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ON THE MAXIMAL DEDEKIND COMPLETION OF A LATTICE ORDERED GROUP

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Let G be a partially ordered group. The group operation will be written additively. We denote by $M_1(G)$ the set of all Dedekind cuts of the partially ordered set $(G, \leqq)$. The set G can be considered as a subset of $M_1(G)$ under the canonical embedding. The set $M_1(G)$ is partially ordered under the set-inclusion. It is possible to define the operation $+$ on $M_1(G)$ such that $M_1(G)$ turns out to be a partially ordered semigroup having the property that G is a subgroup of the semigroup $M_1(G)$. Denote by $M(G)$ the set of all elements of $M_1(G)$ possesing inverses in $M_1(G)$. Then $M(G)$ is the greatest subgroup of the semigroup $M_1(G)$ (cf. Fuchs [6]). If G is an Abelian group, then $M_1(G)$ is a commutative semigroup and so $M(G)$ is an Abelian group.

C. J. Everett [5] has proved the following theorem:

(A) *Let G be a commutative lattice ordered group. Then $M(G)$ is a lattice ordered group.*

In this note it will be shown that the assertion (A) holds true for all lattice ordered groups (without supposing the commutativity).

Let G be an l-group and suppose that G can be expressed as a mixed product ΩA_i $(i \in I)$ of linearly ordered groups A_i. We denote by K the set of all maximal elements of I. It will be proved that $M(G)$ is (up to isomorhisms) the mixed product ΩB_i $(i \in I)$, where $B_i = M(A_i)$ if $i \in K$ and $B_i = A_i$ if $i \in I - K$. A similar result has been proved by J. Jakubík [8] for the maximal Dedekind completion of an Abelian l-group which is the direct product of l-groups. Analogous results concerning the Cantor extension are obtained in [3] and [4].

1. The maximal Dedekind completion $M(G)$ of a lattice ordered group G

In this paragraph there will be constructed the maximal Dedekind completion $M(G)$ of an arbitrary lattice ordered group G.

Let G be a lattice ordered group. Let us denote by $X^u(X^l)$ the set of all upper (lower) bounds of a subset $X \subseteq G$ in G. Let $G^\#$ be the system of all ideals in G of the form $(X^u)^l$, where X is a nonempty and upper bounded subset of G. The system $G^\#$ is partially ordered under the set-inclusion. Then $G^\#$ is a conditionally complete lattice. The lattice operations in $G^\#$ will be denoted by $\wedge$, $\vee$. If a system of sets $\{Z_\lambda\}_{\lambda \in \Lambda} \subseteq G^\#$ has an upper (lower) bound in $G^\#$, then

$$\vee Z_\lambda (\lambda \in \Lambda) = ((\cup U_\lambda)^u)^l (\wedge Z_\lambda (\lambda \in \Lambda) = \cap Z_\lambda (\lambda \in \Lambda).$$

The mapping $\varphi: G \to G^\#$ *defined by* $\varphi(a) = (\{a\}^u)^l$ is one-to-one and it preserves all intersections and joins existing in G. In the next we shall identify a and $\varphi(a)$. Then G is a sublattice of $G^\#$ and the following conditions are satisfied:

(i) Every nonempty subset of G bounded from above (below) has the least upper bound (greatest lower bound) in $G^\#$.

(ii) For each element $z \in G^\#$ there exist nonempty subsets M_1, M_2 of G such that M_1 is bounded from above in G, M_2 is bounded from below in G and $\sup M_1 = z = \inf M_2$ in the partially ordered set $G^\#$.

For an element $z \in G^\#$ we denote

$$U(z) = \{h \in G : h \geqslant z\}, \quad L(z) = \{g \in G : g \leqslant z\}.$$

Let $z_1, z_2 \in G^\#$. From (ii) it follows that the sets $L(z_1)$ and $L(z_2)$ are nonempty and bounded from above in G. Then also the set $Z = \{g_1 + g_2 : g_1 \in L(z_1), g_2 \in L(z_2)\}$ is nonempty and bounded from above in G. By (i) there exists $\sup Z$ in $G^\#$. Define the operation $+$ in $G^\#$ by putting $z_1 + z_2 = \sup Z$. Then $G^\#$ is a semigroup (cf. Fuchs [6]). For each $z \in G^\#$ we have

(1) $$\text{if } z_1 \leqslant z_2, \text{ then } z_1 + z \leqslant z_2 + z, \quad z + z_1 \leqslant z + z_2.$$

If $z_1, z_2 \in G$, then the operation $z_1 + z_2$ in $G^\#$ coincides with the operation $z_1 + z_2$ in G. Thus G is an l-subgroup of $G^\#$. It should be observed that $G^\#$ is not a group in general (cf. [5]).

