Math. Slovaca 31, 1981, No. 2, 133—139

REMARK ON ANALYTIC FUNCTIONS IN ORDERED SPACES

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Many concepts and propositions known for analytic (or holomorphic) functions have been generalized for functions with values in a locally convex topological vector space, cf. e.g. [1; 2]. Further properties can be obtained if the range space of an analytic function has an order structure, cf. [8; 7, App. 2]. In this remark we give a certain generalization to ordered spaces of the classical Vivanti—Pringsheim theorem which asserts that a power series with nonnegative coefficients and radius of convergence equal to a positive number d defines an analytic function that has singularity at the point d, cf. also [8; 7, App. 2]. This generalization permits one to consider, in particular, power series with coefficients from the weakly normal positive cone of the ordered separated locally convex space, cf. [8; 7, App. 2] or from the positive cone of the strong dual of the ordered barrelled locally convex space with a generating positive cone.

1. Definition of a norming triple. Let E be a vector space over K (K being the set of all real or complex numbers) and p a seminorm on it. Assume that there is given a complete seminormed space $(E_p, \|\cdot\|_p)$ and a bilinear form $\langle,\rangle_p$ from the space $E\times E_p$ into K. We shall say that the triple $(E_p, \|\cdot\|_p, \langle,\rangle_p)$ is a norming triple for the seminorm p if

$$p(x)=\sup \{|\langle x, y\rangle_p| : y\in E_p, \|y\|_p\leqq 1\}$$

for all x in E, cf. [1].

Example 1. Let p be a seminorm on the space E. Let E_p be the family of all linear forms y on the space E such that

$$r(y)=\sup \{|\langle x, y\rangle| : x\in E, p(x)\leqq 1\}$$

is finite number. Then $y\rightarrow r(y)$ is a norm on E_p and E_p is a Banach space with the norm $y\rightarrow r(y)=\|y\|_p$ [4, 1.10.6, 1.10.10].

Define the bilinear form by means

$$\langle x, y\rangle_p = \langle x, y\rangle \quad \text{for all} \quad x \in E,\ y \in E_p.$$

From the Hahn-Banach theorem it follows that the triple $(E_p, \|\cdot\|_p, \langle,\rangle_p)$ is norming for the seminorm p.

Example 2. Let E be a sequentially complete locally convex space. Consider the strong dual F of E, $F=(E')_\beta$. Denote by $\mathscr{B}(E)$ the family of all closed absolutely convex bounded subsets in E. If we put, for each B in $\mathscr{B}(E)$ and x' in E',

$$p_B(x') = \sup\{|\langle x, x'\rangle| : x \in B\},$$

then the set $\{p_B: B \in \mathscr{B}(E)\}$ determines the strong topology $\beta(E', E)$ on E', cf. [6, III.2].

Every $B \in \mathscr{B}(E)$ defines a seminorm $\bar{p}_B$ on the closed vector subspace E_B spanned by B. With $\bar{p}_B$ as a seminorm E_B is a complete seminormed space. Moreover $B=\{x: \bar{p}_B(x) \leqq 1\}$. Thus we have

$$p_B(x') = \sup\{|\langle x, x'\rangle| : x \in E_B,\ \bar{p}_B(x) \leqq 1\}$$

for all x' in E'.

Now the triple $(E_B, \|\cdot\|_B, \langle,\rangle_B)$, where $\|x\|_B = \bar{p}_B(x)$, $\langle x, x'\rangle_B = \langle x, x'\rangle$, $x \in E_B$, is a norming triple for the seminorm p_B, for every $B \in \mathscr{B}(E)$.

2. Vector-valued analytic functions. Let E be a sequentially complete separated locally convex topological vector space (biefly — semi-complete separated convex space) over C (complex numbers). E' will denote the topological dual of E. Recall some concepts and facts concerning analytic functions with values in E cf. [1; 2].

Let D be an open subset of C. A function f: $D \to E$ is called analytic (or holomorphic) in D if for every point d in D there are elements a_k in E, $k=0, 1, 2, \ldots$ and a positive number (radius) r such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z-d)^k \quad \text{if} \quad |z-d| < r.$$

The function f: $D \to E$ is analytic if and only if the function f has at every point d in D the derivatives $f^{(k)}(d)$, $k=1, 2, \ldots$ and

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(d)(z-d)^k, \quad |z-d| < \varrho = \text{dist.}(d, \partial D).$$

If a series $\sum_{k=0}^{\infty} a_k z^k$, $a_k \in E$, $k=0, 1, 2, \ldots$ converges in the disk $|z| < \varrho$, then it converges uniformly on every disk $|z| < r$, where $0 < r < \varrho$. The sum $z \to f(z) =$

$=\sum_{k=0}^{\infty} a_k z^k$ is analytic for $|z|<\varrho$. The function $f\colon D\to E$ is analytic if and only if for every $x'\in E'$ the function $z\to\langle f(z), x'\rangle$ is analytic.

