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ON CONVERGENCE SPACES AND GROUPS

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In literature two types of convergence spaces and convergence groups can be found. While in one of them the convergence is defined by means of filters (see [2], [3], [4], [5]), in the other this is done by means of sequences (see [6]). The purpose of the present paper is to study their mutual relations. The first two sections of this paper are devoted to convergence spaces. In the third section it is shown that to each sequential convergence group there corresponds a filter convergence group with the same closure operator. Using an example constructed by J. Novák in [6], three problems given by B. V. Hearsy in [3] are solved.

0.

In order to avoid misunderstandings and to make the paper more self-contained we recall in this section the definitions of both types of convergence spaces and groups and state some of their basic properties (cf. [3], [6]).

Let S be a non-empty set, let $F(S)$ be the set of all filters on S, and let $\exp S$ be the power set of S. A mapping $q: F(S)\to \exp S$ is called a (filter) convergence structure for S if the following conditions are satisfied:

(a) $x\in q(\dot{x})$ for each $x\in S$, where $\dot{x}=\{A\subset S|x\in A$.

(b) If $x\in q(\mathcal{F})$, then $\mathcal{G}\supset\mathcal{F}$ implies $x\in q(\mathcal{G})$.

(c) If $x\in q(\mathcal{F})$, then $x\in q(\mathcal{F}\cap\dot{x})$.

If, moreover,

(d) $x\in q(\mathcal{F})$ and $x\in q(\mathcal{G})$ imply $x\in q(\mathcal{F}\cap\mathcal{G})$ holds true, then q is said to be a Limitierung.

Let q be a convergence structure for S. Denote by $\mathcal{N}(x)$ the intersection filter of all filters $\mathcal{F}\in F(S)$ such that $x\in q(\mathcal{F})$. If q satisfies the following condition:

(e) $x\in q(\mathcal{N}(x))$ for each $x\in S$, then q is called a pretopology.

A pair (S, q), where q is a convergence structure for S, is called a (filter) convergence space and if $x\in q(\mathcal{F})$, then we say that $\mathcal{F}$ q-converges to x. The set of all convergence structures for S is denoted by $C(S)$.

Let L be a non-empty set. Then a multivalued (sequential) convergence for L is usually defined as a set $\mathfrak{L} \subset L^N \times L$ of pairs $(\langle x_n \rangle, x)$ which satisfies the following conditions:

($\mathscr{L}_1$) Constant sequences converge, i.e. $(\langle x \rangle, x) \in \mathfrak{L}$ for each $x \in L$.

($\mathscr{L}_2$) Subsequences of a converging sequence converge, i.e. $(\langle x_n \rangle, x) \in \mathfrak{L}$ implies $(\langle x_n' \rangle, x) \in \mathfrak{L}$ for each subsequence $\langle x_n' \rangle$ of $\langle x_n \rangle$.

The set of all multivalued convergences for L is denoted by $S(L)$. A multivalued convergence is called one-valued, or briefly a convergence, if the additional condition

($\mathscr{L}_0$) A sequence has at most one limit, i.e. $(\langle x_n \rangle, x) \in \mathfrak{L}$ and $(\langle x_n \rangle, y) \in \mathfrak{L}$ imply $x = y$, holds true.

From q, resp. $\mathfrak{L}$, a closure operator k_q, resp. λ, (a closure operator u for X has all the properties of a topological closure operator except that it is not necessarily idempotent, i.e. u: $\exp X \to \exp X$, $\emptyset = u\emptyset$, $A \subset uA$ and $u(A \cup B) = uA \cup uB$ for each A, $B \subset X$) can be derived in a natural way:

(1) For $A \subset S$ we put $k_q A = \{x \in S \mid A \in \mathscr{F},\ x \in q(\mathscr{F})\}$.

(2) For $A \subset L$ we put $\lambda A = \{x \in L \mid (\langle x_n \rangle, x) \in \mathfrak{L},\ \cup(x_n) \subset A\}$.

If $\mathfrak{L}$ is a convergence for L, then $(L, \mathfrak{L}, \lambda)$, or simply (L, λ), is called a (sequential) convergence space.

