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INTEGRALS ON LATTICE-ORDERED GROUPS

JÁN ŠIPOŠ

The notion of an upper integral was introduced for a real valued function by Bourbaki [2] and Topsoe [6]. Integrability with respect to an upper integral was introduced there as well. These methods resemble the measurability of sets with respect to an outer measure. Neither Bourbaki's nor Topsoe's definition of an upper integral is axiomatic. Both utilize the properties which follow from the special methods of constructing the integral. The aim of this paper is to introduce the notion of integrability which will be defined axiomatically. The family of all real functions on any space is an l-group. So we shall build up an integration theory on an l-group.

Integration

We shall be concerned below with lattice ordered groups, or l-groups, in the following sense.

Definition 1. *An l-group S is* (i) *a lattice,* (ii) *a commutative group, in which* (iii) *the inclusion relation* $\leqq$ *induced by the lattice structure is invariant under all group-translations* $x \mapsto a + x$, *i.e. if* $a, b, x \in S$ *and* $a \leqq b$, *then* $a + x \leqq b + x$.

An element a of an l-group S is called positive iff $a \geqq 0$, where 0 is a neutral element of the additive group S. The set of all positive elements of S will be denoted by S^+.

Definition 2. *An upper integral is a function* $\mu \colon S^+ \to \langle 0, \infty \rangle$ *satisfying the following conditions:*

(i) $\mu(0) = 0$,

(ii) *if* $x \leqq y$, *then* $\mu(x) \leqq \mu(y)$ *(monotonity),*

(iii) *for every* $x, y \in S^+$

$$\mu(x + y) \leqq \mu(x) + \mu(y)$$

(subadditivity),

(iv) *if* $x_n \nearrow x$ *(i.e. if* $x = \bigvee_{n=0}^{\infty} x_n$ *and* $x_n \leqq x_{n+1}$*), then*

$$\lim_n \mu(x_n) = \mu(x),$$

(continuity from below).

Example 1. Let

$$S^+ = \{f; f: \langle 0,1 \rangle \to \langle 0, \infty)\},$$

and let

$$\mu(f) = \sup \{f(x); x \in \langle 0,1 \rangle\},$$

then μ is an upper integral. The proofs of (i), (ii), (iii) and (iv) from Definition 2 are trivial.

Example 2. Let X be a locally compact topological space. By $\mathscr{K}(X)$ we denote the space of all real valued continuous functions on X with compact support, and by $\mathscr{I}^+(X)$ we denote the space of all real valued, nonnegative lower semi-continuous functions.

Let μ be a positive Radon measure on $\mathscr{K}(X)$. We define a map $\mu^*: \mathscr{F}^+ \to \langle 0, \infty \rangle$, where $\mathscr{F}^+ = \{f; f: X \to \langle 0, \infty)\}$. If $f \in \mathscr{I}^+(X)$ we put

$$\mu^*(f) = \sup \{\mu(g); g \in \mathscr{K}^+(X), g \leqq f\}.$$

If $f \in \mathscr{F}^+$, we put

$$\mu^*(f) = \inf \{\mu^*(h); h \in \mathscr{I}^+(X), h \geqq f\}.$$

We propose that μ^* is an upper integral on $\mathscr{F}^+$. For the proof see [2].

We note that if we put $0.\ \infty = 0$, then in examples 1 and 2 $\mu(\alpha f) = \alpha\mu(f)$, $\mu^*(\alpha f) = \alpha\mu^*(f)$ for every positive α.

Definition 3. $x \in S^+$ *is said* $(\mu, +)$*-integrable iff*

$$\mu(a) = \mu(x \wedge a) + \mu(a - (x \wedge a))$$

for every $a \in S^+$.

By S_μ^+ we denote the set of all $(\mu, +)$-integrable elements in S^+. Instead of saying that x is $(\mu, +)$-integrable we shall sometimes use only that x is integrable.

It is easy to prove that 0 is always integrable.

Lemma 1. *In any l-group we have*

(i) $x \wedge a + (a - x \wedge a) \wedge y = (x + y) \wedge a$,

(ii) $a - x \wedge a - (a - x \wedge a) \wedge y = a - (x + y) \wedge a$ *for every* $a, x, y \in S^+$.

Proof. Since it is clear that the assumption (i) implies (ii), there remains only (i) to be proved. Since the ordering in the l-group is invariant under all group-translations (see Definition 1), it follows

$$(u + w) \wedge (v + w) = w + (u \wedge v). \tag{1}$$

If we replace in (1) u by $(a-x\wedge a)$, v by y and w by $(x\wedge a)$, we get

$$a\wedge(x\wedge a+y)=(x\wedge a)+(a-(x\wedge a))\wedge y\,.$$

Applying (1) to the first bracket we get

$$(x\wedge a)+(a-(x\wedge a))\wedge y=a\wedge((x+y)\wedge(a+y))=a\wedge(x+y)\,.$$

Proposition 1.

