le 2\varepsilon + \sum_{i=2}^{m-1}\left|\varphi\left(\frac{i}{m+1}\right)\right|\,\left\|\mathrm{E}\left[\chi_{\langle\frac{i-1}{m},\frac{i}{m})}\,|\mathcal{C}_N\right] - \mathrm{E}[\chi_{\langle\frac{i-1}{m},\frac{i}{m})}\,|\mathcal{C}]\right\|.\end{aligned}$$

Thus to prove (3.65) it is sufficient to show that for every $a < b$ from $(0,1)$

$$\lim_{N\to\infty}\|\mathrm{E}[\chi_{\langle a,b)}\,|\mathcal{C}_N] - \mathrm{E}[\chi_{\langle a,b)}\,|\mathcal{C}]\| = 0 \tag{3.66}$$

a. s. But this statement can be proved by means of the fact that according to the Glivenko–Cantelli theorem $\Delta_N = \sup\{|F_N(t)-F(t)|;\, t\in\mathbb{R}^1\} \to 0$ almost surely. □

P r o o f o f T h e o r e m 2.3. (I) First assume that (cf. (2.13))

$$\hat{p}_j = \frac{n_j}{N} \longrightarrow p_j\,,\quad j=1,\ldots,k\,. \tag{3.67}$$

Suppose that $\{U_{j,i}\}_{i=1}^{\infty}$, $j=1,\ldots,k$ are mutually different random variables which are uniformly distributed over $(0,1)$ and in the notation from the introduction of the paper $\{X_{j,i}\}_{i=1}^{\infty}$, $j=1,\ldots,k$, $\{U_{j,i}\}_{i=1}^{\infty}$, $j=1,\ldots,k$ are independent random variables. Put $(X_{r,v}, U_{r,v}) \prec (X_{j,i}, U_{j,i})$ if either $X_{r,v} < X_{j,i}$, or $X_{r,v} = X_{j,i}$ and $U_{r,v} < U_{j,i}$, and for $n_1, \dots, n_k$ fixed let

$$R^*_{j,i} = \#\{(r,v);\ (X_{r,v}, U_{r,v}) \preceq (X_{j,i}, U_{j,i})\,,\ r = 1, \dots, k\,,\ v = 1, \dots, n_r\,\}\,,$$
$$R^{*(N)} = (R^*_{1,1}, \dots, R^*_{1,n_1}, R^*_{2,1}, \dots, R^*_{2,n_2}, \dots, R^*_{k,1}, \dots, R^*_{k,n_k})\,.$$

According to Theorem 29.A from [6] the random vectors $R^{*(N)}$, $\tau_N(Z)$ are independent and $R^{*(N)}$ is uniformly distributed over the set $\mathcal{R}^{(N)}$ of all permutations of the set $\{1, \dots, N\}$. Put

$$\tilde{S}^{*(\varphi)}_{j,N} = \tilde{S}^{*(\varphi)}_{j,N}(R^{*(N)}, \tau_N(Z)) = \sum_{i=1}^{n_j} \tilde{a}^{(\varphi)}_N(R^*_{j,i} | \tau_N(Z))\,. \tag{3.68}$$

According to the proof on p. 130 of [6]

$$\tilde{S}^{(\varphi)}_{j,N} = \tilde{S}^{*(\varphi)}_{j,N}\,. \tag{3.69}$$

Let $c^{(N)}_j(r,v) = 1$ for $r = j$, $v = 1, \dots, n_j$, and $c^{(N)}_j(r,v) = 0$ otherwise. Obviously

$$\tilde{S}^{*(\varphi)}_{j,N} = \sum_{r=1}^{k} \sum_{v=1}^{n_r} c^{(N)}_j(r,v)\tilde{a}^{(\varphi)}_N(R^*_{r,v} | \tau_N(Z))\,. \tag{3.70}$$

