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ON PARTIALLY DIRECTED GEODETIC GRAPHS

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1. Introduction

Geodetic graphs (undirected, directed or mixed) have been studied in several papers [1, 2, 3, 4, 5, 8]. The class of planar geodetic graphs was characterized by Stemple and Watkins [7]. Plesník [6] and Zelinka [9] have dealt with construction of undirected geodetic graphs. In the present paper we construct partially directed geodetic graphs $Z_{t,d}$. Further, we show that for any integer $d \geq 3$ the graph $Z_{3,d}$ is a P-graph that is neither a quasitree nor a graph similar to a T-graph. This implies that the converse to Theorem 10 in [3] is not true and Problem 5 of [2] is solved.

2. Notations and preliminary results

The *graphs* considered in this paper are *partially directed,* i.e., they may contain directed edges as well as undirected ones; in particular, there are studied *mixed graphs,* i.e., they contain at least one directed edge and at least one undirected edge.

For a given graph G, $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively.

A *semitrail* from u to v (or $u-v$ *semitrail*) in a graph G is a finite sequence

$$S=[v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n],$$

where n is a non-negative integer (the length of S); $v_0=u$, v_1, $v_2, \ldots, v_{n-1}$, $v_n=v \in V(G)$; $e_1, e_2, \ldots, e_n$ are mutually different edges of G and v_{i-1}, v_i are the end vertices of $e_i \in E(G)$ for $i=1, 2, \ldots, n$. A semitrail whose vertices are mutually different is called a *semipath.* A *semipath* [*semitrail*] S whose every edge e_i is either undirected or directed from v_{i-1} to v_i is called a *path* [*trail*]. The length of S will be denoted by $|S|$. A *segment* of S between the vertices $v_i=x$ and $v_j=y$ $(i \leq j)$ will be denoted by $S[x, y]$.

A semitrail [trail] from u to v is called a *semicycle* [*cycle*] if it has a positive length and if its vertices are mutually different with the exception of u v.

A graph G is said to be *connected* [*strongly connected*] if for every ordered pair $[u, v]$ of vertices of G there exists a semipath [path] from u to v The distance between the vertices $u, v \in V(G)$ is denoted by $\varrho_G(u \;\; v)$ and it is the length of a shortest u v path of G, if any. The supremum of all distances in G is the *diameter* G and is denoted by $d(G)$. A graph is said to be *geodetic* if two arbitrary vertices are connected by a unique shortest path.

Let C be an even semicycle (i.e., C has an even length) of a graph G and let u, v be two vertices of C. Then we shall say that the vertices u, v are *C-opposite* in G if from the vertices and edges of C there is possible to form two different u v paths each of the length $|C|$ 2.

Theorem 1. (cf. Stemple and Watkins [7, Theorem 2]). *A partially directed graph G is geodetic if and only if G is strongly connected and G contains no even semicycle C such that for some C-opposite pair of its vertices u v we have*

$$\varrho_G(u, v) = |C| \; 2.$$

Proof. Let a graph G be geodetic. Then obviously G is strongly connected. Let there exist an even semicycle C such that there are C opposite vertices u, v and $\varrho_G(u, v) = |C|$ 2. Then there exist two different shortest u v paths of the length $|C| \; 2 = \varrho_G(u, v)$ and this is a contradiction to the definition of a geodetic graph.

Conversely, let us assume that G is strongly connected but not geodetic Then there exist vertices $u, v \in V(G)$ such that there are two distinct shortest $u - v$ paths P_1 and P_2. Let $u = x_0, x_1, x_2, \ldots, x_n = v$ be the vertices of P_1. Then at least one of the vertices $x_1, x_2, \ldots, x_n$ is not on P_2. Let x_i be the first vertex of P_1 that is not on P_2 and let x_k be the first vertex of P_1 occurring in P_1 after x_i such that x_k is on P_2. Then we must have

$$|P [x_{i-1}, x_k]| - |P [x_{i-1}, x_k]|.$$

(If, e.g., $|P_1[x_{i-1}, x_k]| > |P_2[x_{i-1}, x_k]|$, then the path with the vertices u x, x, ..., $x_{i-1} = y_1, y_2, \ldots, y_r$ $x_k, x_{k+1}, \ldots, x_n = v$ would be shorter than P and this is a contradiction; here $y_1, y_2, \ldots, y_r$ are the vertices of the path $P [x_{i-1}, x_k]$.) Let C be the semitrail consisting of the semipaths $P [x_{i-1}, x_k]$ and $S [x_k, x$], where $S [x_k, x_{i-1}]$ is the semipath arisen from the path $P [x_{i-1}, x_k]$ by reversing the order its elements. C is an even semicycle and

$$\varrho_G(x_{i-1}, x_k) - |C| \; 2.$$

Q E.D.