(i) *If x and y are in S_μ^+, then $x+y$ is in S_μ^+.*

(ii) *if x is in S_μ^+, then $\mu(x+a)=\mu(x)+\mu(a)$ for all a in S^+.*

Proof. (i) If x and y are in S_μ^+ and $a\in S^+$, then

(3) $\mu(a)=\mu(x\wedge a)+\mu(a-x\wedge a)$,

(4) $\mu(a-x\wedge a)=\mu((a-x\wedge a)\wedge y)+\mu((a-x\wedge a)-(a-x\wedge a)\wedge y)$.

Substituting (4) in (3) we obtain

$$\begin{aligned}\mu(a)&=\mu(x\wedge a)+\mu((a-x\wedge a)\wedge y)+\\&\quad\mu((a-x\wedge a)-(a-x\wedge a)\wedge y)\\&\geqq\mu(x\wedge a+(a-x\wedge a)\wedge y)+\mu(a-(x+y)\wedge a)\\&\geqq\mu((x+y)\wedge a)+\mu(a-(x+y)\wedge a)\geqq\mu(a).\end{aligned}$$

Here we have used twice the subadditivity of μ and Lemma 1.

(ii) If x is in S_μ^+ and $a\in S^+$, then

(5) $\mu(a)=\mu(x\wedge a)+\mu(a-x\wedge a)$.

If in the equality (5) we replace a by $a+x$, we get

$$\begin{aligned}\mu(a+x)&=\mu(x\wedge(a+x))+\mu(a+x-(x\wedge(a+x)))\\&=\mu(x)+\mu(a).\end{aligned}$$

Proposition 2. *If x and y are in S_μ^+, then $x\wedge y$ is in S_μ^+ too.*

Proof. If x and y are in S_μ^+ and $a\in S^+$, then

(6) $\mu(a)=\mu(x\wedge a)+\mu(a-x\wedge a)$,

and

(7) $\mu(x\wedge a)=\mu(x\wedge a\wedge y)+\mu(x\wedge a-x\wedge a\wedge y)$.

Substituting (7) in (6) we obtain

$$\begin{aligned}\mu(a)&=\mu(x\wedge a\wedge y)+\mu(x\wedge a-x\wedge a\wedge y)+\mu(a-x\wedge a)\\&\geqq\mu(x\wedge a\wedge y+x\wedge a-(x\wedge a\wedge y)+a-x\wedge a)=\mu(a).\end{aligned}$$

Here we used the subadditivity of μ.

Lemma 2. In any l-group S we have

(i) $(a+y)\wedge x=a\wedge(x-y)+y$,

(ii) $a+y-(a+y)\wedge x=a-a\wedge(x-y)$,

for every a, x, $y\in S$.

Proof. If, similarly as in Lemma 1, we replace in the equality (1) u by a, v by $x-y$ and w by y, we get

$$(a+y)\wedge x = a\wedge(x-y)+y.$$

(ii) is a simple consequence of (i).

Proposition 3. *If x and y are in S_μ^+, $x \geqq y$ and $\mu(y) < \infty$, then $x-y$ is in S_μ^+.*

Proof. Let x and y be in S_μ^+, $x \geqq y$ and $\mu(y) < \infty$. From the $(\mu, +)$-integrability of x we have (for $a \in S^+$)

$$\mu(a+y) = \mu((a+y)\wedge x) + \mu(a+y-(a+y)\wedge x).$$

Using (ii) from Proposition 1 and Lemma 2 we have

$$\mu(a)+\mu(y) = \mu((a\wedge(x-y)) + \mu(y) + \mu(a - a\wedge(x-y)).$$

Since $\mu(y) < \infty$, we get

$$\mu(a) = \mu(a\wedge(x-y)) + \mu(a - a\wedge(x-y)).$$

Proposition 4. *If x and y are in S_μ^+ and $\mu(x\wedge y) < \infty$, then $x\vee y$ is in S_μ^+.*

Proof. The proof of this theorem is a simple consequence of the identity

$$x\vee y = x+y-x\wedge y,$$

and the information already gained at this stage.