Further, let

$$S^{*(\tilde{\varphi})}_{j,N} = \sum_{r=1}^{k} \sum_{v=1}^{n_r} c^{(N)}_j(r,v) a^{(\tilde{\varphi})}_N(R^*_{r,v}) \tag{3.71}$$

where (cf. (2.33))

$$a^{(\tilde{\varphi})}_N(i) = \tilde{\varphi}\Big(\frac{i}{N+1}\Big)\,, \qquad \tilde{\varphi}(t) = \mathrm{E}[\varphi \,|\, \mathcal{C}](t)\,. \tag{3.72}$$

Let us consider the difference

$$\begin{aligned} \overline{D}^{(\varphi)}_{j,N} &= \overline{D}^{(\varphi)}_{j,N}(R^{*(N)}, \tau_N(Z)) \\ &= S^{*(\tilde{\varphi})}_{j,N} - \mathrm{E}(S^{*(\tilde{\varphi})}_{j,N}) - \Big[\tilde{S}^{*(\varphi)}_{j,N}(R^{*(N)}, \tau_N(Z)) - N\overline{c}^{(N)}_j \overline{a}^{(\varphi)}_N\Big]\,. \end{aligned} \tag{3.73}$$

Since $\mathrm{E}(S^{*(\tilde{\varphi})}_{j,N}) = N\overline{c}^{(N)}_j \overline{a}^{(\tilde{\varphi})}_N$ and the random vectors $R^{*(N)}$, $\tau_N(Z)$ are independent,

$$\mathrm{E}\Big[\Big(\overline{D}^{(\varphi)}_{j,N}\Big)^2 \Big| \tau_N(Z) = \tau\Big] = \mathrm{E}\Big[\Big(\overline{D}^{(\varphi)}_{j,N}(R^{*(N)}, \tau)\Big)^2\Big] = \mathrm{var}\Big[\overline{D}^{(\varphi)}_{j,N}(R^{*(N)}, \tau)\Big]$$
$$\le \frac{1}{N-1} \sum_{r=1}^{k} \sum_{v=1}^{n_r} \Big(c^{(N)}_j(r,v) - \overline{c}^{(N)}_j\Big)^2 \sum_{i=1}^{N} \Big(a^{(\tilde{\varphi})}_N(i) - \tilde{a}^{(\varphi)}_N(i|\tau)\Big)^2\,.$$

Hence putting

$$\sigma^{*2,\tilde{\varphi}}_{j,N} = \sum_{r=1}^{k} \sum_{v=1}^{n_r} \Big(c^{(N)}_j(r,v) - \overline{c}^{(N)}_j\Big)^2 V_{\tilde{\varphi}} \tag{3.74}$$

one obtains that

$$\mathrm{E}\Big[\frac{\big(\overline{D}^{(\varphi)}_{j,N}\big)^2}{\sigma^{*2,\tilde\varphi}_{j,N}}\,\Big|\,\tau_N(Z)=\tau\Big] \le \frac{2}{V_{\tilde\varphi}}\frac{1}{N}\sum_{i=1}^{N}\Big(a^{(\tilde\varphi)}_N(i)-\tilde a^{(\varphi)}_N(i|\tau)\Big)^2$$

$$\le \frac{2}{V_{\tilde\varphi}}\Big[\,\|a^{(\tilde\varphi)}_N(1+[tN])-\tilde\varphi(t)\|+\|\tilde\varphi(t)-\tilde a^{(\varphi)}_N(1+[tN]|\tau)\|\,\Big]^2 . \tag{3.75}$$

