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Mathematica Slovaca, Vol. 33 (1983), No. 2, 165--180

Persistent URL: `http://dml.cz/dmlcz/136327`

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ON THE FACE-VECTORS OF TRIVALENT CONVEX POLYHEDRA

STANISLAV JENDROĽ

1. Introduction

Let S be a convex polyhedron and let $p_k(S)$, or $v_k(S)$ denote the number of its k-gonal faces, or k-valent vertices, respectively. We shall call the sequence $(p_k(S))$ the *face-vector* of S and the sequence $(v_k(S))$ the *vertex-vector* of S. A polyhedron S is said to be *trivalent* if $v_k(S)=0$ for all $k\neq 3$. Consider a sequence of nonnegative integers (p_k). The present paper deals with necessary conditions for (p_k) to be the face-vector of some trivalent convex polyhedron S, i. e. conditions for the existence of a trivalent convex polyhedron S such that $p_k(S)=p_k$ for all $k\geqq 3$. (Evidently $p_1=p_2=0$).

The well-known Euler formula leads for a trivalent convex polyhedron to the condition

$$3p_3+2p_4+p_5=12+\sum_{i\geqq 6}(i-6)p_k \tag{1}$$

for the terms of the sequence (p_i). The equality (1) gives no information about p_6. Thus the above problem is equivalent to the following **problem**:

Let $p=(p_i\,|\,3\leqq i\neq 6)$ be a sequence of nonnegative integers satisfying (1). *Denote by $P(p)$ the set of all nonnegative integers p_6 such that if p_6 is added to p, then the face-vector of a trivalent convex polyhedron S is obtained. Characterize $P(p)$.*

For any sequence $p=(p_i\,|\,3\leqq i\neq 6)$ of nonnegative integers let

$$\sigma=\Sigma p_i \quad \text{for} \quad 3\leqq i\neq 6$$

and

$$\varrho=\Sigma p_j \quad \text{for} \quad 3<j\not\equiv 0 \pmod 3).$$

As far back as 1891 Eberhard [1] proved the following theorem (cf. Grünbaum [4, p. 254], Jucovič [9, p. 64]):

Theorem 1. *$P(p)$ is nonempty for any sequence of nonnegative integers $p=(p_k\,|\,3\leqq k\neq 6)$ satisfying* (1).

In 1974, Fisher [2] proved the following assertion.

Theorem 2. *For any sequence $p=(p_k\,|\,3\leq k\neq 6)$ of nonnegative integers satisfying* (1) *there exists an integer $d\leq 3\sigma$ such that $P(p)$ contains the number $p_6=d+2t$ for any nonnegative integer t.*

Theorem 3 (Fisher [2, 3]). *For any sequence $p=(p_k\,|\,3\leq k\neq 6)$ of nonnegative integers with $p_5\geq 2$ or $p_4\geq 2$ which satisfies* (1) *there exists an integer $d\leq 3\sigma$ such that $P(p)$ contains every integer $\geq d$.*

Grünbaum [4, p. 272] proved

Theorem 4. *Let $p=(p_k\,|\,3\leq k\neq 6)$ be a sequence of nonnegative integers with $\varrho\leq 2$.*

(i) *If $\sigma\equiv 0$* (mod 2), *then no odd integer is an element of $P(p)$.*

(ii) *If $\sigma\equiv 1$* (mod 2), *then no even integer is an element of $P(p)$.*

For detailed references to results concerning this problem, see the works of Grünbaum [4, 6], Jendroľ—Jucovič [7] and Jucovič [9].

The purpose of the present paper is to prove that this assertion of Grünbaum characterizes *all* sequences $p=(p_k\,|\,3\leq k\neq 6)$ for which the set of nonnegative integers *not belonging* to $P(p)$ is infinite.

More precisely, we shall prove the following

Theorem 5. *Let a sequence $p=(p_k\,|\,3\leq k\neq 6)$ of nonnegative integers satisfy* (1).

