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ON QUASIPARABOLICAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

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We shall be dealing with the initial value problem

$$A_0 \frac{\mathrm{d}^m u}{\mathrm{d}t^m} + \ldots + A_{m-1} \frac{\mathrm{d}u}{\mathrm{d}t} + A_m u = f^*(t) \tag{0.1}$$

$$\left. \frac{\mathrm{d}^r u}{\mathrm{d}t^r} \right|_{t=0} = u_r,\ r = 0, 1, \ldots, m-1 \tag{0.2}$$

with the functions $u: ([0, \infty) \to H)$, $f^*: ([0, \infty) \to H^*)$, the operators $A_r: (H \to H^*)$ and the elements $u_r \in H$, where H is a real Hilbert space and H^* is a dual space to H.

The existence and the uniqueness of a solution of (0.1), (0.2) will be verified in the first part. We shall analyze the behaviour of the solution for $t \to \infty$ in the second part. We restrict ourselves only to the problems of the first and second order. The application on the equation of bending of viscoelastic plate is shown in the third part.

1. Existence and Uniqueness of a Solution

Let H be the real Hilbert space with a scalar product $(.,.)$ and a norm $\|.\|$, H^* be the dual space to H i.e. the space of all linear and bounded functionals over H with a norm $\|.\|_*$. If $f^* \in H$, $h \in H$, then we denote $\langle f^*, h \rangle = f^*(h)$. The Riesz operator $R \in L(H^*, H)$ is defined by

$$(Rf^*, h) = \langle f^*, h \rangle,\ f^* \in H^*,\ h \in H. \tag{1.1}$$

We denote by $C^{(m)}([0, \infty), H)$ the space of all m-times continuously differentiable functions with the domain $[0, \infty)$ and with the values in H and by $C([0, \infty), H^*)$ the space of all continuous functions with the domain $[0, \infty)$ and the values in H^*.

We assume that $A_0, \ldots, A_m$ are linear and bounded operators with the domain H and the values in H i.e.

$$A_r \in L(H, H^*),\ r = 0, 1, \ldots, m \tag{1.2}$$

The operator A_0 is assumed to be coercive i.e.

$$\langle A_0 x, x\rangle \geq \alpha_0 \|x\|^2,\ x \in H,\ \alpha_0 > 0. \tag{1.3}$$

Definition 1.1. *Let $f^* \in C([0, \infty), H^*)$, $A_k \in L(H, H^*)$, $k=0, 1, \ldots, m$, $u_r \in H$, $r=0, 1, \ldots, m-1$. The function $u: ([0, \infty) \to H)$ is a solution of the initial value problem* (0.1), (0.2) *iff*

i) $u \in C^{(m)}([0, \infty), H)$

ii) *u satisfies* (0.1), (0.2).

Theorem 1.1. *Let $f^* \in C([0, \infty), H^*)$, $u_r \in H$, $r=0, 1, \ldots, m-1$, $A_k \in L(H, H^*)$, $k=0, 1, \ldots, m$, A_0 satisfies* (1.3). *Then there exists a unique solution of the problem* (0.1), (0.2).

Proof. Using the Riesz operator R we convert the problem (0.1), (0.2) into the initial value problem for the differential equation in the space H in the same way as in the paper [4]:

$$B_0 \frac{d^m u}{dt^m} + \ldots + B_{m-1} \frac{du}{dt} + B_m u = f(t) \tag{1.4}$$

$$\left.\frac{d^r u}{dt^r}\right|_{t=0} = u_r,\ r=0, 1, \ldots, m-1, \tag{1.5}$$

where

$$B_r = RA_r \in L(H, H),\ r=0, 1, \ldots, m, \tag{1.6}$$

$$(B_0 x, x) \geq \alpha_0 \|x\|^2,\ x \in H,\ \alpha_0 > 0, \tag{1.7}$$

$$f = Rf^* \in C([0, \infty), H). \tag{1.8}$$

The operator R is linear bounded and invertible and hence the problems (0.1), (0.2) and (1.4), (1.5) are equivalent. It is sufficient to show that there exists a solution $u \in C^{(m)}([0, \infty), H)$ of the problem (1.4), (1.5). That solution will be the solution of (0.1), (0.2) too. The initial value problem (1.4), (1.5) is equivalent with the problem