Let $f\colon D\to E$ be an analytic function. A frontier point d of D is called a regular point for f if there is an open neighbourhood V of d and an analytic function on $D\cup V$ into E which coincides with f in D. A frontier point of D is said to be singular for f (or is singularity of f) if it is not regular.

If p is a seminorm on a vector space E and $(E_p, \|\cdot\|_p, \langle,\rangle_p)$ is a norming triple for the seminorm p, then it is easy to prove the inquality

$$|\langle x, y\rangle_p|\leqq p(x)\|y\|_p \quad \text{for} \quad x\in E,\ y\in E_p.$$

It follows that the linear form $x\to\langle x, y\rangle_p$, $y\in E_p$, is continuous and therefore if $f\colon D\to E$, E being a semi-complete convex space, is an analytic function, then the scalar function $z\to\langle f(z), y\rangle_p$, $y\in E_p$, is analytic in the classical sense.

Let D and D_1 be open connected subsets of C such that $D\subset D_1$. We shall essentially make use of the following result, cf. [1].

Theorem. *Let E be a semi-complete separated convex space over C with the topology generated by a family $P=\{p\}$ of seminorms. To every seminorm p let there correspond a norming triple $(E_p, \|\cdot\|_p, \langle,\rangle_p)$. Let $f\colon D\to E$ be an analytic function and assume that for every seminorm p and every y in E_p the analytic function $z\to\langle f(z), y\rangle_p$ has an extension to an analytic function on D_1. Then there exists an analytic function $f_1\colon D_1\to E$ such that $f_1(z)=f(z)$ for all z in D.*

3. Ordered vector spaces [7, Ch. V.; 8]. Let E be a separated topological vector space over R (real numbers). Recall that E is said to be ordered (i.e. partially ordered) if a convex cone E^+ of vertex 0 is specified in E which is closed and proper (i.e. $E^+\cap(-E^+)=\{0\}$). The order relation $x\leqq y$ in E is then defined to mean $y-x\in E^+$ and E^+ is referred to as the positive cone of E.

Let (E, F) be a dual pair over R. If E^+ is a cone in E, the dual cone F^+ in F is the set of all y of F such that $x\in E^+$ implies $\langle x, y\rangle\geqq 0$. A cone E^+ in E is called generating in E if $E=E^+-E^+$. If E is a vector space over C (complex numbers), then E is said to be ordered if its underlying real space E_0 is ordered [cf. 7, Ch. V.]. If (E, F) is a dual pair denote by F_0 the subset of F consisting of all y in F corresponding to the real linear forms on E. Let F^+ be the set of all y in F_0 such that $x\in E^+$ implies $\langle x, y\rangle\geqq 0$.

We shall need the following result.

Lemma. *Let E be an ordered semi-complete separated convex space over K with the topology generated by a family $P=\{p\}$ of seminorms and with a positive cone E^+. To every seminorm p let there correspond a norming triple $(E_p, \|\cdot\|_p, \langle,\rangle_p)$ with a positive cone E_p^+. Suppose that for every p there exists q in P such that*

$E_{p0} \subset E_q^+ - E_q^+$. *Let a_{ij}, $i, j = 1, 2, \ldots$ be elements of E^+ such that for every p in P and every y in E_p there exists a finite limit*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle a_{ij}, y \rangle_p = \lim_{m \to \infty} \sum_{i=1}^{m} \lim_{k \to \infty} \sum_{j=1}^{k} \langle a_{ij}, y \rangle_p.$$

Then for every p in P and every y in E_p the equality holds:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle a_{ij}, y \rangle_p = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \langle a_{ij}, y \rangle_p.$$

Proof. For every p in P and every u in E_p^+ we have

$$\langle a_{ij}, u \rangle_p \geqq 0, \quad i, j = 1, 2, \ldots$$

Hence we have

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle a_{ij}, u \rangle_p = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \langle a_{ij}, u \rangle_p.$$

Since $E_{p0} \subset E_q^+ - E_q^+$ for some q in P, we have for every y in E_{p0} the equality $y = u - v$, where u, v belong to E_q^+ and hence

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle a_{ij}, y \rangle_p = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \langle a_{ij}, y \rangle_p.$$

It follows that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle a_{ij}, y \rangle_p = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \langle a_{ij}, y \rangle_p$$

for all y in E_p because every linear form on E can be written in the form $y = u + iv$, where u and v are uniquely determined real linear forms on E [7, I.7.1].

4. A generalization of the Vivanti—Pringsheim theorem. The generalization that will be established contains as a particular case the generalization given in [8; 7, App. 2] and gives a new proof of this result.

