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ON THE CONTINUITY FOR CONNECTED FUNCTIONS

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The presented paper deals with properties of sets of points of continuity of real functions with a connected graph. These functions will be referred to as connected functions. Garret, Nelms and Kellum [1] defined the connectivity points of a real function and demonstrated that a function f: $R\to R$ is connected if and only if each point of the line R is a point of connectedness. Bruckner and Ceder [2] defined the Darboux points of a real function and noted that it follows from a theorem of Császár [4] that the function f: $R\to R$ possesses the Darboux property if and only if each point of R is a Darboux point. Ceder [3] has demonstrated, among others, that for any set C of G_δ type on R there exists a real function possessing the Darboux property and such that C is the set of continuity points of the function.

The aim of this paper is to deliver the proof of that theorem concerning the set of continuity points of a connected function which is the analogon of Ceder's theorem on Darboux functions.

Notations. If M is a subset of the plane R^2, then the projection of M onto the Ox-axis will be denoted by $(M)_x$. A line perpendicular to the Ox-axis passing through the point $(x, 0)$ will be denoted by l_x. The set of continuity points of a function f will be denoted by $C(f)$ and the graph of f by G_f.

Definition 1. Let $M\subset R^2$. The point (x_0, y_0) is said to be the limit point of M from the right if for any $\delta>0$ the set $\{ x, y)$: $x_0<x<x_0+\delta$, $y_0-\delta<y<y_0+ +\delta\}\cap M\neq\emptyset$. Analogously, if for any $\delta>0$ the set $\{(x, y)$: $x_0-\delta<x<x_0$, $y_0-\delta<y<y_0+\delta\}\cap M\neq\emptyset$, then (x_0, y_0) is said to be the limit point of M from the left.

Definition 2 (Garrett, Nelms, Kellum). The point $z\in R$ is said to be a left-hand (right-hand) connectedness point of a function f if the fact that a and b are two left-hand (right-hand) limit values of f in the point z and that $M\subset R^2$ is a continuum such that $(M)_x$ is a non-degenerated interval with the right-hand (left-hand) end point in z and $M\cap l_z\subset\{(z, y): a<y<b\}$ implies that $M\cap G_f\neq\emptyset$.

Definition 3 (Garrett, Nelms, Kellum). The function f will be referred to as connected in the point z belonging to its domain if $f(z)$ is the right-hand and left-hand limit value of f in z and if z is a left-hand and right-hand connectedness point of f.

Theorem. *If C is a set of the G_δ type on the line R, then there exists a connected function f such that $C = C(f)$.*

Proof. If $C = R$, we take $f(x) = \text{const}$. Then, of course, $C(f) = R = C$. Assume now that $R - C \neq \emptyset$.

Denote the interior of $R - C$ by the letter B. Then $C \cup B$ is a dense G_δ-set and $A = R - (C \cup B)$ is a F_σ-set of the first category. Let F_n be a sequence of closed, pairwise disjoint sets such that $A = \bigcup_n F_n$.

Each F_n is nowhere dense. For each F_n there exists an open set G_n such that $G_n \cap F_n = \emptyset$, $F_n \subset \bar{G}_n$, $F_n \subset \overline{R - (F_n \cup G_n)}$, $R - (F_n \cup G_n)$ is a union of closed non-degenerated intervals and each component of G_n is bounded. Let $G_n = \bigcup_{n=1}^{\infty} (a_i^{(n)}, b_i^{(n)})$ where the open intervals $(a_i^{(n)}, b_i^{(n)})$ are pairwise disjoint. Take $\tau_n(x) = 0$ for $x \in R - G_n$, $\tau_n(x) = 2^{-n}$ for $x = 2^{-1}(a_i^{(n)} + b_i^{(n)})$ and $\tau_n(x)$ linear on the intervals $\langle a_i^{(n)}, 2^{-1}(a_i^{(n)} + b_i^{(n)})\rangle$ and $\langle 2^{-1}(a_i^{(n)} + b_i^{(n)}), b_i^{(n)}\rangle$. Each function τ_n belongs to the Baire class 1, possesses the Darboux property and $C(\tau_n) = R - F_n$. As $0 \leqslant \tau_n \leqslant 2^{-n}$, the series $\tau(x) = \sum_{n=1}^{\infty} \tau_n(x)$ is uniformely convergent and the function $\tau(x)$ is also a Darboux function of the Baire class 1. As the sets F_n are pairwise disjoint, $C(\tau) = \bigcap_{n=1}^{\infty} C(\tau_n) = R - A = C \cup B$.