Let (X, u) be a closure space (i.e. u is a closure operator for X). Then $U \subset X$ is called a neighborhood of $x \in X$ if $x \in X - u(X - U)$. Denote by $\mathscr{N}(x)$ the filter of all neighborhoods of x and define a mapping q: $F(X) \to \exp X$ as follows:

(3) If $\mathscr{F} \supset \mathscr{N}(x)$, then $x \in q(\mathscr{F})$.

Then q is a pretopology and $k_q = u$. Thus we shall make no distinction between a pretopology and the corresponding closure operator.

Both $C(S)$ and $S(L)$ are endowed by a partial order:

(4) $q_1 \leqq q_2$ if for each $\mathscr{F} \in F(S)$ we have $q_1(\mathscr{F}) \supset q_2(\mathscr{F})$,

(5) $\mathfrak{L}_1 \leqq \mathfrak{L}_2$ if $\mathfrak{L}_1 \subset \mathfrak{L}_2$,

and by an equivalence relation:

(6) $q_1 \sim q_2$ if $k_{q_1} = k_{q_2}$,

(7) $\mathfrak{L}_1 \sim \mathfrak{L}_2$ if $\lambda_1 = \lambda_2$.

If $[q]$, resp. $[\mathfrak{L}]$, is the equivalence class containing q, resp. $\mathfrak{L}$, then k_q is the least element of $[q]$ and $[\mathfrak{L}]$ possesses the greatest element $\mathfrak{L}^*$ which is characterized by the following property (sometimes called the Fréchet-Urysohn axiom):

($\mathscr{L}_3$) If each subsequence $\langle x_n' \rangle$ of a sequence $\langle x_n \rangle$ contains a subsequence $\langle x_n'' \rangle$ such that $(\langle x_n'' \rangle, x) \in \mathfrak{L}$, then $(\langle x_n \rangle, x) \in \mathfrak{L}^*$.

Thus there is a one-to-one correspondence between pretopologies, resp. multivalued convergences satisfying ($\mathscr{L}_3$), and equivalence classes of convergence structures, resp. equivalence classes of multivalued convergences.

For our purpose we introduce another axiom for a sequential convergence $\mathfrak{L}$, which is obviously weaker than ($\mathscr{L}_3$):

($\mathscr{M}\mathscr{L}$) Mixed sequences converge, i.e. $(\langle x_n \rangle, x) \in \mathfrak{L}$ and $(\langle y_n \rangle, x) \in \mathfrak{L}$ imply

$(\langle z_n \rangle, x) \in \mathfrak{L}$, where $z_1 = x_1$, $z_2 = y_1$, $z_3 = x_2$, $z_4 = y_2$, ...

Let $(G, +)$ be a group, let q be a convergence structure for G, let $\mathfrak{L}$ be a convergence for G, and let λ be the corresponding closure operator. Then both $(G, q, +)$ and $(G, \mathfrak{L}, \lambda, +)$ are called convergence groups (cf. [5], [6]) if the "operation —" is correspondingly continuous, i.e. if the following conditions are correspondingly satisfied:

(FG) If $x \in q(\mathscr{F})$ and $y \in q(\mathscr{G})$, then $(x-y) \in q(\mathscr{F}-\mathscr{G})$.

(SG) If $(\langle x_n \rangle, x) \in \mathfrak{L}$ and $(\langle y_n \rangle, y) \in \mathfrak{L}$, then there is a subsequence $\langle n_i \rangle$ of $\langle n \rangle$ such that $(\langle x_{n_i} - y_{n_i} \rangle, x-y) \in \mathfrak{L}$, or equivalently, $(\langle x_n \rangle, x) \in \mathfrak{L}$ and $(\langle y_n \rangle, y) \in \mathfrak{L}$ imply $(\langle x_n - y_n \rangle, x-y) \in \mathfrak{L}^*$.

From the context it will be always clear what type of convergence group is in question.

Notice that if (G, q) is a convergence group such that q is a pretopology, then it follows from [1, 19B.4] that (G, q) is a topological group.

1.