Let $M(G)$ be the set of all elements of $G^\#$ that have an inverse in $G^\#$. Then $M(G)$ is a group; $M(G)$ is a maximal subgroup of the semigroup $G^\#$. With respect to (1) $M(G)$ is a patially ordered group. In the following will be shown that $M(G)$ is an l-group.

Let X_1, X_2 be subsets of G such that $z_1 = \sup X_1$, $z_2 = \sup X_2$. In a similar manner as above we get that the set $Z' = \{g_1' + g_2' : g_1' \in X_1, g_2' \in X_2\}$ is nonvoid and bounded from above in G. Hence by (i) there exists $z' = \sup Z'$ in $G^\#$. We intend to show that $\sup Z = \sup Z'$, i. e. that the following statement is true:

1.1. $z_1 + z_2 = z'$.

Proof. The relations $X_1 \subseteq L(z_1)$, $X_2 \subseteq L(z_2)$ imply $Z' \subseteq Z$ and so $z' \leqslant z_1 + z_2$. It remains to prove that $z_1 + z_2 \leqslant z'$, i. e., $U(z') \subseteq U(z_1 + z_2)$. If $u \in U(z')$, then $u \in G$, $u \geqslant z' \geqslant g_1' + g_2'$ for every $g_1' \in X_1$, $g_2' \in X_2$. Hence $-g_1' + u \geqslant g_2'$ and thus $-g_1' + u \geqslant$

$z_2 \geqslant g_2$ for each $g_2 \in L(z_2)$. From $u - g_2 \geqslant g_1'$ we get $u - g_2 \geqslant z_1 \geqslant g_1$, $u \geqslant g_1 + g_2$ for each $g_1 \in L(z_1)$, $g_2 \in L(z_2)$. Therefore $u \geqslant z_1 + z_2$. Then $u \in U(z_1 + z_2)$.

Jakubík [7] introduced the notion of a generalized completion $D_1(G)$ of an l-group G. For the operation $+$ on $D_1(G)$ a relation analogous to 1.1. is valid ([7], Lemma 2.1). Observe that if G is an Abelian l-group, then $D_1(G)$ is an l-subgroup of $M(G)$ (see [8]).

For $H \subseteq G$ denote $-H = \{-g \in G : g \in H\}$. If $z \in G^{\#}$, then by (ii) there exist nonvoid subsets X, Y of G with the property

$$z = \sup X = \inf Y. \tag{2}$$

1.2. *Let $z \in G^{\#}$ and let X, Y be as in (2). If $\wedge(y - x\,;\, x \in X, y \in Y) = 0$ in G, then z has a rightinverse in $G^{\#}$.*

Proof. From (2) it follows that $-Y$ is a nonvoid and bounded from above in G. According to (i) there is $z' \in G^{\#}$, $z' = \sup(-Y)$. We shall show that z' is a rightinverse to z.

By 1.1. we obtain $z + z' = \sup\{x + y : x \in X,\ y \in -Y\} = \sup\{x - y : x \in X,\ y \in Y\}$ in $G^{\#}$. Since $0 = \inf\{y - x\} = -\sup\{x - y\}$ in G, we conclude that $\sup\{x - y\} = 0$ in $G^{\#}$. Hence $z + z' = 0$.

Remark 1. In an analogical way we obtain that $z' = \sup(-Y)$ is a left-inverse to z whenever $\wedge(-x + y\,;\, x \in X,\ y \in Y) = 0$ holds in G.

1.3. *Let $z \in G^{\#}$ and let (2) be fulfilled. Then $z \in M(G)$ if and only if the following conditions are satisfied in G:*

(*a*) $\wedge(y - x\,;\, x \in X, y \in Y) = 0$,

(*b*) $\wedge(-x + y\,;\, x \in X,\ y \in Y) = 0$.