Let dist $(x, \bar{C})$ denote the distance of x from $\bar{C}$. If $C = \emptyset$, we take dist $(x, \bar{C}) = 1$. Let

$$P = \{(x, y) : x \in B,\ \tau(x) < y < \tau(x) + \operatorname{dist}(x, \bar{C})\}.$$

We form a transfinite sequence Γ_ξ, $\xi < \Omega$, of all continua of R^2 such that $\Gamma_\xi \cap P \neq \emptyset$. The projection of the set $\Gamma_\xi \cap P$ onto the Ox-axis will be denoted by Π_ξ. If Int $\Pi_\xi \neq \emptyset$, then we select by means of a transfinite induction the denumerable set of points $S_\xi = \{x_1^{(\xi)}, x_2^{(\xi)}, \dots\}$ dense in Int Π_ξ in such a manner that $S_\xi \cap \left(\bigcup_{\eta<\xi} S_\eta\right) = \emptyset$. We define now a function $f(x)$ such that for $x \in S_\alpha$ the condition $(x, f(x)) \in \Gamma_\alpha \cap P$ holds, whereas for $x \notin \bigcup_{\xi<\Omega} S_\xi$ we take $f(x) = \tau(x)$. We shall show that a function defined in this way satisfies the conditions of the theorem.

Let $x \in B$ and let M be the continuum occurring in the definition of the right-hand connectedness point. Then there exists an $\xi < \Omega$ such that $M = \Gamma_\xi$. In each interval $(x, x + \delta)$ there exists a point $x_i^{(\xi)} \in S_\xi$. According to the definition of the function $(x_i^{(\xi)}, f(x_i^{(\xi)})) \in \Gamma_\xi$ and therefore x is a right-hand connectedness point.* The point $(x, f(x))$ belongs to the closure of P. The graph of f is dense in P. Point x is an interior point of B. It follows therefrom that $f(x)$ is the right-hand and left-hand limit value of f in x.

Let $x \in \bar{C} - C$. If x is a right-hand continuity point of f, then it is also a right-hand connectedness point of f, or otherwise $\overline{\lim_{t \to x+}} f(t) > \underline{\lim_{t \to x+}} f(t)$. As $x \notin B$, dist $(x, \bar{C}) = 0$ and $\overline{\lim_{t \to x+}} f(t) = \overline{\lim_{t \to x+}} \tau(t)$ and $\underline{\lim_{t \to x+}} f(t) = \underline{\lim_{t \to x+}} \tau(t)$. Let M be the continuum occurring in the definition of the right-hand connectedness point of f. The functions $\tau(x)$ and $\tau(x)$ + dist $(x, \bar{C})$ belong to the Baire class 1 and possess the Darboux proprety. According to Kuratowski and Sierpiński [5], the graphs of both these functions are connected sets. Therefore by the theorem of Garrett, Nelms and Kellum [1] each point of its domain is a connectedness point of both these functions. For any $\delta > 0$ there exist point Θ such that $\Theta \in (x, x+\delta)$ and $(\Theta, \tau(\Theta)) \in M$. If even one of these points, say Θ' belongs to $\bar{C}$, then $f(\Theta') = \tau(\Theta')$ and $(\Theta', f(\Theta')) \in M$. Otherwise no point Θ belongs to $\bar{C}$ and hence all Θ belong to B. Then all points Θ'' such that $\Theta'' \in (x, x+\delta)$ and $(\Theta'', \tau(\Theta'')$ + dist $(\Theta'', \bar{C})) \in M$ must also belong to B. We shall show that in this case

$$\text{(i)} \qquad M \cap P \neq \emptyset.$$

For any $t \in \bar{C} \cap (x, x+\delta)$ let $P_t = (t, f(t))$. For any $t \in \bar{C} \cap (x, x+\delta)$ there exists $\varepsilon_t > 0$ such that $M \cap K(P_t, \varepsilon_t) = \emptyset$, where $K(P, \varepsilon_t)$ denotes a sphere with its center in P_t and the radius ε_t. Let $Q = [(x, x+\delta) \times R] \cap [P \cup \bigcup_{t \in \bar{C}} K(P_t, \varepsilon_t)]$. The set Q is open and each point of Q can be connected by means of a segment contained in Q with points belonging to the graph of the function $\tau(t) + 2^{-1}$ dist $(t, \bar{C})$, $(x < t < x+\delta)$, which has a connected graph. Therefore Q is connected.

Assume that (i) does not hold. Then there exists a δ', $0 < \delta' < \delta$ such that for any $x' \in (x, x+\delta)$ there exist points $(x', y_1) \in M$ and $(x'', y_2) \in M$ such that $y_1 \leq f(x') \leq y_2$ and $y_1 < y_2$.

In fact, suppose on the contrary that there exists a sequence $\{x_n\}$ tending from the right-hand side to x such that for any n

$$\text{(ii)} \qquad \begin{cases} \text{either} & M \cap \{(x_n, y): y > f(x_n) = \emptyset \\ \text{or} & M \cap \{(x_n, y): y < f(x_n) = \emptyset \end{cases}$$

Choose $x_{n_0} < x + \delta$. Assume that

$$\text{(iii)} \qquad M \cap \{(x_{n_0}, y): y > f(x_{n_0})\} = \emptyset.$$

*) In a analogous way we can prove that x is a left-hand connectedness point.