Let L be a non-empty set. For $\alpha = (\langle x_n \rangle, x) \in L^N \times L$ denote by $\mathscr{F}(\alpha)$ the filter of sections of α, i.e. $\mathscr{F}(\alpha) = \{F \subset L \mid x \in F, x_n \in F$ for all but finitely many $n\}$. Let $\mathfrak{L}$ be a multivalued convergence for L. We define a mapping $q(\mathfrak{L})\colon F(L) \to \exp L$ in the following way:

(8) $x \in q(\mathscr{F})$ if $\mathscr{F} \supset \mathscr{F}(\alpha)$ for some $\alpha = (\langle x_n \rangle, x) \in \mathfrak{L}$.

Proposition 1.1. *$q(\mathfrak{L})$ is convergence structure for L. If $\mathfrak{L}$ satisfies $(\mathscr{L}_0)$, then each filter $\mathscr{F} \in F(L)$ $q(\mathfrak{L})$-converges to at most one $x \in L$. If $\mathfrak{L}$ satisfies $(\mathscr{M}\mathscr{L})$, then $q(\mathfrak{L})$ is a Limitierung.*

The easy proof is omitted.

Proposition 1.2. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Then $k_{q(\mathfrak{L})} = \lambda$.*

Proof. Let $A \subset L$. If $x \in k_{q(\mathfrak{L})}A$, then, according to (1), there is $\mathscr{F} \in F(L)$ such that $A \in \mathscr{F}$ and $x \in q(\mathscr{F})$. From (8) it follows that $\mathscr{F} \supset \mathscr{F}(\alpha)$ for some $\alpha \in \mathfrak{L}$. Since $A \cap F \neq \emptyset$ for each $F \in \mathscr{F}$, we have either $x \in A$ or $\cup(x'_n) \subset A$ for some subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$ and hence $x \in \lambda A$. Consequently, $k_{q(\mathfrak{L})}A \subset \lambda A$. The converse inclusion is trivial.

Example 1.3. The real line is a sequential convergence space such that $\mathfrak{L}$ satisfies $(\mathscr{L}_3)$ and $q(\mathfrak{L})$ is not a pretopology.

Proposition 1.4. *Let $\mathfrak{L}_1, \mathfrak{L}_2$ be multivalued convergences for a non-empty set L. Then the following statements hold:*

(i) *If $\mathfrak{L}_1 \leqq \mathfrak{L}_2$, then $q(\mathfrak{L}_1) \geqq q(\mathfrak{L}_2)$.*

(ii) *$q(\mathfrak{L}_1) \sim q(\mathfrak{L}_2)$ iff $\mathfrak{L}_1 \sim \mathfrak{L}_2$.*

Proof. (i) is trivial. (ii) follows from Proposition 1.2.

Example 1.5. Let $L = \{1, 1/2, 1/3, ..., 1/n, ..., 0\}$ and let $\mathfrak{L}$ be a convergence for L defined as follows: $(\langle x \rangle, x) \in \mathfrak{L}$ for each $x \in L$ and $(\langle x_n \rangle, 0) \in \mathfrak{L}$ if $\langle x_n \rangle$ is a subsequence of the sequence $\langle 1/n \rangle_{n=2}^{\infty}$ such that $\sum_{n=1}^{\infty} x_n < +\infty$. It is easy to see that $\mathfrak{L}^*$ is the usual convergence for real numbers (restricted to L). Denote by λ the closure operator derived from $\mathfrak{L}$. Then $q(\mathfrak{L}) \neq q(\mathfrak{L}^*)$. For, if we put $\alpha = (\langle 1/n \rangle, 0)$, then $0 \in q(\mathfrak{L}^*)(\mathscr{F}(\alpha))$ but $q(\mathfrak{L})(\mathscr{F}(\alpha)) = \emptyset$. Since $\mathfrak{L}$ satisfies $(\mathscr{ML})$, it follows from Proposition 1.1 that $q(\mathfrak{L})$ is a Limitierung. $q(\mathfrak{L})$ is not a pretopology. Really, from Proposition 1.2 it follows that $k_{q(\mathfrak{L})} = k_{q(\mathfrak{L}^*)} = \lambda$, and hence $q(\mathfrak{L}) \sim q(\mathfrak{L}^*) \sim \lambda$. According to Proposition 1.4 we have $q(\mathfrak{L}) = q(\mathfrak{L}^*)$. Since λ is the unique pretopology in $[q(\mathfrak{L})]$ and $q(\mathfrak{L}) \neq q(\mathfrak{L}^*) \geqq \lambda$, it follows that $q(\mathfrak{L})$ cannot be a pretopology.