Theorem. *Let E be an ordered semi-complete separated convex space over K with the topology generated by a family $P = \{p\}$ of seminorms and with a positive cone E^+. To every seminorm p let there correspond a norming triple (E_p, $\|\cdot\|_p$, $\langle , \rangle_p$) with a positive cone E_p^+. Suppose that for every p in P there exists q in P such that $E_{p0} \subset E_q^+ - E_q^+$. Let $r > 0$ be the radius of convergence of a power series*

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in E^+, \quad n = 0, 1, 2, \ldots$$

Then the point $z = r$ is singular for the function f defined by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < r.$$

Proof. One may suppose that r is equal to 1. If $z = 1$ were a regular point for f, there would exist an open disk D with the center in 1 and an analytic function g:

$U \cup D \to E$ which coincides with f in $U = \{z: |z| < 1\}$. Since g is analytic in D there are points x of D with $0 < x < 1$ and $z = 1 + d$ of D, $d > 0$, such that

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(x)(z-x)^k.$$

Since

$$g^{(k)}(x) = f^{(k)}(x), \quad k = 0, 1, 2, \ldots,$$

we have

(1)
$$g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x)(z-x)^k.$$

But we have

$$\frac{f^{(k)}(x)}{k!} = \sum_{n=k}^{\infty} \binom{n}{k} a_n x^{n-k}, \quad k = 0, 1, 2, \ldots$$

Hence the series (1) has the form

$$g(z) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} a_n x^{n-k} (z-x)^k, \quad z = 1 + d.$$

By the lemma for every p of P and every y of E_p we have

$$\langle g(z), y \rangle_p = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} \langle a_n, y \rangle_p x^{n-k} (z-x)^k =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \langle a_n, y \rangle_p \binom{n}{k} (z-x)^k x^{n-k} =$$

$$= \sum_{n=0}^{\infty} \langle a_n, y \rangle_p [(z-x)+x]^n = \sum_{n=0}^{\infty} \langle a_n, y \rangle_p z^n.$$

In this way for every p of P and every y of E_p the series $\sum_{n=0}^{\infty} \langle a_n, y \rangle_p z^n$ is convergent for $z = 1 + d$ and hence for all z, $|z| \leqq 1 + d$. It follows that the analytic function $z \to \langle f(z), y \rangle_p$, z in U, has an extension to an analytic function on the disk U_1 with the centre 0 of the radius greater than 1 for every p of P and every y of E_p. But then there would exist an analytic function f_1 on U_1 into E such that $f_1(z) = f(z)$ for all z of U according to Theorem in Section 2. This contradicts, of course, the assumption that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ is equal to 1, and the theorem is proved.

Corollary 1. *Let E be an ordered semi-complete separated convex space over K with a positive cone E^+ and a dual cone E'^+ such that $E'_0 = E'^+ - E'^+$. Let $r < \infty$ be the radius of convergence of a power series*

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in E^+, \quad n = 0, 1, 2, \ldots,$$

of one complex variable. Then the point $z=r$ is singular for f,

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad |z|<r.$$

The proof of Corollary 1 follows from Example 1. If $E'_0=E'^+-E'^+$, the cone E^+ is said to be weakly normal [7, V.3.3]. We have obtained another proof of the result from [8, Th. 1].

Corollary 2. *Let E be an ordered Banach space with a positive normal cone E^+. If $r<\infty$ is the radius of convergence of a power series*

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in E^+, \quad n=0, 1, 2, \ldots,$$

then the point $z=r$ is singular for f,

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad |z|<r.$$

This corollary is a particular case of the preceding one since for the normed spaces normality of the cone E^+ is equivalent to $E'_0=E'^+-E'^+$, i.e. to the weak normality of E^+ [7, V.3.3].

Corollary 3. *Let F be an ordered semi-complete barrelled convex space over K with the generating positive cone F^+ and let E be its dual endowed with the strong topology, $E=(F')_\beta$. Let $r<\infty$ be the radius of convergence of a power series*

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in E^+, \quad n=0, 1, 2, \ldots$$

Then the point $z=r$ is singular for f,

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad |z|<r.$$

The proof of this corollary follows from Example 2. Recall that since F is barrelled, the space E is semi-complete (even quasi-complete) [5, p. 218]. F^+ is generating, for example, if the topology of F is decomposable in the sense [3, p. 61]. Note that in this corollary we may take for F a quasi-complete bornological space since such a space is barrelled [7, II.8.4].

Corollary 4. *Let F be an ordered semi-complete bornological convex space with a $\mathcal{B}$-strict positive cone F^+. Let E be its strong dual. Then the same assertion as* in Corollary 3 *is true.*

The proof of this corollary follows from Example 2. The space E is complete in the strong topology [5, p. 223]. Since the cone F^+ is $\mathcal{B}$-strict, the topology of F is decomposable, hence F^+ is generating [3, p. 67].

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Received March 3, 1979

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ЗАМЕЧАНИЕ ОБ АНАЛИТИЧЕСКИХ ФУНКЦИЯХ В УПОРЯДОЧЕННЫХ ПРОСТРАНСТВАХ

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Резюме

В работе дано обобщение на функции со значениями из упорядоченного отделимого локально выпуклого векторного пространства классической теоремы Виванти–Прингсхейма, по которой степенной ряд с неотрицательными коэффициентами и радиусом сходимости один определяет аналитическую функцию, для которой $x = 1$ является особой точкой. Это обобщение охватывает как степенные ряды с коэффициентами из слабо нормального положительного конуса, так и из положительного конуса сильного сопряженного бочечного пространства с порождающим конусом.