$$\frac{d^m u}{dt^m} + \sum_{r=1}^{m} B_0^{-1} B_r \frac{d^{m-r} u}{dt^{m-r}} = B_0^{-1} f(t) \tag{1.9}$$

$$\left.\frac{d^r u}{dt^r}\right|_{t=0} = u_r,\ r=0, 1, \ldots, m-1, \tag{1.10}$$

where the inverse operator $B_0^{-1} \in L(H, H)$ exists due to (1.7). The existence and the uniqueness of a solution of (1.9), (1.10) is proved in [9] under more general assumptions. It is sufficient to consider the problem (1.9), (1.10) as the initial value problem of the first order in the space $\mathscr{H} = H^m$. A solution $u \in C([0, \infty), H)$ of (1.9), (1.10) is then a unique solution of the problem (1.4), (1.5) and (0.1), (0.2).

Remark. We can consider in (0.1) under suitable assumptions also the families of nonlinear operators $A_r(t)$. The assumptions must be similar as in [4] for the problem of the first order.

2. Behaviour of a Solution for $t \to \infty$

We restrict our consideration only to the equations of the first and second order.

The initial value problem (0.1), (0.2) for $m=1$ has the form

$$A_0 \frac{\mathrm{d}u}{\mathrm{d}t} + A_1 u = f^*(t) \tag{2.1}$$

$$u(0) = u_0 \tag{2.2}$$

We assume that A_0, A_1 satisfy the assumptions (1.2), (1.3) and moreover A_0 is symmetric and A_1 is coercive i.e.

$$\langle A_0 x, y \rangle = \langle A_0 y, x \rangle, \; x, y \in H \tag{2.3}$$

$$\langle A_1 x, x \rangle \geqslant \alpha_1 \, \|x\|^2, \; x \in H, \; \alpha_1 > 0. \tag{2.4}$$

The following theorem expresses the asymptotic behaviour of the solution of the problem (2.1), (2.2).

Theorem 2.1. *Assume that the function f^*, the element u_0 the operators A_0, A_1 satisfy the assumptions of* Theorem 1.1 *and A_0, A_1 satisfy* (2.3), (2.4). *Then the next estimate for a solution $u \in C^{(1)}([0, \infty), H)$ of* (2.1), (2.2) *holds with the constants M, $\nu > 0$ depending only on A_0, A_1*

$$\|u(t)\| \leqslant M \,.\, \mathrm{e}^{-\nu t} \left(\|u_0\| + \int_0^t \mathrm{e}^{\nu \tau} \, \|f^*(\tau)\|_* \, \mathrm{d}\tau\right), \; t \geqslant 0 \tag{2.5}$$

If there exists such an element $f^*_\infty \in H^*$ that

$$\lim_{t \to \infty} \|f^*(t) - f^*_\infty\|_* = 0, \tag{2.6}$$

then

$$\lim_{t \to \infty} \|u(t) - u_\infty\| = 0, \tag{2.7}$$

where u_∞ is a solution of the equation

$$A_1 u_\infty = f^*_\infty \tag{2.8}$$

Proof. The initial value problem (2.1), (2.2) is equivalent with the initial value problem

$$\frac{du}{dt}+B_0^{-1}B_1u=B_0^{-1}f(t) \tag{2.9}$$

$$u(0)=u_0 \tag{2.10}$$

with the operators B_0, B_1 and the function f defined in (1.6) and (1.8). The operators B_r are coercive i.e.

$$(B_rx, x)\geq \alpha_r \|x\|^2, \ x \in H, \ \alpha_r>0, \ r=0, 1 \tag{2.11}$$

and B_0 is symmetric i.e.