Since the conditional expectation is linear and (A 3) holds, in proving convergence of (3.75) to zero one may assume that φ is non-decreasing. Further, put

$$\underline{a}=\sup\{\,F(z);\,F(z)<a\}\,,\quad \overline{a}=\inf\{\,F(z);\,F(z)\ge a\}\,,\quad \mathcal{G}=\{(\underline{a},\overline{a});\,a\in(0,1)\}\,,$$

and define the function of the argument $t\in(0,1)$ by the formula

$$\tilde\varphi(t)=\begin{cases}\frac{1}{M-m}\int\limits_{(m,M)}\varphi(t)\,\mathrm{d}t & \text{if } t\in\langle m,M),\ (m,M)\in\mathcal{G},\\ \varphi(t) & \text{otherwise.}\end{cases} \tag{3.76}$$

Since this function fulfills (2.33) for every interval $A\in\mathcal{C}$, we may assume that $\tilde\varphi$ is the function from (3.72). Thus $\tilde\varphi$ is non-decreasing and an application of Lemma 1 on p. 195 of [5] yields that the first term in the bracket on the right-hand side of (3.75) converges to zero and since the second term is $o_P(1)$ owing to (3.63),

$$\mathrm{E}\Big[\frac{\big(\overline{D}^{(\varphi)}_{j,N}\big)^2}{\sigma^{*2,\tilde\varphi}_{j,N}}\,\Big|\,\tau_N=\tau_N(Z)\Big]=o_P(1)\,,\qquad \frac{\big(\overline{D}^{(\varphi)}_{j,N}\big)^2}{\sigma^{*2,\tilde\varphi}_{j,N}}=o_P(1)\,,$$

because the validity of (3.64) means that the right-hand side of (3.75) can be dominated by a positive real number. Hence by (3.73)

$$\frac{\tilde S^{*(\varphi)}_{j,N}(R^{*(N)},\tau_N(Z))-n_j\overline{a}^{(\varphi)}_N}{(\sigma^{*2,\tilde\varphi}_{j,N})^{1/2}}=\frac{S^{*(\tilde\varphi)}_{j,N}(R^{*(N)},\tau_N(Z))-\mathrm{E}(S^{*(\tilde\varphi)}_{j,N})}{(\sigma^{*2,\tilde\varphi}_{j,N})^{1/2}}+o_P(1)\,. \tag{3.77}$$

Let

$$d^{(2,\varphi)}_{j,N}=\frac{1}{N-1}\sum_{r=1}^{k}\sum_{v=1}^{n_r}\Big(c^{(N)}_j(r,v)-\overline{c}^{(N)}_j\Big)^2\sum_{i=1}^{N}\Big(\tilde a^{(\varphi)}_N(i\big|\tau_N(Z))-\overline{a}^{(\varphi)}_N\Big)^2 .$$

Then Lemma 3.3 and the property $|\overline{a}^{(\varphi)}_N-\overline{\varphi}|=o(1)$ following from Lemma 1 on p. 195 of [5] imply that

$$\left(\frac{d^{(2,\varphi)}_{j,N}}{\sigma^{*2,\tilde\varphi}_{j,N}}\right)^{1/2}=1+o_P(1)\,. \tag{3.78}$$

Since according to the assumptions $V_{\tilde\varphi}$ is a positive real number, by means of Theorem 1 on p. 194 of [5]

$$\frac{S^{*(\tilde\varphi)}_{j,N}-\mathrm{E}(S^{*(\tilde\varphi)}_{j,N})}{(\sigma^{*2,\tilde\varphi}_{j,N})^{1/2}}\longrightarrow N(0,1) \tag{3.79}$$

in distribution and by (3.77), (3.78)

$$\frac{\tilde{S}^{*(\varphi)}_{j,N} - n_j \overline{a}^{(\varphi)}_N}{(d^{2,\varphi}_{j,N})^{1/2}} = \frac{\tilde{S}^{*(\varphi)}_{j,N} - n_j \overline{a}^{(\varphi)}_N}{(\sigma^{*2,\tilde{\varphi}}_{j,N})^{1/2}} + o_P(1)\,. \tag{3.80}$$