Lemma 1. *Let C be an even cycle of graph G of the length $2n$. Let any maximal undirected subpath of C have the length $<n$. Then there are no C-opposite vertices in G.*

P r o o f. Let there be in G C-opposite vertices x, y. Then there exist two distinct $x-y$ paths P_1 and P_2 such that

$$|P_1| = |P_2| = |C|/2,$$

and C can be composed of the paths P_1 and P_2 by reversing one of them. As P_1 and P_2 are paths and all directed edges are directed in the same direction, they must be all contained either in P_1 or P_2. Let them be contained, e.g., in P_1. Then P_2 is an undirected path of length $|C|/2 = n$ and this is a contradiction, as P_2 or the reverse of P_2 is a subpath of C. Q.E.D.

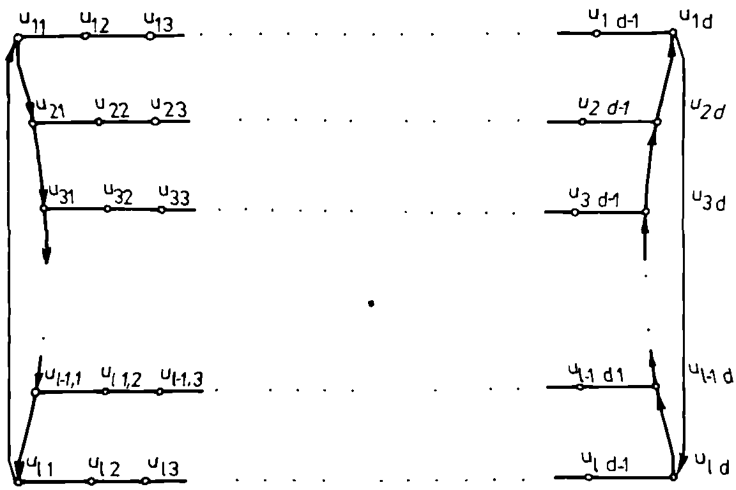

Fig. 1

3. Construction of $Z_{l,d}$ graphs

For given positive integers d and l we construct a graph $Z_{l,d}$ as follows (see Fig. 1):

$$V(Z_{l,d}) = \{u_{ij} \mid i = 1, 2, \ldots, l;\ j = 1, 2, \ldots, d\}$$

$$E(Z_{l,d}) = \{\overrightarrow{u_{i1}u_{j1}} \mid j - i \equiv 1 \pmod{l}\} \cup \{\overrightarrow{u_{id}u_{jd}} \mid i - j \equiv 1 \pmod{l}\} \cup \bigcup_{i=1}^{l} \{u_{ij}u_{i,j+1} \mid j = 1, 2, \ldots, d-1\}.$$

Evidently, the graph $Z_{l,d}$ is strongly connected for any d and l. The graphs $Z_{3,1}$ and $Z_{3,2}$ are drawn in Fig. 2.

Theorem 2. *The graph $Z_{l\,d}$ is geodetic if and only if l is odd.*

Proof. Let $G=Z_{l\,d}$ be geodetic and let $l = 2k$, where k is an integer >1. Let us consider a semicycle $C=[u_{11}, u_{21}, \ldots, u_{k1}, u_{k+1\,1}, u_{k+1\,2}, \ldots, u_{k+1\,d}, u_{k+2\,d}, \ldots, u_{2k\,d}, u_{1d}, \ldots, u_{11}]$ (see Fig. 3). Evidently, $|C| = 2k+2(d-1)$. Obviously, the vertices u_{11} and $u_{k+1\,d}$ are C-opposite. As

$$\varrho_G(u_{11}, u_{k+1\,d}) = k+d-1 = |C|/2,$$

Theorem 1 implies that $Z_{l\,d}$ is not geodetic.