(i) *If $\varrho\leq 2$ and $\sigma\equiv 0$* (mod 2), *then there exists an integer d such that $P(p)$ contains every even integer $\geq d$ and no odd integer.*

(ii) *If $\varrho\leq 2$ and $\sigma\equiv 1$* (mod 2), *then there exists an integer d such that $P(p)$ contains every odd integer $\geq d$ and no even integer.*

(iii) *If $\varrho\geq 3$, then there exists an integer d such that $P(p)$ contains every integer $\geq d$.*

The existence part of the proof comprises the construction of a planar map with a trivalent 3-connected graph and the prescribed number p_k of k-gonal faces. The existence of a convex polyhedron combinatorially equivalent to such a map is guaranteed by the Steinitz theorem (see [5, p. 235] or [9, p. 30]).

2. Basic construction elements and some existence lemmas

In this chapter we prove some existence lemmas which are valid for all maps with the 3-connected graph and on the orientable surface of genus g for any $g\geq 0$ (i. e. not only for planar maps with a trivalent graph).

Consider such a map M with sequences $q=(q_i\,|\,i\geq 3)$ and $v=(v_i\,|\,i\geq 3)$ as a face-vector and vertex-vector, respectively. From the trivial equality $\sum_{i\geq 3} iv_i=\sum_{i\geq 3} iq_i$ there follows a useful relation

$$v_3 = \frac{1}{3}\left(\sum_{i\geq 3} iq_i - \sum_{i\geq 4} iv_i\right). \qquad (2)$$

Basic construction elements: The face-aggregate of a map M as in Fig. 1a (or its mirror image), or 2a, or 3a, called *configuration* A_m, or B_m, or C_m (*conf* A_m, *conf* B_m, *conf* C_m in the sequel) consists of an m-gon, $m \geq 6$, two hexagons and one quadrangle, or of an m-gon, $m \geq 6$, two hexagons and two quadrangles, or of an m-gon, $m \geq 6$, two hexagons and three quadrangles, respectively. (We note, that i, j, k, m, n, t, w mean nonnegative integers in the sequel.)

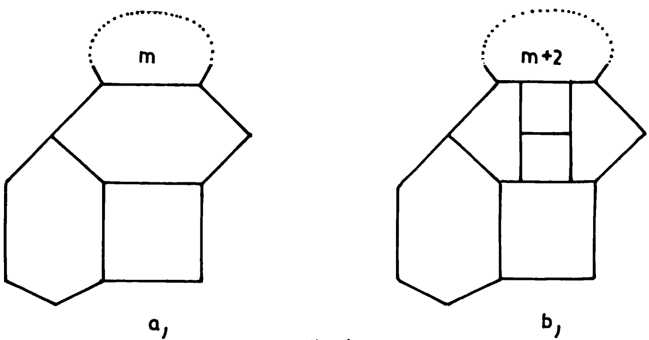


Fig. 1

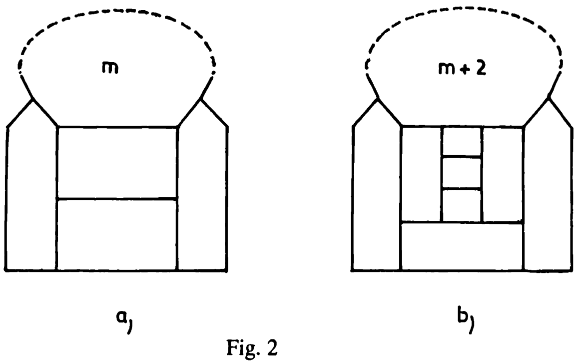


Fig. 2

Basic construction steps: The number of edges of the m-gon in *conf* A_m of M is increased by inserting new edges into the "middle" hexagon so that two edges are divided to form three edges, see Fig. 1b. This gives rise to a conf B_{m+2} or $(m+2)$-gon and a conf B_6 (considering the "bottom" hexagon). Two new hexagons appear in the map M. If it is necessary to increase the number of edges of the $(m+2)$-gon, then conf B_{m+2} is used in further constructions; otherwise we use conf B_6.