Proof. If $z \in G^{\#}$ and if both conditions (*a*) and (*b*) are fulfilled, then 1.2 and Remark 1 imply that $z' = \sup(-Y)$ is an inverse to z, hence $z \in M(G)$. Conversely, let $z \in M(G)$. We shall show that (*a*) holds true. The assumption implies that $0 \leqslant y - x$ for each $x \in X$, $y \in Y$. Let $g \in G$, $0 < g \leqslant y - x$ for every $x \in X$, $y \in Y$. Hence $g + x \leqslant y$. From (2) it follows $g + x \leqslant z$ and by (1) we get $x \leqslant -g + z$. Then $z \leqslant -g + z$ because of (2). From the hypothesis $z \in M(G)$ we conclude that there exists an inverse to z in $G^{\#}$. Hence by (1) we have $0 \leqslant -g$, a contradiction. The proof of (*b*) is analogous.

The question of the independence of the conditions (*a*) and (*b*) remains open.

Everett [5] proved the assertion 1.3 under the assumption that (i) G is commutative and (ii) $X = L(z)$, $Y = U(z)$.

1.4. *If $z \in M(G)$, then $z \wedge 0 \in M(G)$ (the operation $\wedge$ being considered with respect to $G^{\#}$).*

Proof. Suppose that $z \in M(G)$ and let X, Y be is in (2). Since $G^{\#}$ is a lattice, $z \wedge 0 \in G^{\#}$. First we prove that $\wedge(y \wedge 0 - x \wedge 0;\ x \in X,\ y \in Y) = 0$ in G. Using 1.3 and the assumption we get $\wedge(y - x;\ x \in X,\ y \in Y) = 0$ in G. It is clear that $0 \leqslant y \wedge 0 - x \wedge 0$. Let there exist $g \in G$. such that $0 < g \leqslant y \wedge 0 - x \wedge 0$ for each $x \in X$,

$y \in Y$. Applying the result from [1] (p. 296) we get $0 < g \leq y \wedge 0 - x \wedge 0 \leq y - x$ for each $x \in X$, $y \in Y$, a contradiction. Thus $\wedge(y \wedge 0 - x \wedge 0) = 0$ in G. Then from the relations $z \wedge 0 = \sup L(z \wedge 0) = \inf U(z \wedge 0)$, $\{x \wedge 0\}_{x \in X} \subseteq L(z \wedge 0)$, $\{y \wedge 0\}_{y \in Y} \subseteq U(z \wedge 0)$ it follows $\wedge(y_1 - x_1;\, x_1 \in L(z \wedge 0),\, y_1 \in U(z \wedge 0)) = 0$ in G. In a similar way it can be proved that $\wedge(-x_1 + y_1;\, x_1 \in L(z \wedge 0),\, y_1 \in U(z \wedge 0)) = 0$ in G. Then 1.3 completes the proof.

From 1.4 we infer that the partially ordered group $M(G)$ is an l-subgroup of $G^{\#}$. Hence G is an l-subgroup of $M(G)$. The l-group $M(G)$ will be called the maximal Dedekind completion of G.

2. The maximal Dedekind completion of the mixed product of linearly ordered groups

Jakubík [8] studied the maximal Dedekind completion of an Abelian l-group G, which is a direct product of l-groups. In this section there will be investigated the maximal Dedekind completion of an l-group (without assuming commutativity) that is a mixed product of linearly ordered groups.

The concept of the mixed product of partially ordered groups is a common generalization of the concepts of the complete direct product and the lexicographic product (see Fuchs [6], Conrad [2]). Let us recall the definition of the mixed product.

Let $I \neq \emptyset$ be a partially ordered set and let A_i be a partially ordered group for each $i \in I$. Form the system H of all mappings $f: I \to \cup A_i$ $(i \in I)$ such that $f(i) \in A_i$ for each $i \in I$. We denote by G the set of all $f \in H$ such that the set $\sigma(f) = \{i \in I : f(i) \neq 0\}$ fulfils the descending chain condition. If for each $f, g \in G$ and each $i \in I$ we put $(f+g)(i) = f(i) + g(i)$, then G is a group. The set of all minimal elements of the partially ordered set $\sigma(f, g) = \{i \in I : f(i) \neq g(i)\}$ will be denoted by $\min \sigma(f, g)$. Further, we denote $\sigma(f) = \sigma(f, 0)$. We put $f < g$ if and only if $f(i) < g(i)$ for each $i \in \min \sigma(f, g)$; then G is a partially ordered group. It is said to be the mixed product of partially ordered groups A_i and it is denoted by $G = \Omega A_i$ $(i \in I)$.

If I is a trivially ordered set, then the mixed product is the direct product of partially ordered groups A_i. If $I = \{1, 2\}$, then for the direct product we shall use the symbol $G = A_1 \times A_2$.