Proposition 5. *If $\{x_n\}$ is an increasing sequence of elements in S_μ^+ for which $\vee_n x_n = x \in S$, then $x \in S_\mu^+$, and*

$$\mu(x) = \lim_n \mu(x_n).$$

Proof. If $a \in S^+$, then for every n

$$\mu(a) = \mu(x_n\wedge a) + \mu(a - x_n\wedge a)$$

and

$$\mu(a) = \lim_n \mu(x_n\wedge a) + \lim_n \mu((a-x_n)\wedge a).$$

Since $a - x_n\wedge a \geqq a - x\wedge a$ and $x_n\wedge a \nearrow x\wedge a$, using continuity μ from below, and subadditivity we get

$$\mu(a) \geqq \mu(x\wedge a) + \mu(a - x\wedge a) \geqq \mu(a).$$

The desired identity of the proposition is an immediate consequence of Definition 2 (iv).

Proposition 6. *If $\{x_n\}$ is a decreasing sequence of elements in S_μ^+ ($\mu(x_n) < \infty$ for some n) for which $\wedge_n x_n = x \in S^+$, then $x \in S_\mu^+$ and*

$$\mu(x) = \lim_n \mu(x_n).$$

Proof. If $\mu(x_m) < \infty$, then for $n \geqq m$ $\mu(x_n) \leqq \mu(x_m) < \infty$, and therefore $\mu(x) < \infty$. If $a \in S^+$, then for every n

$$\mu(a)=\mu(x_n\wedge a)+\mu(a-x_n\wedge a).$$

Since $a-(x_n\wedge a)\nearrow a-(x\wedge a)$ and $x_n\wedge a\geqq x\wedge a$, we have

$$\begin{aligned}\mu(a)&\geqq\mu(x\wedge a)+\lim_n\,\mu(a-x_n\wedge a)\\&=\mu(x\wedge a)+\mu(a-x\wedge a)\geqq\mu(a).\end{aligned}$$

It follows from the integrability of x_n that

$$\mu(x_m)=\mu(x_m\wedge x_n)+\mu(x_m-x_m\wedge x_n)$$

and

$$\begin{aligned}\mu(x_m)&=\lim_n\,\mu(x_n)+\lim_n\,\mu(x_m-x_n)\\&=\lim_n\,\mu(x_n)+\mu(x_m-x)\\&\geqq\lim_n\,\mu(x_n)+\mu(x_m)-\mu(x).\end{aligned}$$

From this $\mu(x)\geqq\lim_n\,\mu(x_n)$. The opposite inequality is clear.

Definition 4. *An element x in S is μ-integrable iff there exist elements x_1 and x_2 in S_μ^+ such that $x=x_1-x_2$ and $\mu(x_1)$, $\mu(x_2)<\infty$. The number $\mu(x_1)-\mu(x_2)$ is the μ-integral of x, we shall denote it by $\mu(x)$.*

It is clear that the definition of μ is correct. We shall denote by S_μ the set of the μ-integrable elements in S.

Theorem 1. *The integrable elements of S form an l-group S_μ.*

Proof. It follows from the Proposition 1 that S_μ is a group. Let x be in S_μ, $x=x_1-x_2$, where x_1 and x_2 are in S_μ^+ and $\mu(x_1)$, $\mu(x_2)<\infty$. Since $x^+=x_1-x_1\wedge x_2$ and $x^-=x_2-x_1\wedge x_2$ it follows that x^+ and x^- are in S_μ. The following identities show that S_μ is a lattice

$$x\vee y=(y-x)^++x,\quad x\wedge y=-(-x\vee -y).$$

Theorem 2. *If $\{x_n\}$ is a sequence of non-negative μ-integrable elements for which $\mu\left(\sum_{k=1}^{n}x_k\right)\leqq c<\infty$ for every n, then $x=\sum_{k=1}^{\infty}x_k$ is μ-integrable if $\sum_{k=1}^{\infty}x_k$ exists and*

$$\mu(x)=\sum_{k=1}^{\infty}\mu(x_k).$$

Proof. Since $\sum_{k=1}^{n}x_k\nearrow x$, it follows that $x\in S_\mu^+$. On other hand

$$\mu(x)=\lim_n\mu\left(\sum_{k=1}^{n}x_k\right)=\lim_n\sum_{k=1}^{n}\mu(x_k)=\sum_{k=1}^{\infty}\mu(x_k).$$

Finally $\mu(x^+)\leqq c$ and $\mu(x^-)=0$, it follows that x is in S_μ.