Analogously we obtain a conf C_{m+2} (or an $(m+2)$-gon and conf C_6) and three new haxagons from conf B_m; this transformation is shown in Fig. 2b. Finally, Fig. 3b shows how to transform conf C_m into conf A_{m+2} (or $(m+2)$-gon and conf A_6) with one additional conf C_6. Six new hexagons appear in the map.

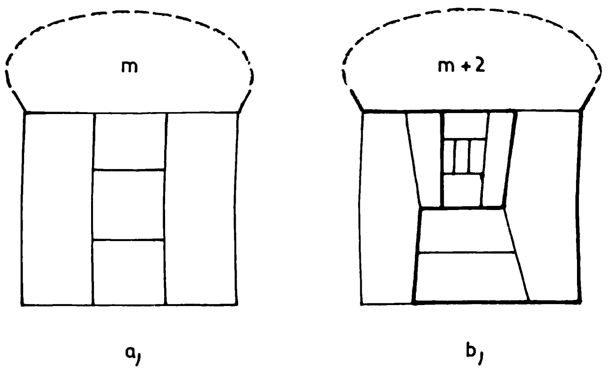


Fig. 3

If it is necessary to change an m-gon in conf A_m to an i-gon, $i \geq m+6$, it can be done by constructing gradually a conf B_{m+2}, conf C_{m+4}, conf A_{m+6} etc. In the sequel we shall call this transition from conf A_m to conf A_{m+6} (in the course of which an m-gon is changed into an $(m+6)$-gon, one conf A_6 and ten new hexagons are created) an A-*step*. Analogously a B-*step* (C-*step*) consists in increasing by six the number of edges of an m-gon in conf B_m (conf C_m) with a conf C_6 and ten hexagons as a by-product.

Let $M = M(q, v, g, a, b, c)$ be a map on the orientable surface of genus g having the following properties:

(i) *Its graph is* 3-*connected.*

(ii) *Sequences $q = (q_i \mid i \geq 3)$ and $v = (v_i \mid i \geq 3)$ are the face-vector and the vertex-vector, respectively, of M.*

(iii) *M contains as submaps at least a configurations A_6, $a \geq 0$, b configurations B_6, $b \geq 0$, and c configurations C_6, $c \geq 0$. Mentioned configurations are pairwise disjoint.*

Lemma 1.α. (Insertion of an j-gon, $j \geq 7$.) *If there exists a map $M(q, v, g, a, b, c)$, then there exists a map $M(q', v', g, a', b', c')$ with $q' = (q'_i \mid q'_i = q_i + s_i)$,*

$$v' = \left(v'_i \mid v'_i = v_i \text{ for all } i \geq 4;\ v'_3 = \frac{1}{3}\left(\sum_{i \geq 3} i q'_i - \sum_{i > 4} i v'_i\right)\right),$$

where $s_i = 0$ for all $i \neq 3, 4, 5, 6, j$; $j \geq 7$, $s_j = 1$ and for the values j, s_3, s_4, s_5, s_6, a', b', c' see Table 1, *lines* 1—9 *if $a \neq 0$, or lines* 10—18 *if $b \neq 0$, or lines* 19—27 *if $c \neq 0$, respectively.*