There exists a point $(\xi, \eta) \in M$ such that $x < \xi < x_{n_0}$ and $\eta > f(\xi)$. If this were not the case, then in girdle $\{(t, y): x < t < x_{n_0}\}$ the entire continuum M would be situated below Q and hence not above the graph of τ. Assume that it is not situated above. Then there exists a sequence of points $t_n \in (x, x_{n_0})$ such that $\lim_{n\to\infty} \tau(t_n) = \underline{\lim}_{t\to x+} \tau(t)$. For the points $(t_n, u_n) \in M$ the condption $u_n \leq \tau(t_n)$ holds. Select from t_n a subsequence t_{i_n} such that u_{i_n} is convergent and let $u = \lim_{n\to\infty} u_{i_n}$. Then $(x, u) \in M$, but $u \leq \underline{\lim}_{t\to x+} \tau(t) = \lim_{t\to x+} f(t)$. This contradicts the choice of the continuum M. From assumption (iii) there follows therefore the existence of (ξ, η). Choose a point x_{n_1} such that $x < x_{n_1} < \xi$. If $M \cap \{(x_{n_1}, y): y < f(x_{n_1})\} = \emptyset$, then the set

$$\{(x_{n_0}, y): y > f(x_{n_0})\} \cup \{(x_{n_1}, y): y < f(x_{n_1})\} \cup P$$

divides the continuum M into two non-empty parts, no one of these parts has a limit point belonging to the other part, which contradicts the definition of the continuum. Hence from the alternative (ii) the condition $M \cap \{(x_{n_1}, y) : y > f(x_{n_1})\} = \emptyset$ remains. However, in this case by (iii) the set

$$\{(x_{n_0}, y): y > f(x_{n_0})\} \cup \{(x_{n_1}, y): y > f(x_{n_1})\} \cup P$$

divides M into two parts, as above, which is impossible. Thus we come to the conclusion that the number δ' exists.

For any $t \in (x, x+\delta')$ there exist points $(t, w) \in M$ and $(t, v) \in M$ such that $w \leq \tau(t)$, $\tau(t) + \text{dist}(t, \bar{C}) \leq v$. Reasoning as in the case of point (x, u) we come to the conclusion that there exist the points $(x, w') \in M$ and $(x, v') \in M$ such that $w' \leq \underline{\lim}_{t\to x+} \tau(t) = \underline{\lim}_{t\to x+} f(t)$ and $v' \geq \overline{\lim}_{t\to x+} \tau(t) = \overline{\lim}_{t\to x+} f(t)$, which contradicts the choice of M. therefore (i) holds.

Hence there exists an ξ such that $M = \Gamma_\xi$. If the measure of $(\Gamma_\xi)_x \cap (x, x+\delta)$ is positive, then there exists a point $x_i^{(\xi)} \in (\Gamma_\xi)_x$ and then $(x_i^{(\xi)}, f(x_i^{(\xi)})) \in \Gamma_\xi = M$. If the measure of $(\Gamma_\xi)_x \cap (x, x+\delta)$ is equal to zero, then the set $H = \Gamma_\xi \cap P \cap \{(t, y) : x \leq t \leq x+\delta\}$ is a sum of segments I parallel to the Oy-axis and such that at least one end point of these segments belongs to G_τ or $G_{\tau'}$ where $\tau' = \tau(t) + \text{dist}(t, \bar{C})$. There exists among them a segment I such that $I = \{(c, y) : \tau(c) < y < \tau(c) + \text{dist}(c, \bar{C})\}$, $x_0 < c < x_0 + \delta$, $c \in B$. Indeed, should I not exist, then $H = H_1 \cup H_2$, where $H_1 \cap H_2 = \emptyset$, H_1 is the sum of intervals I possessing common points only with the graph of the function $\tau(x) + \text{dist}(x, \bar{C})$ and H_2 only with the graph of $\tau(x)$.

The set $Q^* = Q - H$ is connected. By the definition of H we have $Q^* \cap \Gamma_\xi = \emptyset$. This, however, leads in a similar manner as the foregoing case with the assumption that $P \cap M = \emptyset$ to a contradition with the assumed property of M. Thus the existence of I has been proved.

Clearly $(c, f(c)) \in I \subset M$. The point x is therefore a right-hand connectedness point of f.*) The number $f(x) = \tau(x)$ is the left-hand and the right-hand limit value of τ. In fact, for any sequence $\{x_n\}$, $x_n \to x \in \bar{C}$ the condition $\tau(x_n) \leqslant f(x_n) \leqslant \tau(x_n) + \operatorname{dist}(x_n, \bar{C})$ is satisfied. As $\lim_{n\to\infty} \tau(x_n) = \tau(x) = \lim_{n\to\infty} [\tau(x_n) + \operatorname{dist}(x_n, \bar{C})]$, $\lim_{n\to\infty} f(x_n) = \tau(x) = f(x)$. Point $x \in \bar{C} - C$ is therefore also a connectedness point of f, which completes the proof.

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О НЕПРЕРЫВНОСТИ СВЯЗНЫХ ФУНКЦИЙ

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Резюме

Дж. Сидер [3] доказал, что в пространстве вещественных чисел для всякого множества E типа G_δ существует функция $f\colon R \to R$ обладающая свойством Дарбу, непрерывная во всех точках множества E и разрывная в остальных точках. Пользуясь гипотезой континуума авторы этой статьи доказывают, что функция f может быть выбрана так, что ее график связен.

*) In a similar way we can prove that x is a left-hand connectedness point of f.