2.1. *If G is a linearly ordered group, $z_1, z_2 \in G^{\#}$, $z_1 < z_2$, then there exists $g \in G$ with the property $z_1 < g \leq z_2$.*

Proof. Let G be a linearly ordered group. From the relation $z_1 = \sup L(z_1)$, $z_2 = \sup L(z_2)$, $z_1 < z_2$ it follows that $L(z_1)$ is a proper subset of $L(z_2)$. Hence there exists $g \in L(z_2)$, $g \notin L(z_1)$. Linearity of G implies $z_1 < g \leq z_2$.

If G is an l-group, the assertion 2.1 need not hold.

Example. Let C, Q and R be additive groups of all integers, rational and real numbers (with the natural order), respectively. If $G = C \times Q$, then in view of [8] (Theorem 2.7) and [5] (Theorem 7) we obtain $M(G) = M(C) \times M(Q) = C^{\#} \times Q^{\#} = C \times R$. It suffices to set $z_1 = (0, \sqrt{2})$, $z_2 = (1, \sqrt{2})$.

Let I be a partially ordered set and let A_i be a linearly ordered group for each $i \in I$, $A_i \neq \{0\}$. Suppose that G is an l-group such that

$$G = \Omega A_i \qquad (i \in I).$$

2.2. *Let $i_0 \in I$, $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq G$, $\wedge x_\lambda = 0$. Then there exists $\lambda \in \Lambda$ with the property $x_\lambda(i) = 0$ for each $i \in I$, $i < i_0$.*

Proof. Assume that for each $\lambda \in \Lambda$ there exists $i \in I$, $i < i_0$ such that $x_\lambda(i) \neq 0$. Then $i_0 \notin \min \sigma(x_\lambda)$. There are $a \in A_{i_0}$, $a > 0$ and $g \in G$ such that $g(i_0) = a$, $g(j) = 0$ for each $j \in I$, $j \neq i_0$. Therefore $0 < g \leq x_\lambda$ for each $\lambda \in \Lambda$, contrary to $\wedge x_\lambda = 0$.

From 2.2 it follows that the set $\Lambda(i_0) = \{\lambda \in \Lambda : x_\lambda(i) = 0 \text{ for each } i \in I, i < i_0\}$ is nonempty.

Denote by K the set of all maximal elements of I.

2.3. *Let $i_0 \in K$, $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq G$, $\wedge x_\lambda (\lambda \in \Lambda) = 0$. Then $\wedge x_\lambda(i_0)$ $(\lambda \in \Lambda(i_0)) = 0$.*

Proof. The assumption implies that $x_\lambda \geq 0$ for each $\lambda \in \Lambda$. We have either $x_\lambda(i_0) = 0$ or $i_0 \in \min \sigma(x_\lambda)$ for each $\lambda \in \Lambda(i_0)$. Hence $0 \leq x_\lambda(i_0)$ for each $\lambda \in \Lambda(i_0)$. If there exists $\lambda \in \Lambda(i_0)$ such that $x_\lambda(i_0) = 0$ the statement is evident. Let there exist $a \in A_{i_0}$ such that $0 < a \leq x_\lambda(i_0)$ *for each* $\lambda \in \Lambda(i_0)$. If g is as in 2.2, in the same way as in 2.2 we arrive at a contradiction with $\wedge x_\lambda (\lambda \in \Lambda) = 0$.

2.4. *Let $j \in I - K$, $z \in M(G)$. Then for each $i \in I$, $i \leq j$ there exists $a_i \in A_i$ with the following properties:*

(a) There exist elements $g \in L(z)$, $h \in U(z)$ such that $g(i) = h(i) = a_i$ for each $i \in I$, $i \leq j$.

(b) If $g_1 \in L(z)$, $h_1 \in U(z)$, $g_1(i) = h_1(i)$ for each $i \in I$, $i \leq j$, then $g_1(i) = a_i$ for each $i \in I$, $i \leq j$.