Table 1

α	j	s_3	s_4	s_5	s_6	a'	b'	c'
1.	$6k$	0	$3k-3$	0	$10k-10$	a	b	$c+k-1$
2.	$6k+1$	1	$3k-4$	0	$10k-8$	$a-1$	b	$c+k-1$
3.	$6k+1$	0	$3k-3$	1	$10k-9$	$a-1$	b	$c+k-1$
4.	$6k+2$	0	$3k-2$	0	$10k-8$	$a-1$	$b+1$	$c+k-1$
5.	$6k+3$	1	$3k-3$	0	$10k-5$	a	b	$c+k-1$
6.	$6k+3$	0	$3k-2$	1	$10k-8$	$a-1$	$b+1$	$c+k-1$
7.	$6k+4$	0	$3k-1$	0	$10k-5$	$a-1$	b	$c+k$
8.	$6k+5$	1	$3k-2$	0	$10k+1$	$a-1$	$b+1$	$c+k-1$
9.	$6k+5$	0	$3k-1$	1	$10k-5$	$a-1$	b	$c+k-1$
10.	$6k$	0	$3k-3$	0	$10k-10$	a	b	$c+k-1$
11.	$6k+1$	1	$3k-4$	0	$10k-7$	$a+1$	$b-1$	$c+k-1$
12.	$6k+1$	0	$3k-3$	1	$10k-10$	a	$b-1$	$c+k-1$
13.	$6k+2$	0	$3k-2$	0	$10k-7$	a	$b-1$	$c+k$
14.	$6k+3$	1	$3k-2$	0	$10k-1$	a	b	$c+k-1$
15.	$6k+3$	0	$3k-2$	1	$10k-7$	a	$b-1$	$c+k-1$
16.	$6k+4$	0	$3k-1$	0	$10k-2$	$a+1$	$b-1$	$c+k$
17.	$6k+5$	1	$3k-2$	0	$10k$	a	$b-1$	$c+k$
18.	$6k+5$	0	$3k-1$	1	$10k-3$	a	$b-1$	$c+k$
19.	$6k$	0	$3k-3$	0	$10k-10$	a	b	$c+k-1$
20.	$6k+1$	1	$3k-4$	0	$10k-4$	a	$b+1$	$c+k-2$
21.	$6k+1$	0	$3k-3$	1	$10k-10$	a	b	$c+k-2$
22.	$6k+2$	0	$3k-2$	0	$10k-5$	$a+1$	b	$c+k-1$
23.	$6k+3$	1	$3k-3$	0	$10k-3$	a	b	$c+k-1$
24.	$6k+3$	0	$3k-2$	1	$10k-6$	a	b	$c+k-1$
25.	$6k+4$	0	$3k-1$	0	$10k-3$	a	$b+1$	$c+k-1$
26.	$6k+5$	1	$3k-2$	0	$10k-1$	$a+1$	b	$c+k-1$
27.	$6k+5$	0	$3k-1$	1	$10k-3$	a	b	$c+k-1$

Proof. To obtain the map $M(q', v', g, a', b', c')$, the required j-gon, $j \geq 7$, is inserted into one of the configurations A_6 (in the cases $\alpha = 1, 2, \ldots, 9$), or of the configurations B_6 (in the cases $\alpha = 10, \ldots, 18$), or of the configurations C_6 (in the cases $\alpha = 19, \ldots, 27$) of the map $M(q, v, g, a, b, c)$, respectively. We use only basic constructions described in the previous part.

A $6k$-gon, $k \geq 1$, is inserted into conf A_6, conf B_6, or conf C_6 by $(k-1)$ repetitions of an A-step, B-step, or C-step, respectively. The starting step for constructing a $(6k+2)$-gon, or a $(6k+4)$-gon, $k \geq 1$, is the insertion of an 8-gon or a 10-gon into the appropriate configuration. This is followed by the necessary number of A-steps, B-steps, or C-steps.

A $(2m+1)$-gon, $m \geq 3$, is inserted into conf A_6 (conf B_6, conf C_6) as follows: we start by inserting a $2m$-gon which will appear in conf A_{2m}, conf B_{2m}, or conf C_{2m}. By adding edges as in Figs. 4, 5, or 6, respectively, we obtain the $(2m+1)$-gon. Figures "a" are considered if $s_5=0$; figures "b" are taken in the opposite case.