Corollary. *If $\{x_n\}$ is an increasing sequence of elements in S_μ, $x_n\nearrow x$ and $\mu(x_n)\leqq c<\infty$, then $x\in S_\mu$ and*

$$\mu(x) = \lim_n \mu(x_n).$$

Theorem 3. *If $\{x_n\}$ is a sequence of μ-integrable elements which converges to x (i.e. $\limsup_n x_n = \liminf_n x_n = x$) and if y is a μ-integrable element such that $|x_n| \leqq |y|$, then x is μ-integrable and*

$$\lim_n \mu(x_n) = \mu(x).$$

Proof. Let us denote $z_n^m = \bigvee_{i=n}^{m} x_i$, then z_n^m is in S_μ and $z_n^m \nearrow \bigvee_{i=n}^{\infty} x_i$ $(m \to \infty)$. Since $\mu(z_n^m) \leqq \mu(y) < \infty$ it follows from Corollary of Theorem 2 that $z_n = \bigvee_{i=n}^{\infty} x_i$ is in S_μ. Similarly $z_n' = \bigwedge_{i=n}^{\infty} x_i$ is in S_μ. Further $x = \bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} x_i \leqq \bigvee_{i=n}^{\infty} x_i = z_n$ and $x = \bigvee_{n=1}^{\infty} \bigwedge_{i=n}^{\infty} x_i \geqq \bigwedge_{i=n}^{\infty} x_i = z_n'$. Finally for every n

$$-y \leqq z_n' \leqq z_{n+1}' \leqq x \leqq z_{n+1} \leqq z_n \leqq y.$$

Clearly $z_n' \nearrow x$, $z_n \searrow x$, $\mu(z_n') \nearrow \mu(x)$, $\mu(z_n) \searrow \mu(x)$ and $x \in S_\mu$. Since $\mu(z_n') \leqq \mu(x_n) \leqq \mu(z_n)$ hence

$$\lim_n \mu(x_n) = \mu(x).$$

Theorem 4. *If $\{x_n\}$ is a sequence of non-negative μ-integrable elements which converges to x and $\mu(x_n) \leqq c < \infty$, then x is μ-integrable and $\mu(x) \leqq c$.*

Proof. Let us denote $z_n = \bigwedge_{i=n}^{\infty} x_i$. Clearly $z_n \nearrow x$ and $z_n \leqq x_n$. It follows from the Corollary of Theorem 2 that $x \in S_\mu$ and $\mu(x) = \lim_n \mu(z_n) \leqq c$, because $\mu(z_n) \leqq \mu(x_n) \leqq c$ for every n.

Definition 5. We put

$$\varrho_\mu(x, y) = \mu(|x - y|).$$

Theorem 5. *If S is an l-group, which is also a conditionally σ-complete lattice and if μ is an upper integral on S^+, then (S_μ, ϱ_μ) is a complete pseudometric space.*

Proof. Since $|x - y| = x \vee y - x \wedge y$ and $\varrho_\mu(x, y) = \mu(x \vee y) - \mu(x \wedge y)$, the proof follows from [5].

Applications

This theory has sense mainly in the case when the l-group S is a linear subspace of the set of all real functions $f\colon X \to R$ (where X is some space). If this is the case and we require the positive homogenity of μ i.e. $\mu(\alpha f) = \alpha \mu(f)$, $\alpha \geqq 0$ (where $0 \cdot \infty = 0$). Then the following is true.

Proposition 7. *The set of all μ-integrable functions S_μ is a linear subspace of S, and is also an l-group.*

Proof. Let $f \in S_\mu, f = f_1 - f_2, f_1, f_2 \in S_\mu^+$ and $\mu(f_1), \mu(f_2) < \infty$. Let $\alpha > 0$ and $h \in S^+$, clearly $\alpha^{-1}h \in S^+$ because S is a linear space. From the $(\mu, +)$-integrability of f_i we have (for $i = 1, 2$)

$$\mu(\alpha^{-1}h) = \mu(f_i \wedge (\alpha^{-1}h)) + \mu(\alpha^{-1}h - (\alpha^{-1}h) \wedge f_i).$$

From this we get

$$\mu(h) = \mu(\alpha f_i \wedge h) + \mu(h - (\alpha f_i) \wedge h),$$

using the homogenity of μ. That means $\alpha f_i \in S_\mu^+$ $i = 1, 2$ hence $\alpha f \in S_\mu$. Since S_μ is an l-group, $\alpha f \in S$ for every α.

It is an interesting fact that this theory may give us very poor results. In example 1 the only integrable functions are the constants.

If X is a topological space, then it is a natural question whether the continuous functions on X are μ-integrable. However, the answer to this question is negative even in the case when $X = R$ and μ is a Lebesgue measure (the identity map from R to R is not integrable). We shall therefore discuss the question whether the non-negative continuous functions on X are $(\mu, +)$-integrable.