Proof. From $z \in M(G)$ and from 1.3 we get $\wedge(h - g\,;\, h \in U(z),\, g \in L(z)) = 0$. There exists $j' \in I$, $j' > j$. According to 2.2 there are $h \in U(z)$, $g \in L(z)$ such that $(h - g)(i) = 0$ and so $g(i) = h(i)$ for each $i \in I$, $i \leq j$. For the elements g, h with the mentioned property and for each $i \leq j$ denote $a_i = g(i) = h(i)$. Thus (a) is valid. Let g_1 and h_1 fulfil the assumption of the condition (b). Suppose that there exist $i' \in I$, $i' \leq j$ such that $g_1(i') \neq a_{i'}$. Hence $g(i') = h(i') \neq g_1(i') = h_1(i')$. Let $i_0 \in I$, $i_0 \leq i'$, $i_0 \in \min \sigma(g_1, h)$. Then $i_0 \in \min \sigma(g, h_1)$. Since $g_1 \leq h$, $g \leq h_1$, we have $g_1(i_0) < h(i_0) = g(i_0)$, $g(i_0) < h_1(i_0) = g_1(i_0)$, a contradiction.

From (b) it follows that for each $i \in I - K$ the element a_i is uniquely determined by $z \in M(G)$ (it does not depend on $j \in I$).

Let $z \in M(G)$, $i_0 \in I$ and suppose that i_0 is not minimal in I. Denote

$L^{i_0}(z)=\{g \in L(z): g(i)=a_i$ for each $i \in I$, $i<i_0\}$, $U^{i_0}(z) = \{h \in U(z): h(i)=a_i$ for each $i \in I$, $i<i_0\}$.

Now let i_0 be a minimal element of I. We define

$L^{i_0}(z)=\{g \in L(z): g(i_0)=a_{i_0}$, $U^{i_0}(z)=\{h \in U(z): h(i_0)=a_{i_0}\}$

if i_0 is not maximal in I and

$L^{i_0}(z)=L(z)$, $\quad U^{i_0}(z)=U(z)$

if i_0 is a maximal element of I. Further, for any $i_0 \in I$ denote

$L^{i_0}(z)\,(i_0)=\{u \in A_{i_0}$: there exists $g \in L^{i_0}(z)$, $g(i_0)=u\}$, $U^{i_0}(z)\,(i_0) = \{v \in A_{i_0}$: there exists $h \in U^{i_0}(z)$, $h(i_0)=v\}$.

From 2.4 we infer that $L^{i_0}(z)\neq\emptyset$, $U^{i_0}(z)\neq\emptyset$ and so $L^{i_0}(z)\,(i_0)\neq\emptyset$, $U^{i_0}(z)\,(i_0)\neq\emptyset$. Because of $u\leqslant v$ for each $u \in L^{i_0}(z)\,(i_0)$, $v \in U^{i_0}(z)\,(i_0)$, we have that $L^{i_0}(z)\,(i_0)$ ($U^{i_0}(z)\,(i_0)$) is a set bounded from above (below). Hence there exist $c \in A^{\#}_{i_0}$ and $d \in A^{\#}_{i_0}$, $c=\sup L^{i_0}(z)\,(i_0)$, $d=\inf U^{i_0}(z)\,(i_0)$ in $A^{\#}_{i_0}$. Clearly $c\leqslant d$. According to 1.3 we obtain $\wedge(h-g\,;\, g \in L(z),\, h \in U(z))=0$.

Let i_0 be a maximal element of I. Using the definition of the sets $L^{i_0}(z)$ and $U^{i_0}(z)$ we obtain that the equality $(h-g)\,(i)=0$ is valid for each $i \in I$, $i<i_0$ and for each $g \in L^{i_0}(z)$, $h \in U^{i_0}(z)$. We conclude from 2.3 that $\wedge(h(i_0)-g(i_0);\, g \in L^{i_0}(z),$ $h \in U^{i_0}(z))=0$. Similarly we get $\wedge(-g(i_0)+h(i_0);\, g \in L^{i_0}(z), h \in U^{i_0}(z))=0$. Using 1.3 it is easily verified that $c \in M(A_{i_0})$. Analogously it can be proved that $d \in M(A_{i_0})$. We intend to show that $c=d$. If $c<d$, i. e., $d-c>0$, then by 2.1 there exists $a \in A_{i_0}$, $0<a\leqslant d-c\leqslant h(i_0) - g(i_0)$ for each $g\geqslant L^{i_0}(z)$, $h \in U^{i_0}(z)$, a contradiction. Let us denote $a^*_{i_0}=c=d$. The definition of $a^*_{i_0}$ implies that $a^*_{i_0} \in M(A_{i_0})$,

$$a^*_{i_0}=\sup L^{i_0}(z)\,(i_0)=\inf U^{i_0}(z)\,(i_0). \tag{3}$$

From (3) we conclude that for each $i_0 \in K$ the elements $a^*_{i_0}$ are uniquely determined by $z \in M(G)$.