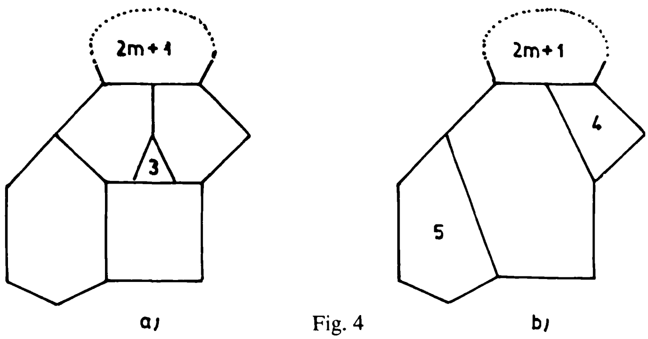


Fig. 4

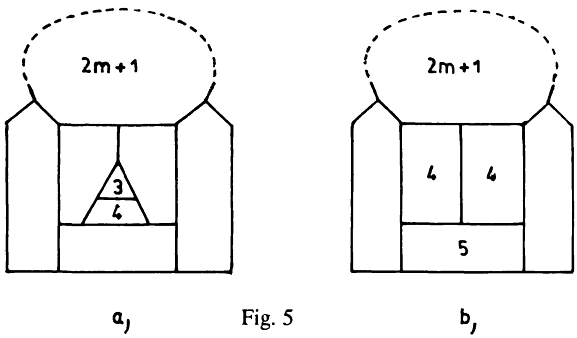


Fig. 5

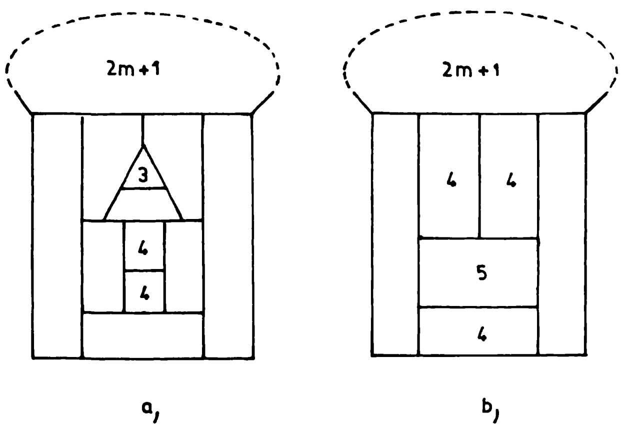


Fig. 6

Lemma 2.α. (Insertion of a pair of odd-gons.) *Let $m \geq 7$, $n \geq 7$. If there exists a map $M(q, v, g, a, b, c)$, then there exists a map $M(q', v', g, a', b', c')$ with*

$q' = (q'_i \mid q'_i = q_i$ *for all* $i \neq 4,6, m, n$; $q'_4 + s_4$, $q'_6 = q_6 + s_6$, $q'_m = q_m + 1)$, $q'_n = q_n + 1$, *if* $m \neq n$, *or*

$q' = (q'_i \mid q'_i = q_i$ *for all* $i \neq 4,6, m$; $q'_4 = q_4 + s_4$, $q'_6 = q_6 + s_6$, $q'_m = q_m + 2)$, *if* $m = n$, *and*

$$v' = \left(v'_i \mid v'_i = v_i \text{ for all } i \neq 3,\ v_3 = \frac{1}{3}\left(\sum_{i \geq 3} i q'_i - \sum_{i \geq 4} i v'_i\right)\right).$$