In accordance with (3.72) and (2.27) put

$$\sigma^{2,\tilde{\varphi}}_N = \frac{1}{N-1}\sum_{i=1}^{N}\left(a^{(\tilde{\varphi})}_N(i) - \overline{a}^{(\tilde{\varphi})}_N\right)^2 .$$

An application of Lemma 1 on p. 195 of [6] yields that $\sigma^{2,\tilde{\varphi}}_N / V_{\tilde{\varphi}} \to 1$ as $N \to \infty$, and as the weakly convergent random variables in (3.79) are bounded in probability,

$$\frac{S^{*(\tilde{\varphi})}_{j,N} - \mathrm{E}(S^{*(\tilde{\varphi})}_{j,N})}{(N\hat{p}_j(1-\hat{p}_j)\sigma^{2,\tilde{\varphi}}_N)^{1/2}} = \frac{S^{*(\tilde{\varphi})}_{j,N} - \mathrm{E}(S^{*(\tilde{\varphi})}_{j,N})}{(\sigma^{*2,\tilde{\varphi}}_{j,N})^{1/2}} + o_P(1)\,. \tag{3.81}$$

Combine (3.77) – (3.81) to show that

$$\frac{\tilde{S}^{*(\varphi)}_{j,N} - n_j \overline{a}^{(\varphi)}_N}{(n_j \tilde{\sigma}^{2,\varphi}_N)^{1/2}} = \frac{S^{*(\tilde{\varphi})}_{j,N} - n_j \overline{a}^{(\tilde{\varphi})}_N}{(n_j \sigma^{2,\tilde{\varphi}}_N)^{1/2}} + o_P(1)\,, \tag{3.82}$$

and for the same reasons

$$\frac{\tilde{S}^{*(\psi)}_{j,N} - n_j \overline{a}^{(\psi)}_N}{(n_j \tilde{\sigma}^{2,\psi}_N)^{1/2}} = \frac{S^{*(\tilde{\psi})}_{j,N} - n_j \overline{a}^{(\tilde{\psi})}_N}{(n_j \sigma^{2,\tilde{\psi}}_N)^{1/2}} + o_P(1)\,. \tag{3.83}$$

To prove the convergence (2.40) put

$$\xi^{*(\tilde{\varphi})}_{j,N} = \left(S^{*(\tilde{\varphi})}_{j,N} - n_j \overline{a}^{(\tilde{\varphi})}_N\right) / \sqrt{n_j \sigma^{2,\tilde{\varphi}}_N}\,, \qquad \xi^{*(\tilde{\psi})}_{j,N} = \left(S^{*(\tilde{\psi})}_{j,N} - n_j \overline{a}^{(\tilde{\psi})}_N\right) / \sqrt{n_j \sigma^{2,\tilde{\psi}}_N}\,.$$

Let the random vector $\mathbf{Z}^*_N = \left(\xi^{*(\tilde{\varphi})}_{1,N}, \dots, \xi^{*(\tilde{\varphi})}_{k,N}, \xi^{*(\tilde{\psi})}_{1,N}, \dots, \xi^{*(\tilde{\psi})}_{k,N}\right)'$. Then according to Theorem 3.1 of [12] the weak convergence of probabilities

$$\mathcal{L}(\mathbf{Z}^*_N) \longrightarrow N_{2k}(\mathbf{0}, \mathbf{K}_{\tilde{\mathbf{V}}} \otimes \mathbf{A}(\mathbf{p})) \tag{3.84}$$

holds, here (cf. (2.35) and (2.36))

$$\mathbf{K}_{\tilde{\mathbf{V}}} = \begin{pmatrix} 1 & \tilde{\rho} \\ \tilde{\rho} & 1 \end{pmatrix}, \quad \tilde{\rho} = \frac{V_{\tilde{\varphi},\tilde{\psi}}}{(V_{\tilde{\varphi}} V_{\tilde{\psi}})^{1/2}}\,,$$

$$\mathbf{A}(\mathbf{p}) = \mathbf{I_k} - \sqrt{\mathbf{p}}(\sqrt{\mathbf{p}})', \;\; \sqrt{\mathbf{p}} = (\sqrt{p_1}, \dots, \sqrt{p_k})'\,.$$