2.4′. *Let $j \in I-K$, $z \in M(G)$ and let X, Y be as in (2). Then the following conditions are valid.*

(a′) There exist elements $x \in X$, $y \in Y$ such that $x(i)=y(i)=a_i$ for each $i\leqslant j$.

(b′) If $x_1 \in X$, $y_1 \in Y$, $x_1(i)=y_1(i)$ for each $i\leqslant j$, then $x_1(i)=a_i$ for each $i\leqslant j$.

The proof of this assertion is analogous to that of 2.4.

If the symbols X^{i_0}, Y^{i_0}, $X^{i_0}(i_0)$, $Y^{i_0}(i_0)$ have an analogical meaning with $L^{i_0}(z)$, $U^{i_0}(z)$, $L^{i_0}(z)\,(i_0)$, $U^{i_0}(z)\,(i_0)$, in the same way as above we get the following statemant.

2.5. $a^*_{i_0}=\sup X^{i_0}(i_0)=\inf Y^{i_0}(i_0)$ *for each* $i_0 \in K$.

2.6. *a_i is the greatest (least) element of the set $L^i(z)\,(i)$ ($U^i(z)\,(i)$) for each $i \in I-K$.*

Proof. Let $i \in I-K$. By 2.4 there exist elements $g \in L(z)$, $h \in U(z)$, $g(j)=h(j)=a_j$ for each $j \in I$, $j\leqslant i$. Since $g \in L^i(z)$, $h \in U^i(z)$, we have $a_i=g(i) \in$

$L^i(z)(i)$, $a_i = h(i) \in U^i(z)(i)$. Therefore $a_i \leq v$ for each $v \in U^i(z)(i)$, $a_i \geq u$ for each $u \in L^i(z)(i)$ and the proof is complete.

Let X, Y be as in (2). Since $X \subseteq L(z)$, $Y \subseteq U(z)$, with respect to 2.6 the following assertion is valid.

2.7. *a_i is the greatest (least) element of the set $X^i(i)$ ($Y^i(i)$) for any $i \in I \backslash K$.*

2.8. *There exists an element $a \in G$ such that $a(i) = a_i$ for each $i \in I - K$.*

Proof. Let us denote $A = \{i \in I - K : a_i \neq 0\}$. We have to show that each nonempty set $I_1 \subseteq A$ contains a minimal element. If $i_0 \in I_1$ is not minimal in I_1, then $I_2 = \{i \in I_1 : i < i_0\} \neq \emptyset$. By 2.4 there exists $g \in L(z)$, $g(i) = a_i$ for each $i < i_0$ and we have $I_2 \subseteq \sigma(g)$. From the fact $g \in G$ it follows that every nonempty subset of $\sigma(g)$ has a minimal element. Consequently, I_2 contains a minimal element i'. Hence i' is a minimal element of I_1, too.

Let us form $B = \Omega B_i$ $(i \in I)$, where $B_i = A_i$ for each $i \in I - K$ and $B_i = M(A_i)$ for each $i \in K$. In view of 2.8 there exist elements $z_1, z_2 \in B$ such that $z_1(i) = a_i$, $z_2(i) = 0$, whenever $i \in I - K$ and $z_1(i) = 0$, $z_2(i) = a_i^*$ whenever $i \in K$. Hence $z_1 + z_2 = z' \in B$,

$$z'(i) = a_i \text{ if } i \in I - K \text{ and } z'(i) = a_i^* \text{ if } i \in K. \tag{4}$$

Let X, Y be as in (2). Because of $A_i \subseteq M(A_i)$, we have $X \subseteq B$, $Y \subseteq B$.