$$(B_0x, y)=(x, B_0y), \ x, y \in H \tag{2.12}$$

The solution $u(t)$ of (2.9), (2.10) can be expressed (see [1]) in the form

$$u(t)=e^{-B_0^{-1}B_1t}u_0+\int_0^t e^{-B_0^{-1}B_1(t-\tau)}B_0^{-1}f(\tau)\,d\tau, \tag{2.13}$$

where $e^{-B_0^{-1}B_1t}$ is the abstract exponential-operator function with values in the Banach space $L(H, H)$ of all linear and bounded operators in H. The Bochner integral in H ([10]) is considered in (2.13). In order to estimate (2.13) we shall use the spectral theory. For this purpose we extend the space H to the complex Hilbert space which we denote again H. We extend the operators B_r over the complex space too. It can be verified easily that $B_0=B_0^*$ and B_1 satisfy the inequalities

$$(B_0x, x)\geq \alpha_0 \|x\|^2, \ \alpha_0>0 \tag{2.14}$$

$$\mathrm{Re}\,(B_1x, x)\geq \alpha_1 \|x\|^2, \ \alpha_1>0, x \in H. \tag{2.15}$$

Let $\sigma(-B_0^{-1}B_1)$ be the spectrum of $-B_0^{-1}B_1$. If there exists such a constant $\nu>0$, that

$$\mathrm{Re}\,\lambda<\nu, \ \lambda \in \sigma(-B_0^{-1}B_1) \tag{2.16}$$

then there exists such a constant N depending only on B_0, B_1 that

$$\|e^{-B_0^{-1}B_1t}\|\leq N\,e^{-\nu t}, \ t \in [0, \infty) \tag{2.17}$$

This assertion is proved in ([6], Th. 1.2). If (2.17) holds with $\nu>0$ then the conclusions of the theorem follow easily from (2.13). We show at first that $\mathrm{Re}\,\lambda<0$ for all $\lambda \in \sigma(-B_0^{-1}B_1)$. It is sufficient to show that the relation $0 \in \sigma(\lambda B_0+B_1)$ implies $\mathrm{Re}\,\lambda<0$.

Let $\mathrm{Re}\,\lambda\geq0$. We denote by

$$T(\lambda)^*=\bar{\lambda}B_0+B_1^* \tag{2.18}$$

the operator adjoint to $T(\lambda)=\lambda B_0+B_1$. The operators $T(\lambda)$ and $T(\lambda)^*$ satisfy the relations

$$\text{(2.19)} \qquad \operatorname{Re}\,(T(\lambda)x, x) = \operatorname{Re}\,(T(\lambda)^*x, x) \geq \alpha_1\, \|x\|^2, \quad \alpha_1 > 0,\ x \in H.$$

Using the Schwarz inequality and the corollary from ([10], VII) we obtain

$$\text{(2.20)} \qquad R(T(\lambda)) = H,\ N(T(\lambda)) = \{\bar{0}\},$$

where $R(T(\lambda))$ is the range of the operator $T(\lambda)$ and $N(T(\lambda)) = \{x \in H,\ T(\lambda)x = \bar{0}\}$. Hence $\lambda \notin \sigma(-B_0^{-1}B_1)$ and so $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(-B_0^{-1}B_1)$.

It remains to show the existence of a number $\nu > 0$ such that $\operatorname{Re}\lambda < -\nu$ for all $\lambda \in \sigma(-B_0^{-1}B_1)$. Assume that there does not exist such a number. Then there exists a sequence $\lambda_n \in \sigma(-B_0^{-1}B_1)$ such that $\lim\limits_{n\to\infty} \operatorname{Re}\lambda_n = 0$. All points of $\sigma(-B_0^{-1}B_1)$ lie in the circle $|\lambda| \leq \|-B_0^{-1}B_1\|$ We can choose such a subsequence λ_{n_k}, that $\lim\limits_{k\to\infty} \lambda_{n_k} = \lambda_0$ and $\operatorname{Re}\lambda_0 = 0$. The spectrum $\sigma(-B_0^{-1}B_1)$ is closed in the complex plane ([7]) and thereby $\lambda_0 \in \sigma(-B_0^{-1}B_1)$, $\operatorname{Re}\lambda_0 = 0$ that is a contradiction which completes the proof of (2.16). As we remarked above (2.16) implies (2.17) Then due to (1.8), (2.13), (2.14) the estimate (2.5) holds with the constant $M = N \max(1, \alpha^{-1})$.