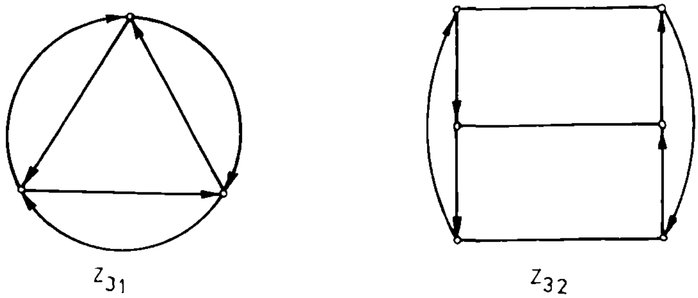

Fig. 2

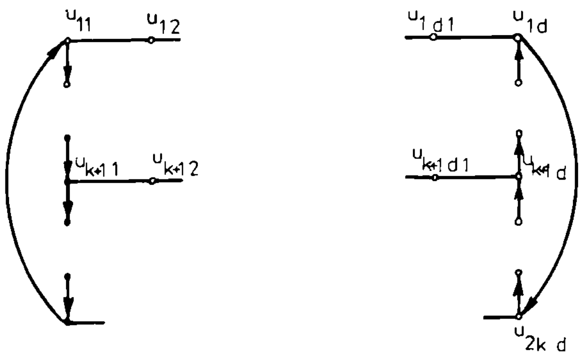

Fig. 3

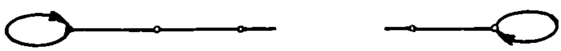

Fig. 4

Conversely, let $l=2k+1$ and $k>0$. Then any even semicycle of $Z_{l\,d}$ satisfies the conditions of Lemma 1 and, as $Z_{l\,d}$ is strongly connected, Theorem 1 implies that $Z_{l,d}$ is geodetic. (Especially if $l=1$, the $Z_{1\,d}$ has no semicycle with the exception of directed loops (see Fig. 4), and is evidently geodetic.) Q.E.D.

Theorem 3. *The graph* $Z_{l,d}$ *has the diameter*

$$k=[l/2]+d-1.$$

Proof. Put $G=Z_{l,d}$. Evidently, $\varrho_G(u_{11}, u_{[(l+2)/2]d}) = [l/2]+d-1$. Therefore it is sufficient to prove

$$\varrho_G(u, v)\leq[l/2]+d-1$$

for any u, $v\in V(G)$. Let $u=u_{ij}$, $v=u_{IJ}$, $1\leq i$, $I\leq l$, $1\leq j$, $J\leq d$. (See Fig. 5.) If $i=I$, then

$$\varrho_G(u, v)=|J-j|\leq d-1\leq[l/2]+d-1.$$

Otherwise, there exist two paths, namely $u_{ij}, \ldots, u_{i1}, \ldots, u_{I1}, \ldots, u_{IJ}$, and $u_{ij}, \ldots, u_{id}, \ldots, u_{Id}, \ldots, u_{IJ}$. The sum of their lengths is $l+2(d-1)$ so that at least one of them has the length $\leq[l/2]+d-1$. Q.E.D.

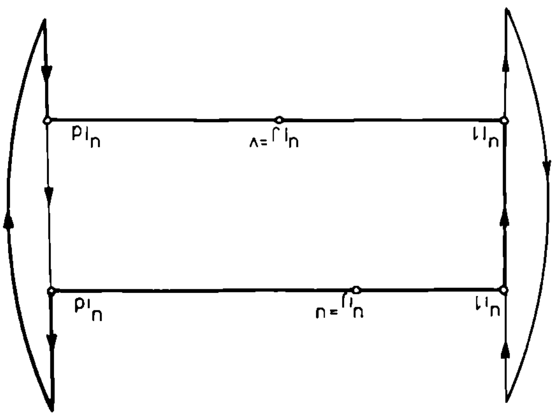

Fig. 5

4. *P*-graphs and *T*-graphs

A partially directed graph G is said to be a *T-graph* [*P-graph*] if for each ordered pair $[u, v]$ of vertices of G there exists in G exactly one trail [path, respectively] from u to v of a length not greater than the diameter of G.

A graph G is said to be a *quasitree* if for each ordered pair $[u, v]$ of vertices of G there exists exactly one path from u to v.

Lemma 2. (Bosák [1, Theorem 6]) *A graph G is a quasitree if and only if G is connected and every block of G is isomorphic to* K_2, C_1 *or a directed cycle.*

Lemma 3. (Bosák [3, Theorem 10])

(1) *Every quasitree is a P-graph.*

(2) *Every graph similar to a T-graph is a P-graph.*

(The graphs G and H are said to be *similar* if deleting all the loops and replacing every undirected edge by a pair of oppositely directed edges in both G and H yields two isomorphic directed graphs.)

J. Bosák in [1, 2, 3] has suggested the following problem: Is the converse of Theorem 10 in [3] true in the sense that every P-graph is either a quasitree or similar to a T-graph?

In the case of undirected graphs the answer to this problem is obviously positive as then we have:

Lemma 4. (see Bosák [1, Lemma 8]) *Let G be a loopless undirected graph. Then G is a P-graph if and only if G is a T-graph.*

In the case of mixed or directed graphs it will be proved that the converse of Lemma 3 is not true (see Corollary to Theorem 5 below).