2.9. *$z' = \sup X = \inf Y$ in B.*

Proof. We intend to show that $z' = \sup X$ in B. Pick out any $x \in X$. If $x = z'$, then in view of (4), 2.7 and (3) z' is the greatest element of X and the assertion follows. Let $x \neq z'$, $i_0 \in \min \sigma(x, z')$. Hence $x(i) = z'(i) = a_i$ whenever $i \in I$, $i < i_0$. Since $x \in X^{i_0}$, we get $x(i_0) \in X^{i_0}(i_0)$. If $i_0 \in I - K$, we infer from 2.7 that $x(i_0) < a_{i_0} = z'(i_0)$. If $i_0 \in K$, by using (3) and 2.5 we obtain $z'(i_0) = a_{i_0}^* = \sup X^{i_0}(i_0)$ and thus $x(i_0) < z'(i_0)$. Therefore $x \leq z'$. Let $u \in B$. $u \geq x$ for each $x \in X$ and let $i_0 \in \min \sigma(u, z')$. If $i_0 \in I - K$, by 2.4′ there is $x \in X$, $x(i) = a_i$ for each $i \leq i_0$. Hence $x \in X^{i_0}$ and $i_0 \in \min \sigma(u, x)$. Then $u(i_0) > x(i_0) = a_{i_0} = z'(i_0)$. If $i_0 \in K$, then either $u(i_0) = x(i_0)$ or $i_0 \in \min \sigma(u, x)$. Thus $u(i_0) \geq x(i_0)$. This inequality is valid for each $x \in X^{i_0}$. From $a_{i_0}^* = \sup X^{i_0}(i_0)$ it follows that $u(i_0) > a_{i_0}^* = z'(i_0)$. Thus $u \geq z'$. The proof of the relation $z' = \inf Y$ is analogous.

Denote $A = \{g \in G : g \leq z'\}$.

2.10. $L(z) = A$.

Proof. Since $z = \sup L(z)$ in $M(G)$, by 2.9 we get $z' = \sup L(z)$ in B. Hence $L(z) \subseteq A$. Let $g \in A$. Because of $z = \inf U(z)$ in $M(G)$, by 2.9 we obtain $z' = \inf U(z)$ in B. Thus $g \leq h$ for each $h \in U(z)$. Then $g \leq z$, i. e. $g \in L(z)$.

2.11. *If $z_1, z_2 \in M(G)$, then $z_1' + z_2' = \sup Z$ in B, where $Z = \{g_1 + g_2 : g_1 \in L(z_1), g_2 \in L(z_2)\}$.*

Proof. From $z_1 = \sup L(z_1)$, $z_2 = \sup L(z_2)$ in $M(G)$ and from 2.9, we infer that $z_1' = \sup L(z_1)$, $z_2' = \sup L(z_2)$ in B. Hence $z_1' \geq g_1$, $z_2' \geq g_2$ for every $g_1 \in L(z_1)$, $g_2 \in L(z_2)$. Thus $z_1' + z_2' \geq g_1 + g_2$, i. e. $z_1' + z_2'$ is an upper bound of Z in B. Let

$b \in B, b \geqslant g_1 + g_2$ for each $g_1 \in L(z_1)$, $g_2 \in L(z_2)$ and let $i_0 \in \min \sigma(b, z_1' + z_2')$. For z_n $(n = 1, 2)$ let a_{ni} and a_{ni}^* have an analogous meaning as a_i and a_i^* have for the element z. If $i_0 \notin K$, then by 2.4 and (4) there exists $g_1 \in L(z_1)$, $g_2 \in L(z_2)$ such that $g_1(i) = a_{1i} = z_1'(i)$, $g_2(i) = a_2 i = z_2'(i)$ for each $i \leqslant i_0$. We will show that $b(i_0) > (z_1' + z_2')(i_0) = z_1'(i_0) + z_2'(i_0)$. If $b(i_0) < z_1'(i_0) + z_2'(i_0) = g_1(i_0) + g_2(i_0)$, then because of $i_0 \in \min \sigma(b, g_1 + g_2)$ we obtain $b \ngeqslant g_1 + g_2$, which is imposible. Now we prove that $b(i_0) > z_1'(i_0) + z_2'(i_0)$ for $i_0 \in K$. Suppose that $b(i_0) < z_1'(i_0) + z_2'(i_0)$. According to (4) we get $z_1'(i_0) - a_{1i_0}^* = \sup L^{i_0}(z_1)(i_0)$, $z_2'(i_0) = a_{2i_0}^* = \sup L^{i_0}(z_2)(i_0)$ in $M(A_{i_0})$. The definition of the operation $+$ in $M(A_{i_0})$ and 1.1 imply $b(i_0) < z_1'(i_0) + z_2'(i_0) = \sup \{g_1(i_0) + g_2(i_0) \cdot g_1 \in L^{i_0}(z_1), g_2 \in L^{i_0}(z_2)\}$ in $M(A_{i_0})$. From the fact that A_{i_0} is a linearly ordered set it follows that we can find $g_1' \in L^{i_0}(z_1) \subseteq L(z_1)$, $g_2' \in L^{i_0}(z_2) \subseteq L(z_2)$ with $b(i_0) < g_1'(i_0) + g_2'(i_0)$. From $g_1' \in L^{i_0}(z_1)$, $g_2' \in L^{i_0}(z_2)$ we conclude that $g_1'(i) - a_{1i} - z_1'(i)$ $g'(i) = a_{2i} = z_2'(i)$ for each $i \in I$, $i < i_0$. Then $i_0 \in \min \sigma(b, g_1' + g')$. Thus $b \ngeqslant g_1' + g$, a contradiction.