We verify now the second part of the theorem. Assume that (2.6) holds. The existence of u_∞ in (2.8) is secured because $A_1 \in L(H, H^*)$ and (2.4) holds. Let us denote

$$\text{(2.21)} \qquad v(t) = u(t) - u_\infty$$

$$\text{(2.22)} \qquad v_0 = u_0 - u_\infty$$

$$\text{(2.23)} \qquad g^*(t) = f^*(t) - f^*_\infty$$

The function v is a solution of the initial value problem

$$\text{(2.24)} \qquad A_0 \frac{dv}{dt} + A_1 v = g^*(t)$$

$$\text{(2.25)} \qquad v(0) = v_0$$

Using (2.5) we obtain

$$\text{(2.26)} \qquad \|v(t)\| \leq M\, e^{-\nu t}\Big(\|v_0\| + \int_0^t e^{\nu\tau}\, \|g^*(\tau)\|_*\, d\tau\Big).$$

If $\lim\limits_{t\to\infty} \int_0^t e^{\nu\tau}\, \|g^*(\tau)\|_*\, d\tau < \infty$ then

$$\text{(2.27)} \qquad \lim_{t\to\infty} \|v(t)\| = 0,$$

otherwise we use L'Hospitale rule and we obtain (2.27) from (2.6) too. Comparing (2.21) and (2.27) we arrive at (2.7) which completes the proof.

Let us consider now the initial value problem of the second order

$$A_0 \frac{d^2u}{dt^2} + A_1 \frac{du}{dt} + A_2 u = f^*(t) \tag{2.28}$$

$$u(0) = u_0 \tag{2.29}$$

$$\left.\frac{du}{dt}\right|_{t=0} = u_1 \tag{2.30}$$

We assume that A_0, A_1, A_2 satisfy the assumptions of Theorem 1.1, A_0, A_1 satisfy (2.3), (2.4) and A_2 is symmetric and coercive i.e.

$$\langle A_2 x, y\rangle = \langle A_2 y, x\rangle, \quad x, y \in H \tag{2.31}$$

$$\langle A_2 x, x\rangle \geqslant \alpha_2 \,.\, \|x\|^2, \quad \alpha_2 > 0, x \in H. \tag{2.32}$$

The following theoren expresses the asymptotic behaviour of a solution of the problem (2.28), (2.29), (2.30).

Theorem 2.2. *Assume that the funnction f^*, the elements u_0, u_1, the operators A_0, A_1, A_2 satisfy the assumptions of* Theorem 1.1 *and A_0, A_1, A_2 satisfy* (2.3), (2.4), (2.31), (2.32). *Then a solution $u \in C^{(2)}([0, \infty), H)$ of* (2.28), (2.29), (2.30) *satisfies the estimate*

$$(\|u(t)\|^2 + \|u'(t)\|^2)^{1/2} \tag{2.33}$$

$$M\, e^{-\nu . t}[(\|u_0\|^2 + \|u_1\|^2)^{1/2} + \int_0^t e^{\nu\tau}\, \|f^*(\tau)\|_*\, d\tau]$$

with the constants M, $\nu > 0$ depending only on A_0, A_1, A_2.

If there exists such a functional $f^* \in H$ that

$$\lim_{t\to\infty} \|f^*(t) - f^*_\infty\|_* = 0, \tag{2.34}$$

then

$$\lim_{t\to\infty} (\|u(t) - u_\infty\| + \|u'(t)\|) = 0, \tag{2.35}$$

where $u_\infty \in H$ is a solution of the equation

$$A_2 u_\infty = f^*_\infty. \tag{2.36}$$

Proof. We can consider instead of (2.28) the equation

$$\frac{d^2u}{dt^2}+B_0^{-1}B_1\frac{du}{dt}+B_0^{-1}B_2=B_0^{-1}f(t) \tag{2.37}$$

with the operators $B_r=RA_r$, $r=0, 1, 2$ and the function $f=Rf^*$. The initial value problem (2.28), (2.29), (2.30) is equivalent with the problem (2.37), (2.29), (2.30). The operators B_r are coercive

$$(B_rx, x)\geq\alpha_r\,\|x\|^2,\quad \alpha_r>0,\ r=0, 1, 2,\ x\in H. \tag{2.38}$$