Theorem 4. *Let l and d be positive integers. The graph $Z_{l\,d}$ is a P-graph if and only if at least one of the following conditions hold:*

(1) $l-1$

(2) $l=3$

(3) *l is odd and* $d-1$.

Proof. It is easy to verify that the graph $Z_{1\,d}$ [$Z_{l\,1}$] is a P-graph for every d [for every odd l, respectively]. We prove that $Z_{3\,d}$ is a P-graph for every d. From Theorem 3 it follows that the diameter of $Z_{3\,d}$ is d. Therefore it is sufficient to prove that for any $u, v \in V(G)$ there exists at most one u v path of the length not greater than d. Let $u = u_{ij}$, $v = u_{IJ}$, $1 < i$, $I < 3$, $1 \leq j$ $J < d$. If $i = I$, then the length of a path P_1 is

$$|P_1| = \varrho_G(u, v) = |J - j| \leq d - 1.$$

For any other $u - v$ path P_2, the length of P_2 is

$$|P_2| > d - 1 + 2 > d.$$

If $i \neq I$, then for any two distinct $u - v$ paths P_1, P_2 we have

$$|P\ | + |P_2| > 2(d-1) + 3 = 2d + 1$$

so that at most one of them has the length $> d$. (See Fig. 6.)

Conversely, let $Z_{l,d}$ be a P-graph. Then $Z_{l\,d}$ is geodetic and by Theorem 2 l is odd. Let $l \geq 5$ and $d > 2$. According to theorem 3 the diameter of $Z_{l\,d}$ is

$$[l\ 2] + d - 1 > [5\ 2] + d - 1 = d + 1.$$

But there exist in $Z_{l,d}$ two different paths from u_{11} to u_{1d} (see Fig. 1):

$$P_1 = [u_{11}, u_{12}, \ldots, u_{1d}], \ |P_1| = d-1 \quad \text{and}$$
$$P_2 = [u_{11}, u_{21}, u_{22}, \ldots, u_{2d}, u_{1d}], \ |P_2| = d+1.$$

The length of both paths is less than or equal to the diameter of $Z_{l,d}$ so that $Z_{l,d}$ is not a P-graph. Q.E.D.

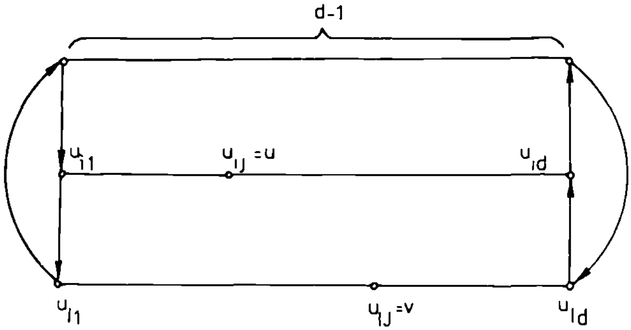

Fig. 6

Theorem 5.
(A) *The graph $Z_{l,d}$ is a quasitree if and only if $l=1$.*
(B) *The graph $Z_{l,d}$ is similar to a T-graph if and only if one of the following cases occurs:*
(1) $l=1$
(2) $l=3$, $d=2$
(3) *l is odd and $d=1$.*

Proof. (A) follows from Lemma 2, (B) from Theorem 4 and Lemma 3 as $Z_{3,d}$ for $d \geq 3$ contains a cycle of length 3, which is less than the diameter of $Z_{3,d}$. Q.E.D.

Corollary. *The graph $Z_{3,d}$ for $d \geq 3$ is a mixed P-graph of diameter d that is neither a quasitree nor a graph similar to a T-graph.*

Proof follows from Theorems 3—5.

Remark. To get a directed example, replace in $Z_{3,d}$ each undirected edge by a pair of opposite directed edges.

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О ЧАСТИЧНО ОРИЕНТИРОВАННЫХ ГЕОДЕЗИЧЕСКИХ ГРАФАХ

Павол Хиц

Резюме

Частично ориентированный граф G называется геодезическим, если для каждых двух вершин существует единственный кратчайший путь между ними. Частично ориентированный граф G называется T-графом [P графом], если для всякой упорядоченой пары $[u, v]$ его вершин существует в G точно одна цепь [один путь] длины, не превышающей диаметр графа G. Автор дает конструкцию геодезических графов $Z_{i,d}$ (рис. 1) и далее показывает, что для каждого натурального числа $d > 3$, $Z_{3,d}$ является P графом и не является ни T-графом, ни квазидеревом