2.

Let (S, q) be a convergence space. Denote by $\mathfrak{L}(q)$ a subset of $S^N \times S$ defined as follows:

(9) $\alpha = (\langle x_n \rangle, x) \in \mathfrak{L}(q)$ if $x \in q(\mathscr{F}(\alpha))$.

Notice that if q is a pretopology, then (9) is equivalent to the usual definition of the convergence of sequences in a closure space (cf. [1]). For, if $\mathscr{N}(x)$ is the neighborhood filter of x and $\alpha = (\langle x_n \rangle, x)$, then $x \in q(\mathscr{F}(\alpha))$ iff $\mathscr{F}(\alpha) \supset \mathscr{N}(x)$. But $\mathscr{F}(\alpha) \supset \mathscr{N}(x)$ holds iff for each $O \in \mathscr{N}(x)$ we have $x_n \in O$ for all but finitely many n.

Proposition 2.1. *$\mathfrak{L}(q)$ is a multivalued convergence for S. If each filter $\mathscr{F} \in F(S)$ q-converges to at most one point $x \in S$, then $\mathfrak{L}(q)$ satisfies $(\mathfrak{L}_0)$. If q is a pretopology, then $\mathfrak{L}(q)$ satisfies $(\mathscr{L}_3)$. If q_1 and q_2 are convergences for S and $q_1 \geqq q_2$, then $\mathfrak{L}(q_1) \leqq \mathfrak{L}(q_2)$.*

The easy proof is omitted.

Example 2.2. Let (P, u) be a topological non Fréchet space and let λ be the closure operator derived from $\mathfrak{L}(u)$. Then $\lambda \neq u$.

Example 2.3. Let (P, u) be the Čech-Stone compactification of N (the space of natural numbers) and let v be the discrete topology for P. Then $u \neq v$, but $\mathfrak{L}(u) = \mathfrak{L}(v)$.

Example 2.4. Let $(N \cup (\infty), u)$ be the topological space of all ordinals less or equal than ω_0 with the usual order topology. Let q be a convergence structure for $N \cup (\infty)$ defined as follows: $x \in q(\dot{x})$ for each $x \in N \cup (\infty)$, $\infty \in q(\mathscr{F})$ if either $\mathscr{F}$ is a free ultrafilter for N, or $\mathscr{F} = (\mathscr{F}' \cap \dot{\infty})$, where $\mathscr{F}'$ is a free ultrafilter for N. Then $u = k_q$, but, since $(\langle n \rangle, \infty) \in \mathfrak{L}(u)$ and $\langle n \rangle$ is totally $\mathfrak{L}(q)$-divergent, $\mathfrak{L}(q)$ and $\mathfrak{L}(u)$ are not equivalent.

Proposition 2.5. *Let (S, q) be a convergence space. Then the following statements hold:*

(i) *If $\langle x_n \rangle \in S^N$ and $(\langle x_{n+1} \rangle, x) \in \mathfrak{L}(q)$, then $(\langle x_n \rangle, x) \in \mathfrak{L}(q)$.*

(ii) *If* $(\langle x_n \rangle, x) \in \mathfrak{L}(q)$ *and* $f\colon N \to N$ *is one-to-one and onto, then* $(\langle x_{f(n)} \rangle, x) \in \mathfrak{L}(q)$.

(iii) *If* q *is a Limitierung, then* $\mathfrak{L}(q)$ *satisfies* $(\mathcal{ML})$.

The easy proof is omitted.