The operators B_0, B_2 are symmetric

$$(B_rx, y)=(x, B_ry),\ r=0, 2;\ x, y\in H. \tag{2.39}$$

The problem (2.37), (2.29), (2.30) can be formulated as the initial value problem of the first order in the space $\mathscr{H}=H\otimes H$

$$\frac{dU}{dt}+\mathscr{B}U=F(t) \tag{2.40}$$

$$U(0)=U_0 \tag{2.41}$$

with

$$F(t)=(\bar{0}, B_0^{-1}f(t))^T \tag{2.42}$$

$$U_0=(u_0, u_1)^T \tag{2.43}$$

$$\mathscr{B}=\begin{pmatrix}0 & -I\\ B_0^{-1}B_1 & B_0^{-1}B_2\end{pmatrix} \tag{2.44}$$

A solution $U\in C^{(1)}([0, \infty), \mathscr{H})$ of (2.40), (2.41) has the form

$$U(t)=e^{-\mathscr{B}t}U_0+\int_0^t e^{-\mathscr{B}(t-\tau)}F(\tau)\,d\tau. \tag{2.45}$$

We shall use the spectral theory in the same way as in the proof of Theorem 2.1. We extend the space H to the complex Hilbert space. The extended operators B_0, B_1, B_2 remain coercive and B_0, B_2 remain symmetric. Our aim is to verify the existence of a constant $\nu>0$ such that

$$\operatorname{Re}\lambda<-\nu,\quad \lambda\in\sigma(-\mathscr{B}) \tag{2.46}$$

If (2.46) holds, then there exists a constant N depending only on $\mathscr{B}$ such that

$$\|e^{-\mathscr{B}t}\|_{\mathscr{H}}\leq N\,e^{-\nu t},\quad t\in[0, \infty). \tag{2.47}$$

Comparing with the proof of Theorem 2.1 we can see, that it is sufficient to verify that $\operatorname{Re}\lambda<0$ for all $\lambda\in\sigma(-\mathscr{B})$. It can be verified easily that $\lambda\in\sigma(-\mathscr{B})$ iff $0\in\sigma(D(\lambda))$, where

(2.48) $$D(\lambda)=\lambda^2B_1+\lambda B_1+B_2.$$

If $\lambda=0$, then $D(\lambda)=B_2$. The operator B_2 is coercive and hence $0\in\sigma(D(\lambda))$. Let $\operatorname{Re}\lambda\geqslant 0$, $\lambda\neq 0$. We consider, instead od $D(\lambda)$, the operator ([5])

(2.49) $$T(\lambda)=\lambda^{-1}D(\lambda)=\lambda B_0+B_1+\lambda^{-1}B_2.$$

The adjoint operator $T(\lambda)^*$ has the form

(2 50) $$T(\lambda)^*=\bar{\lambda}B_0+B_1^*+\bar{\lambda}^{-1}B_2$$

The operators $T(\lambda)$, $T(\lambda)^*$ satisfy the relations

(2.51) $$\begin{aligned}&\operatorname{Re}(T(\lambda)x,x)=\operatorname{Re}(T(\lambda)^*x,x)=\\&=\operatorname{Re}\lambda(B_0x,x)+\operatorname{Re}(B_1x,x)+\operatorname{Re}\lambda\,|\lambda|^{-2}(B_2x,x)\geqslant\\&\geqslant\operatorname{Re}(B_1x,x)\geqslant\alpha_1\,\|x\|^2,\ \alpha_1>0,\,x\in H.\end{aligned}$$