Theorem 6. *Let (X, τ) be a topological space, let S be an l-group of all real functions defined on the space X. Let μ be an upper integral defined on S^+. If the characteristic functions of open sets are $(\mu, +)$-integrable then the non-negative continuous functions are $(\mu, +)$-integrable too.*

Proof. Let $f: X \to R$ be continuous and let $c < a < b$ be elements of R; then

$$\chi_{f^{-1}\langle a, b)} = \chi_{f^{-1}(c, b)} - \chi_{f^{-1}(c, a)}.$$

On the right hand side we have the characteristic functions of open sets, and so from Proposition 7 and from the hypothesis of this theorem about the $(\mu, +)$-integrability it follows that $\chi_{f^{-1}\langle a, b)}$ is a μ-integrable function. Since $\chi_{f^{-1}\langle a, b)} \geqq 0$, we have $\chi_{f^{-1}\langle a, b)} \in S_\mu^+$. Let

$$g_n = \sum_{k=0}^{\infty} (k/2^n) \cdot \chi_{f^{-1}\langle k/2^n, (k+1)/2^n)}.$$

From the first part of the proof and from Proposition 7 it follows that the g_n are $(\mu, +)$-integrable. Clearly $g_n \nearrow f$. From Proposition 5 we get that g is $(\mu, +)$-integrable.

The reverse of this theorem is not valid, as shown by the following example.

Example 3. Let (X, τ) be a regular Hausdorff topological space on which every continuous function is constant (see [4]). Let $\mathscr{F}^+ = \{f; f: X \to \langle 0, \infty)\}$ and let

$$\mu(f) = \sup\{f(x); x \in X\}.$$

Clearly every continuous function, i.e. every constant function, is $(\mu, +)$-integrable, while the characteristic functions of open sets are not integrable.

In some special case (for example if (X, τ) is metrizable) the reverse of Theorem 6 is true.

Definition 6. *A normal topological space (X, τ) is said to be perfectly normal* ([3]) *iff every closed set is G_δ.*

Lemma 3. *Let the topological space (X, τ) be perfectly normal. If $A \subset X$ is closed; then there exists a sequence of continuous functions f_n: $X \to \langle 0, \infty)$ such that $f_n \searrow \chi_A$.*

Proof. If A is closed then there exists a sequence of open sets $\{G_n\}$ such that $A = \cap_n G_n$, $G_n \supseteqq G_{n+1}$. Let f_n be a continuous function f_n: $X \to \langle 0,1 \rangle$ such that $f_n(x) = 1$ for $x \in A$ and $f_n(x) = 0$ for $x \in X - G_n$. Then $f_n \to \chi_A$.

Theorem 7. *If the topological space (X, τ) is perfectly normal and the continuous functions from X to $\langle 0, \infty)$ are $(\mu, +)$-integrable, then the characteristic functions of open sets are $(\mu, +)$-integrable.*

Proof. The proposition of this theorem is a consequence of Lemma 3 and Proposition 6.

Corollary. *If (X, ϱ) is a metric space, then from the $(\mu, +)$-integrability of continuous functions it follows the $(\mu, +)$-integrability of the characteristic functions of open sets.*

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ИНТЕГРАЛ НА СТРУКТУРНО-УПОРЯДОЧЕННОЙ ГРУППЕ

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Резюме

В статье построена аксиоматическая теория интегрирования.

Пусть S^+ множество неотрицательных элементов структурно-упорядоченной группы S. Функцию $\mu: S^+ \to \langle 0, \infty \rangle$ назовем верхним интегралом, если (i) $\mu(0)=0$; (*ii*) если $x \leqq y$, то $\mu(x) \leqq \mu(y)$; (iii) $\mu(x+y) \leqq \mu(x)+\mu(y)$; (iv) если $x_n \nearrow x$, то $\mu(x_n) \nearrow (x)$.

Элемент $x \in S^+$ является $(\mu, +)$ – интегрируемый если

$$\mu(a)=\mu(x \wedge a)+\mu(a-x \wedge a)$$

для каждого $a \in S^+$.

Элемент $x \in S$ является μ-интегрируемым, если существуют такие $(\mu, +)$-интегрируемые элементы $x_1, x_2 \in S^+$, что верно $\mu(x_1), \mu(x_2) < \infty$, и $x = x_1 - x_2$.

В статье доказывается, что множество всех μ-интегрируемых элементов S_μ есть структурно-упорядоченная группа. Если S – σ-полная, то (S_μ, ϱ_μ) – полное метрическое пространство $(\varrho_\mu(x, y) = \mu(|x-y|))$.