Proposition 2.6. *Let* $(L, \mathfrak{L}, \lambda)$ *be a convergence space. Then we have* $\mathfrak{L} \leqq \mathfrak{L}(q(\mathfrak{L})) \leqq \mathfrak{L}^*$, *and hence* $\mathfrak{L} \sim \mathfrak{L}(q(\mathfrak{L}))$.

Proof. Since we have $k_{q(\mathfrak{L})} = \lambda$, $\mathfrak{L}^* = \mathfrak{L}(\lambda)$, and $q(\mathfrak{L}) \geqq k$, it follows from the last statement of Proposition 2.1 that $\mathfrak{L}(q(\mathfrak{L})) \subset \mathfrak{L}^*$. The inclusion $\mathfrak{L} \subset \mathfrak{L}(q(\mathfrak{L}))$ is trivial.

Example 2.7. Let $(L, \mathfrak{L}, \lambda)$ be the convergence space from Example 1.5. Then $\mathfrak{L} \neq \mathfrak{L}(q(\mathfrak{L})) \neq \mathfrak{L}^*$. This follows from the fact that $(\langle 1/n^2 \rangle_{n=1}^{\infty}, 0) \in \mathfrak{L}(q(\mathfrak{L})) - \mathfrak{L}$ and $(\langle 1/n \rangle_{n=1}^{\infty}, 0) \in \mathfrak{L}^* - \mathfrak{L}(q(\mathfrak{L}))$.

Proposition 2.8. *Let* (S, q) *be a convergence space. Then* $q(\mathfrak{L}(q)) \geqq q$.

The easy proof is omitted.

Example 2.9. Let $(N \cup (\infty), u)$ be the topological space from Example 2.4. Then it can be easily proved that $u \neq k_{q(\mathfrak{L}(u))}$, and hence u and $q(\mathfrak{L}(u))$ are not equivalent.

3.

Throughout this section we shall use the following notation. If $(G, +)$ is a group, then for each $x \in G$ the symbol $\pm 1x$ means the same as $\pm x$. Let $(G, \mathfrak{L}, \lambda, +)$ be a convergence group. We define a mapping $q\colon F(G) \to \exp G$ as follows:

(10) $x \in q(\mathcal{F})$ if there exist a natural number m, sequences $\alpha_k = (\langle x_{nk} \rangle, x_k) \in \mathfrak{L}$ and numbers $a_k = \pm 1$, $k = 1, 2, \ldots, m$, such that $\sum_{k=1}^{m} a_k x_k = x$, $\mathcal{F} \supset \sum_{k=1}^{m} a_k \mathcal{F}(\alpha_k)$, where the filter $\sum_{k=1}^{m} a_k \mathcal{F}(\alpha_k)$ is generated by the sets of the form $a_1 f_1 + a_2 F_2 + \ldots + a_m F_m$, $F_k \in \mathcal{F}(\alpha_k)$.

Proposition 3.1. $(G, q, +)$ *is a convergence group.*

The straightforward proof is omitted.

Proposition 3.2. *If* $x \in q(\mathcal{F})$, *then* $x \in \lambda F$ *for each* $F \in \mathcal{F}$.

Proof. Let $x \in q(\mathcal{F})$. Then, according to (10), there are a natural number m, sequences $\alpha_k = (\langle x_{nk} \rangle, x_k) \in \mathfrak{L}$ and numbers $a_k = \pm 1$, $k = 1, \ldots, m$, such that $\sum_{k=1}^{m} a_k x_k = x$, $\mathcal{F} \supset \sum_{k=1}^{m} a_k \mathcal{F}(\alpha_k)$. We shall prove that in each set $F \in \mathcal{F}$ there is a sequence $\langle y_i \rangle$ such that $(\langle y_i \rangle, x) \in \mathfrak{L}$, and hence $x \in \lambda F$.