(2.51) implies, in the same way as (2 19) in the proof of Theorem 2.1, that $0\ \sigma(T(\lambda))$ i. e. $\lambda\notin\sigma(-\mathscr{B})$. Hence $\operatorname{Re}\lambda<0$ for all $\lambda\in\sigma(-\mathscr{B})$ and as we remarked there exists $\nu>0$ such that (2.46) holds. (2.46) implies (2.47). Combining (2.45) and (2.47) we obtain

(2.52) $$\|U(t)\|_{\mathscr{H}}\leq N\,\mathrm{e}^{-\nu t}(\|U_0\|_{\mathscr{H}}+\int_0^t \mathrm{e}^{\nu\tau}\,\|F(\tau)\|_{\mathscr{H}}\,\mathrm{d}\tau)$$

Using (2.40), (2.42), (2.43), (2.52) we obtain the inequality (2.33) with the constant $M=N\max(1,\alpha_0^{-1})$.

The second assertion of the theorem can be verified in the same way as (2.7) in Theorem 2.1. It is sufficient to use (2.33).

3. Bending of Viscoelastic Plates

The previous theory can be applied to the mixed problems which express bending of viscoelastic plates ([2], [3]).

(3.1) $$\sum_{r=0}^{m}K_{ijkl}^{(r)}\frac{\mathrm{d}^{m-r}}{\mathrm{d}t^{m-r}}u_{,ijkl}=f^*(x_1,x_2,t)$$

(3.2) $$\left.\frac{\mathrm{d}^r u}{\mathrm{d}t^r}\right|_{t=0}=u_r,\ \ r=0,1,\dots,m-1$$

(3.3) $$u=\frac{\partial u}{\partial n}=0\quad\text{on}\quad\partial\Omega$$

or

$$u=0 \quad \text{on}$$

$$(3.4) \qquad Nu=\sum_{r=0}^{m} K^{(r)}_{ijkl}\frac{\mathrm{d}^{m-r}}{\mathrm{d}t^{m-r}}\, u_{,ij}\cos(n, x_k)\cos(n, x_e)=0 \quad \text{on} \quad \partial\Omega$$

We use the notation $u_{,ijkl}=\dfrac{\partial^4 u}{\partial x_i\,\partial x_j\,\partial x_k\,\partial x_l}$, $i, j, k, l \in \{1,2\}$. Summation over repeated subscripts $i.\, j.\, k.\, l$ is implied. The plate has the form of a bounded domain $\Omega\subset E_2$ with Lipschitzian boundary $\partial\Omega$ (def. [8]). The coefficients $K^{(r)}_{ijkl}$ are symmetric i. e.

$$(3.5) \qquad K^{(r)}_{ijkl}=K^{(r)}_{jikl}=K^{(r)}_{ijlk}=K^{(r)}_{klij}$$

and positive definite i.e.

$$(3.6) \qquad \begin{gathered} K^{(r)}_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\geqslant c_r(\varepsilon_{11}^2+\varepsilon_{12}^2+\varepsilon_{22}^2),\\ c_r>0,\ r=0, 1, \ldots, m,\ (\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22})\in E_3. \end{gathered}$$

The classical solutions of the problems (3.1), (3.2), (3.3) or (3.4) can be established only for sufficiently smooth boundary $\partial\Omega$ ([2]). We introduce therefore a weak solutions of the problems.

We denote by $H^2(\Omega)$ the Sobolev space of all functions from $L_2(\Omega)$, whose generalized derivatives up to the 2-nd order are in $L_2(\Omega)$. The scalar product in $H^2(\Omega)$ is defined by

$$(3.7) \qquad (u, v)_2=\sum_{|i|\leqslant 2}\int_\Omega D^i u D^i v \,\mathrm{d}\Omega$$

$$\left(D^i u=\frac{\partial^{|i|}u}{\partial x_1^{i_1}\,\partial x_2^{i_2}},\quad i=i_1+i_2\right).$$