Define a mapping $\varphi: M(G) \to B$ by the rule $\varphi(z)$ z'. With respect to 2.10 we have $L(z_1) = \{g \in G : g \leqslant z_1'\}$, $L(z_2) = \{g \in G . g \leqslant z_2'\}$. Then $z_1' = z_2'$ if and only if $L(z_1) = L(z_2)$. Hence φ is a one-to-one mapping. Since z_1 z_2 if and only if $L(z_1) \subseteq L(z_2)$, by 2.9 and 2.10 we obtain $z_1 \leqslant z_2$ if and only if $z_1' \leqslant z_2'$. Now we show that φ is a mapping $M(G)$ onto B. Let $b \in B$, $B_1 = \{g$ $G : g \leqslant b\}$. Since $b(i) \in M(A_i)$ for each $i \in I$, the sets $\{a \in A_i : a \leqslant b(i)\}$, $\{a \in A_i : a \geqslant b(i)\}$ are nonempty for any $i \in I$. There are elements $g, h \in G$ such that $g(i) = h(i) = b(i) \in A_i$ for $i \in I - K$ and $g(i) = u_i$, where $u_i \in A_i$; $u_i \leqslant b(i)$, $h(i) = v_i \in A_i$, $v_i \geqslant b(i)$ for $i \in K$. Then $B_1 \neq \emptyset$, $B_1^u \neq \emptyset$, since $g \in B_1$, $h \in B_1^u$. Hence by (i) there is $z \in G^\#$, $z = \sup B_1$. Now we show that $z \in M(G)$. Denote $U_1 = \{u \in G : u(i) = b(i)$ for each $i \in I - K$ and $u(i) \in A_i$, $u(i) \leqslant b(i)$ for each $i \in K\}$, $V_1 = \{v \in G : v(i) = b(i)$ for each $i \in I - K$ and $v(i) \in A_i$, $v(i) \geqslant b(i)$ for each $i \in K\}$. Therefore $u(i) - v(i) = 0$ $(u \in U_1, v \in V_1$ for each $i \in I - K$. Since $b(i) \in M(A_i)$ for each $i \in K$, according to 1.3 we obtain $\wedge(u(i) - v(i);\ u \in U_1,\ v \in V_1) = 0$. Then $\wedge(u - v;\ u \in U_1,\ v \in V_1) = 0$. From $U_1 \subseteq U(z)$, $V_1 \subseteq L(z)$, $z = \sup L(z) = \inf U(z)$ and from 1.3 we conclude $z \in M(G)$ In view of 2.9 we obtain $z' = \sup B_1 = b = \varphi(z)$. It is easily seen that φ preserves the group operation. In fact, using 2.9 and 2.11 from $z_1 + z_2 = \sup Z$ in $M(G)$ it follows that $(z_1 + z_2)' = \sup Z = z_1' + z_2'$ in B.

We have proved that the following theorem is true.

Theorem. *Let G be a lattice ordered group that can be written as a mixed product $G = \Omega A_i$ $(i \in I)$, where A_i is linearly ordered for each $i \in I$. Put $B_i = M(A_i)$ if i is maximal in I and $B_i = A_i$ otherwise. Then there exists an isomorphism φ of $M(G)$ onto ΩB_i $(i \in I)$ such that $\varphi(g) = g$ for each $g \in G$.*

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МАКСИМАЛЬНОЕ ДЕДЕКИНДОВО ПОПОЛНЕНИЕ СТРУКТУРНО УПРЯДОЧЕННОЙ ГРУППЫ

Штефан Чернак

Резюме

Эверетт доказал, что максимальное дедекиндово пополнение коммутативной структурно упорядоченной группы есть структурно упорядоченная группа. В этой статье результат Эверетта обобщается для всех структурно упорядоченных групп. Доказаны некоторы свойства максимального дедекиндового пополнения смешанного произведения линейно упорядоченных групп.