Consider the sets $N^* = N \cup (\infty)$, $\Gamma = \{g\colon \{1, \ldots, m\} \to N^*\}$, $\Phi = \{f\colon \{1, \ldots, m\} \to N\} \subset \Gamma$, which are ordered in the natural way, i.e. $n < \infty$ for each $n \in N$,

$g_1 < g_2$ if $g_1(k) < g_2(k)$ for each $k \in \{1, \ldots, m\}$. If we define $x_{k\sim} = x_k$, then the following statement holds true:

(11) $\forall_{F \in \mathcal{F}} \ \forall_{f \in \Phi} \ \exists_{g \in \Gamma, g > f} : \sum_{k=1}^{m} a_k x_{kg(k)} \in F.$

This follows from the fact that the negation of (11) contradicts the statements: $\left(\bigsqcup_{g > f_0} \left(\sum_{k=1}^{m} a_k x_{kg(k)} \right) \right) \in \sum_{k=1}^{m} a_k \mathcal{F}(\alpha_k)$ and $\mathcal{F} \supset \sum_{k=1}^{m} a_k \mathcal{F}(\alpha_k)$. Now, let $F \in \mathcal{F}$. To reach our final goal we shall construct by induction a sequence $\langle g_n \rangle$, $g_n \in \Gamma$, as follows. According to (11), for $f_1 \in \Phi$, $f_1(i) = 1$, $i = 1, \ldots, m$, we can choose $g_1 \in \Gamma$ such that $f_1 < g_1$ and $\sum_{k=1}^{m} a_k x_{kg_1(k)} \in F$. Suppose that we have already constructed $g_1, \ldots, g_n \in \Gamma$. Denote by $I_n = \{k \in \{1, \ldots, m\}, g_n(k) \in N\}$ and put $g_{n+1} = g_n$ if $I_n = \emptyset$. Otherwise put $f_{n+1} = \sup\{g_n(k), \ k \in I_n\}$ and choose $g_{n+1} \in \Gamma$ such that $g_{n+1} > f_{n+1}$ and $\sum_{k=1}^{m} a_k x_{kg_{n+1}(k)} \in F$. This defines the sequence $\langle g_n \rangle$. Now, there are two possibilities:

a) $g_n(1) = \sim$ for infinitely many n. Then let $\langle g_n^1 \rangle$ be a subsequence of $\langle g_n \rangle$ such that $g_n^1(1) = \sim$.

b) There exists a natural number n_1 such that for each $n > n_1$ we have $g_n(1) < g_{n+1}(1)$. Then let $\langle g_n^1 \rangle$ be a subsequence of $\langle g_n \rangle$ such that $g_n^1 = g_{n_1+n}$. In a similar way we construct a subsequence $\langle g^{k+1} n \rangle$ of $\langle g_n^k \rangle$, $k = 1, \ldots, m$, and finally we obtain a subsequence $\langle g_n^m \rangle$ of $\langle g_n \rangle$ which is for each fixed $k \in \{1, \ldots, m\}$ either constant and $g_n^k(k) = \sim$, or increasing and $\lim g_n^m(k) = \sim$. Put $h_n = g_n^m$. Since $(\langle x_{kh_n(k)} \rangle, x_k) \in \mathfrak{L}$, $k = 1, \ldots, m$, it follows from (SG) that there is a subsequence $\langle h_n' \rangle$ of $\langle h_n \rangle$ such that $\left(\left\langle \sum_{k=1}^{m} a_k x_{kh'_n(k)} \right\rangle, \sum_{k=1}^{m} a_k x_k \right) \in \mathfrak{L}$. From $y_n = \sum_{k=1}^{m} a_k x_{kh'_n(k)} \in F$ and $\sum_{k=1}^{m} a_k x_k = x$ it follows that $x \in \lambda(\cup(y_n)) \in \lambda F$. This completes the proof.

Proposition 3.3. $k_q = \lambda$.

Proof. Let $X \subset G$. If $x \in \lambda X$, then there is a sequence $\langle x_n \rangle$ in X such that $\alpha = (\langle x_n \rangle, x) \in \mathfrak{L}$. From (10) it follows that $x \in q(\mathcal{F}(\alpha))$. For $\mathcal{F} = \{F \subset G \mid x_n \in F$ for all but finitely many $n\}$ we have $\mathcal{F} \supset \mathcal{F}(\alpha)$ and $X \in \mathcal{F}$. From (b) and (1) it follows that $x \in k_q X$. On the other hand if $x \in k_q X$, then there is a filter $\mathcal{F}$ such that $x \in q(\mathcal{F})$ and $X \in \mathcal{F}$. From Proposition 3.2 it follows that $x \in \lambda X$.