Let $\mathscr{D}(\Omega)$ be the set of all arbitrarily differentiable functions with a compact support in Ω and $H^2_0(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ in the space $H^2(\Omega)$. It is well known ([8]) that $H^2_o(\Omega)$ is a Hilbert space with a scalar product

$$(3.8) \qquad (u, v)_0=\sum_{|i|\leqslant 2}\int_\Omega D^i u D^i v\,\mathrm{d}\Omega$$

and a norm

$$(3.9) \qquad \|u\|_0=\Big(\sum_{|i|=2}\int_\Omega (D^i u)^2\,\mathrm{d}\Omega\Big)^{1/2}$$

which is equivalent with the original norm in $H^2(\Omega)$. We denote further by $H^{-2}(\Omega)$

the space of all linear functionals over $\mathring{H}^2(\Omega)$. It can be verified with the help of Fridrichs inequality ([8]), that $L_2(\Omega)\subset H^{-2}(\Omega)$ and if $f^* \in L_2(\Omega)$, $h \in \mathring{H}^2(\Omega)$,

then $\langle f^*, h\rangle = f^*(h) = \int_\Omega f^*h \, \mathrm{d}\Omega$.

Definition 3.1. *Let $f^* \in C([0, \infty), H^{-2}(\Omega))$, $u_r \in \mathring{H}^2(\Omega)$, $r=0, 1, \ldots, m-1$. The function $u \in C^{(m)}([0, \infty), \mathring{H}^2(\Omega))$ which is for each $h \in \mathring{H}^2(\Omega)$ a solution of the initial value problem*

$$\sum_{r=0}^{m} \frac{\mathrm{d}^{m-r}}{\mathrm{d}t^{m-r}} \int_\Omega K^{(r)}_{ijkl} u_{,ij}(t) h_{,kl} \, \mathrm{d}\Omega = \langle f^*(t), h\rangle \tag{3.10}$$

$$\left.\frac{\mathrm{d}^r u}{\mathrm{d}t^r}\right|_{t=0} = u_r, \quad r=0, 1, \ldots, m-1, \tag{3.11}$$

is a weak solution of the problem (3.1), (3.2), (3.3).

Theorem 3.1. *There exists a unique weak solution of the problem* (3.1), (3.2), (3.3).

Proof. It is sufficient to use Theorem 1.1. In this case the operators $A_r : (\mathring{H}^2(\Omega) \to H^{-2}(\Omega))$ are defined with the help of the duality

$$\langle A_r u, h\rangle = \int_\Omega K^{(r)}_{ijkl} u_{,ij} h_{,kl} \, \mathrm{d}\Omega, \tag{3.12}$$

$$u, h \in \mathring{H}^2(\Omega), \quad r=0, 1, \ldots, m.$$

The operators A_r are linear and bounded i. e. $A_r \in L(\mathring{H}^2(\Omega), H^{-2}(\Omega))$. Using (3.5), (3.6), (3.9) we obtain that A_r are symmetric and coercive and hence all assumptions of Theorem 1.1 are fulfilled. The initial value problem (0.1), (0.2) is in this case equivalent with the problem (3.10), (3.11) and the proof is complete.

In the case of the problem (3.1), (3.2), (3.4) we define a weak solution which satisfies only the essential boundary condition $u=0$ on $\partial\Omega$. We denote by $\dot{H}^2(\Omega)$ the subspace of $H^2(\Omega)$ which consists of all functions vanishing on $\partial\Omega$:

$$\dot{H}^2(\Omega) = \{u \in H^2(\Omega), \ u=0 \text{ on } \partial\Omega\}. \tag{3.13}$$

Due to the theorem on traces ([8]) $\dot{H}^2(\Omega)$ is the closed subspace of $H^2(\Omega)$ and hence $\dot{H}^2(\Omega)$ is the Hilbert space with the scalar product (3.7). It can be verified with the help of Friedrichs and Poincaré inequalities ([8]) that $(.,.)_0$ defined in (3.8) is the scalar product on $\dot{H}^2(\Omega)$ and the norm $\|\cdot\|_0$ in (3.9) is equivalent with the original norm $\|\cdot\|_2 = (.,.)^{1/2}$. Let us denote by $\dot{H}^{-2}(\Omega)$ the dual space of $\dot{H}^2(\Omega)$.