Use the notation (2.30) and put

$$\tilde{\xi}^{(\varphi)}_{j,N} = \left(\tilde{S}^{(\varphi)}_{j,N} - n_j \overline{a}^{(\varphi)}_N\right) / \sqrt{n_j \tilde{\sigma}^{2,\varphi}_N}, \qquad \tilde{\xi}^{(\psi)}_{j,N} = \left(\tilde{S}^{(\psi)}_{j,N} - n_j \overline{a}^{(\psi)}_N\right) / \sqrt{n_j \tilde{\sigma}^{2,\psi}_N},$$

$$\tilde{\mathbf{Z}}_N = \begin{pmatrix} \tilde{\xi}^{(\varphi)}_N \\ \tilde{\xi}^{(\psi)}_N \end{pmatrix}, \qquad \tilde{\xi}^{(\varphi)}_N = (\tilde{\xi}^{(\varphi)}_{1,N}, \dots, \tilde{\xi}^{(\varphi)}_{k,N})', \qquad \tilde{\xi}^{(\psi)}_N = (\tilde{\xi}^{(\psi)}_{1,N}, \dots, \tilde{\xi}^{(\psi)}_{k,N})'\,.$$

Then by (3.69), (3.82) and (3.83)

$$\tilde{\mathbf{Z}}_N = \mathbf{Z}^*_N + o_P(1)\,. \tag{3.85}$$

Employ (2.30) to define the matrix by the formula

$$\hat{\mathbf{K}} = \begin{pmatrix} 1 & \hat{\rho} \\ \hat{\rho} & 1 \end{pmatrix}, \qquad \hat{\rho} = \frac{\tilde{\sigma}_N^{2,\varphi,\psi}}{\left(\tilde{\sigma}_N^{2,\varphi}\tilde{\sigma}_N^{2,\psi}\right)^{1/2}}\,. \tag{3.86}$$

Since Lemma 1 on p. 195 of [5] and Lemma 3.3 hold,

$$\hat{\mathbf{K}} = \mathbf{K}_{\tilde{\mathbf{V}}} + o_P(1)\,. \tag{3.87}$$

In accordance with (2.13) put $\hat{\mathbf{p}} = (\hat{p}_1, \ldots, \hat{p}_k)'$. Then (3.87) together with (3.67) means that $\hat{\mathbf{K}}^{-1} \otimes \mathbf{A}(\hat{\mathbf{p}}) = {\mathbf{K}_{\tilde{\mathbf{V}}}}^{-1} \otimes \mathbf{A}(\mathbf{p}) + o_P(1)$ and taking into account (3.85) one obtains that

$$\tilde{\mathbf{Z}}'_N(\hat{\mathbf{K}}^{-1} \otimes \mathbf{A}(\hat{\mathbf{p}}))\tilde{\mathbf{Z}}_N = {\mathbf{Z}^*_N}'({\mathbf{K}_{\tilde{\mathbf{V}}}}^{-1} \otimes \mathbf{A}(\mathbf{p}))\mathbf{Z}^*_N + o_P(1) \tag{3.88}$$

where

$$\mathcal{L}\Big({\mathbf{Z}^*_N}'({\mathbf{K}_{\tilde{\mathbf{V}}}}^{-1} \otimes \mathbf{A}(\mathbf{p}))\mathbf{Z}^*_N\Big) \longrightarrow \chi^2_{2(k-1)} \tag{3.89}$$

in distribution; this can be easily proved by means of (3.84) and Theorem 9.2.3 on p. 173 of [11], because ${\mathbf{K}_{\tilde{\mathbf{V}}}}^{-1} \otimes \mathbf{A}(\mathbf{p})$ is the Moore–Penrose inverse of $\mathbf{K}_{\tilde{\mathbf{V}}} \otimes \mathbf{A}(\mathbf{p})$. But after some computation one obtains that $\tilde{\mathbf{Z}}'_N(\hat{\mathbf{K}}^{-1} \otimes \mathbf{A}(\hat{\mathbf{p}}))\tilde{\mathbf{Z}}_N = \tilde{T}_{n_1,\ldots,n_k}$ is the statistic (2.38), and the convergence (2.40) follows from (3.88) and (3.89).