Proposition 3.4. *Let $(G, q, +)$ be a (filter) convergence group. Then $(G, \mathfrak{L}(q), \lambda, +)$ is a (sequential) convergence group.*

The easy proof is omitted.

In [3] B. V. Hearsey put forward the following three problems:

Problem 1. Let $(G, q, +)$ be a convergence group. Let $t(q)$ be the finest of all topological closure operators coarser than k_q. Is $(G, t(q), +)$ a topological group?

Problem 2. Is each convergence group weakly regular?

•Problem 3. Is each convergence group t-regular?

Recall that a convergence space (G, q) is t-regular if $t(q)$ is a regular topology and (G, q) is weakly regular if for each $x \in G$ and each k_q-closed set $A \subset G$ such that $x \notin A$ the neighborhood filters $\mathcal{N}(x)$ of x and $\mathcal{N}(A)$ of A are disjoint (i.e. $U \cap V = \emptyset$ for some $U \in \mathcal{N}(x)$, $V \in \mathcal{N}(A)$, where $\mathcal{N}(A) = \cap\{\mathcal{N}(y), y \in A\}$).

We shall show via counterexample that the answer to all three questions is no. Since the topology in a T_1 topological group is always completely regular and each t-regular convergence space is weakly regular (cf. Theorem 3 in [3]), it is sufficient to construct a T_1 convergence group $(G, q, +)$ such that (G, q) is not weakly regular.

Denote by $(G, \mathfrak{L}, \lambda, +)$ the minimal completion of the group of rational numbers constructed by J. Novák in [6]. $(G, \mathfrak{L}, \lambda, +)$ is a sequential convergence group such that G is the set of all real numbers, $+$ is the usual addition, $\mathfrak{L}$ is a convergence for G weaker that the usual metric, and λ is a Hausdorff topology. In (G, λ) a sequence $\langle x_n \rangle$ of rational numbers $\mathfrak{L}^*$-converges to a real number $x \in G$ iff $\lim |x_n - x| = 0$. Denote $z_n = \sqrt{2}/n$, $n \in N$, and $A = \cup(z_n)$. Then no subsequence of $\langle z_n \rangle \mathfrak{L}^*$-converges in G, A is closed, and $0 \notin A$. Let q be the convergence structure for G defined by (10). According to Proposition 3.1 and Proposition 3.3 $(G, q, +)$ is a convergence group and $k_q = \lambda = t(q)$. For each $U \in \mathcal{N}(0)$ there is $\varepsilon > 0$ such that the set $\{x \mid x \text{ rational}, |x| < \varepsilon\}$ is contained in U. Similarly, for each $V \in \mathcal{N}(A)$ and for each $z_n \in A$ there is $\varepsilon_n > 0$ such that the set $\{x \mid x \text{ rational}, |x - z_n| < \varepsilon_n\}$ is contained in V. Consequently, the filters $\mathcal{N}(0)$ and $\mathcal{N}(A)$ are not disjoint and (G, q) is not weakly regular. Since $(G - A) \in \mathcal{N}(0)$ and for each $U \in \mathcal{N}(0)$ we have $\lambda U \cap A \neq \emptyset$, it also follows directly that (G, λ) is irregular.

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О ПРОСТРАНСТВАХ И ГРУППАХ СХОДИМОСТИ

Роман Фрич

Резюме

В статье рассматриваются отношения пространств и групп сходимости двух типов. В первом случае сходимость задача посредством последовательностей а во втором случае посредством фильтров. Вводятся в рассмотрение операторы сопоставляющие каждому пространству одного типа некоторое пространство типа другого. Доказывается, что для каждой группы сходимости первого типа существует группа сходимости второго типа так, что замыкание множеств не изменяется. Это позволяет решить некоторые проблемы классификации групп сходимости второго типа.