Definition 3.2. *Let* $f^* \in C([0, \infty), \dot{H}^{-2}(\Omega))$, $u_r \in \dot{H}^2(\Omega)$, $r = 0, 1, \ldots, m-1$. *The function* $u \in C^{(m)}([0, \infty), \dot{H}^2(\Omega))$ *which is for each* $h \in \dot{H}^2(\Omega)$ *a solution of the initial value problem*

$$\sum_{r=0}^{m} \frac{\mathrm{d}^r}{\mathrm{d}t^r} \int_{\Omega} K^{(r)}_{ijkl} u_{,ij}(t) h_{,kl} \, \mathrm{d}\Omega = \langle f^*(t), h \rangle \tag{3.14}$$

$$\left. \frac{\mathrm{d}^r u}{\mathrm{d}t^r} \right|_{t=0} = u_r, \quad r = 0, 1, \ldots, m-1 \tag{3.15}$$

is a weak solution of the problem (3.1), (3.2), (3.4).

Theorem 3.2. *There exists a unique weak solution of the problem* (3.1), 3.2), (3.4).

Proof. It is sufficient to use Theorem 1.1 in the same way as in the proof of Theorem 3.1. The operators $A_r: (\dot{H}^2(\Omega) \to \dot{H}^{-2}(\Omega))$ are of the form (3.12) for $u, h \in \dot{H}^2(\Omega)$. They are symmetric due to (3.5) and coercive due to (3.6) and (3.9).

Theorem 3.3. *Assume that* $m = 1$, *or* $m = 2$. *Let* $f^* \in C([0, \infty), H^{-2}(\Omega))$, $f^*_\infty \in H^{-2}(\Omega)$ $(f^* \in C([0, \infty), \dot{H}^{-2}(\Omega))$, $f^*_\infty \in \dot{H}^{-2}(\Omega)))$, $\lim\limits_{t \to \infty} \|f^*(t) - f^*_\infty\|_* - 0$. *If* $u \in C^{(m)}([0, \infty), H^2_0(\Omega))$ $(u \in C^{(m)}([0, \infty), \dot{H}^2(\Omega))$ *is a weak solution of the problem* (3.1), (3.2), (3.3), ((3.4)) *then*

$$\lim_{t \to \infty} \|u(t) - u_\infty\| = 0, \tag{3.16}$$

where $u_\infty \in H^2_0(\Omega)$ $(u_\infty \in \dot{H}^2(\Omega))$ *is for each* $h \in H^2_0(\Omega)$ $(h \in \dot{H}^2(\Omega))$ *a solution of the problem*

$$\int_{\Omega} K^{(m)}_{ijkl} u_{,ij} h_{,kl} \, \mathrm{d}\Omega = \langle f^*_\infty, h \rangle. \tag{3.17}$$

Proof. Due to the symmetry and the coerciveness of the operators A_r in (3.12) Theorems 2.1 and 2.2 can be applied. The result follows directly.

Remark. Theorem 3.3 expresses the fact that a solution of viscoelastic problems (3.1), (3.2), (3.3), or (3.4) behaves for great time values as a solution of ela tic problem (3.17).

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ПСЕУДОПАРАВОЛИЧЕСКИЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ БЫСШЕГО ПОРЯДКА

Игор Бок

Резюме

В этой работе изучается начальная задача (0.1), (0.2) в пространстве Гильберта H с операторами $A_r \in L(H, H^*)$. Если оператор A_0 коэрцивный, потом для любой непрерывнои функции и любых элементов $u_r \in H$ существует единственное решение начальной задачи (0.1), (0.2). Если эта начальная задача первого или второго порядка, операторы A, A_1, A_2 коэрцивные, операторы A_0, A_2 симетрические и $\lim_{t\to\infty} \|f^*(t) - f^*_\infty\|_* = 0$, $f^*_\infty \in H^*$, то $\lim_{t\to\infty} \|u(t) - u_\infty\| = 0$, где $u_\infty \in H$ решение уравнения $A_m v = f^*_\infty$. Полученные результаты используются для решениа смешанных проблем, которые определяют изгибы вязкоупругих пластин