(II) Now do not assume that the convergence (3.67) holds. Since from every bounded sequence of vectors from $\mathbb{R}^k$ one can choose a convergent subsequence, the convergence (2.40) can be easily proved by means of (I). □

P r o o f o f L e m m a 2.1. For $t \in \langle F_{j-1}, F_j)$ put ${\boldsymbol{\xi}^*}_j = (\varphi^*_j\,,\, \psi^*_j)'$. If $\mathcal{D} \subset \mathcal{C}$, then $\mathrm{E}\Big[\mathrm{E}[\xi|\mathcal{C}]\,|\,\mathcal{D}\Big] = \mathrm{E}\Big[\xi\,|\,\mathcal{D}\Big]$ and therefore is is sufficient to prove that the covariance matrix $\mathbf{V}^*$ of the random vector $\boldsymbol{\xi}^*$ is regular.

Suppose that $\det(\mathbf{V}^*) = 0$. Then there exists $(\alpha\,,\,\beta) \neq (0,0)$ such that (cf. (2.34))

$$\alpha\varphi^*_j + \beta\psi^*_j = \gamma\,,\;\; \gamma = \alpha\overline{\varphi} + \beta\overline{\psi}\,,\;\; j = 1,2,3,4\,. \tag{3.90}$$

Since the case $F(z_2) > \frac{1}{2}$ can be handled analogously, assume that

$$F(z_2) \leq \frac{1}{2}\,. \tag{3.91}$$

(I) Then $(\alpha+\beta)\varphi^*_1 = (\alpha+\beta)\varphi^*_2$. In the case that $\alpha+\beta \neq 0$ we obtain that $\varphi^*_1 = \varphi^*_2$ which is a contradiction, because φ is strictly increasing. If $\alpha+\beta = 0$, then $0 = \alpha\overline{\varphi} + \beta\overline{\psi} = \alpha(\overline{\varphi} - \overline{\psi})$, $\overline{\varphi} = \overline{\psi}$ and therefore

$$\int_{\frac{1}{2}}^{1} \varphi(t)\,\mathrm{d}t = \frac{1}{4}\,. \tag{3.92}$$

This together with (2.42) implies that $\int_0^{\frac{1}{2}} \varphi(t)\,\mathrm{d}t = \frac{1}{4} = \int_{\frac{1}{2}}^1 \varphi(t)\,\mathrm{d}t$, which contradicts the strict monotonicity of φ. Therefore (3.91) cannot hold which means that $F(z_2) > > \frac{1}{2}$. Then $\psi_j^* = 1 - \varphi_j^*$ for $j = 3, 4$ and

$$\gamma = (\alpha - \beta)\varphi_j^* + \beta\,, \qquad j = 3, 4\,. \tag{3.93}$$

Hence if $\alpha = \beta$, then we may assume that $\alpha = \beta = \gamma = 1$, but in such case $1 = \overline{\varphi} + \overline{\psi} = 2\int_0^{\frac{1}{2}} \varphi(t)\,\mathrm{d}t + \frac{1}{2}$. Thus (3.92) holds and as we have already proved, this yields a contradiction. However, if $\alpha \neq \beta$, then from (3.93) we obtain that $\varphi_3^* = \varphi_4^*$ which contradicts the strict monotonicity of φ. This means that the matrix $\mathbf{V}^*$ cannot be singular.

(II) The proof of this part is left to the reader. □

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