Table 2

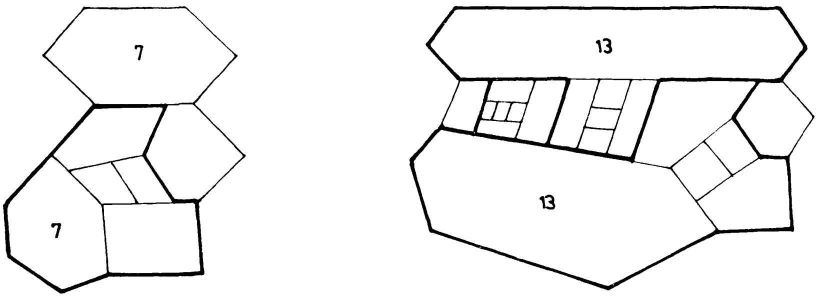
α	m	n	s_4	a'	b'	c'
1.	$6t+1$	$6w+1$	$3(t+w)-5$	$a-1$	$b+1$	$c+t+w-2$
2.	$6t+1$	$6w+3$	$3(t+w)-4$	$a-1$	b	$c+t+w-1$
3.	$6t+1$	$6w+5$	$3(t+w)-3$	a	b	$c+t+w-1$
4.	$6t+3$	$6w+3$	$3(t+w)-3$	a	b	$c+t+w-1$
5.	$6t+3$	$6w+5$	$3(t+w)-2$	$a-1$	$b+1$	$c+t+w-1$
6.	$6t+5$	$6w+5$	$3(t+w)-1$	$a-1$	b	$c+t+w$
7.	$6t+1$	$6w+1$	$3(t+w)-5$	a	$b-1$	$c+t+w-1$
8.	$6t+1$	$6w+3$	$3(t+w)-4$	$a+1$	$b-1$	$c+t+w-1$
9.	$6t+1$	$6w+5$	$3(t+w)-3$	a	b	$c+t+w-1$
10.	$6t+3$	$6w+3$	$3(t+w)-3$	a	b	$c+t+w-1$
11.	$6t+3$	$6w+5$	$3(t+w)-2$	a	$b-1$	$c+t+w$
12.	$6t+5$	$6w+5$	$3(t+w)-1$	$a+1$	$b-1$	$c+t+w$
13.	$6t+1$	$6w+1$	$3(t+w)-5$	$a+1$	b	$c+t+w-2$
14.	$6t+1$	$6w+3$	$3(t+w)-4$	a	$b+1$	$c+t+w-2$
15.	$6t+1$	$6w+5$	$3(t+w)-3$	a	b	$c+t+w-1$
16.	$6t+3$	$6w+3$	$3(t+w)-3$	a	b	$c+t+w-1$
17.	$6t+3$	$6w+5$	$3(t+w)-2$	$a+1$	b	$c+t+w-1$
18.	$6t+5$	$6w+5$	$3(t+w)-1$	a	$b+1$	$c+t+w-1$

For the values m, n, s_4, a', b', c' see Table 2 lines 1—6 if $a \neq 0$, or lines 7—12 if $b \neq 0$ or lines 13—18 if $c \neq 0$ (in the second case consider $m = n$ in the Table 2), s_6 is a constant depending on m and n.

Proof. Inserting into the one from among the configurations A_6 (cases $\alpha = 1, 2, \ldots, 6$), or configurations B_6 (cases 7, ..., 12). or configurations C_6 (cases 13, ..., 18) of the map $M(q, v, g, a, b, c)$ a pair of odd-gons we obtain a map $M(q', v', g, a', b', c')$. Insertion of a pair $(6t+x)$-gon, $(6w+y)$-gon, $t \geq 1$, $w \geq 1$, $x = 1,3$, or 5, $y = 1,3$, or 5 into conf A_6, conf B_6, or conf C_6 is described in Jendroľ—Jucovič

[7]; we shall therefore give only a sketch of their construction. If $t=1$, or $w=1$ we start by inserting a $(6+x)$-gon and a $(6+y)$-gon in such a way that the $(6+y)$-gon (or the $(6+x)$-gon if $t\neq 1$) was a part of conf A_{6+y}, conf B_{6+y} or conf C_{6+y} (conf A_{6+x}, conf B_{6+x} or conf C_{6+x}) and that only hexagons with at most some configurations C_6 are formed.

If $t\geqslant 2$ and $w\geqslant 2$, we start by inserting a $(12+x)$-gon and a $(12+y)$-gon in such a way that a conf C_{12+x}, one of the conf A_{12+y}, conf B_{12+y} and conf C_{12+y} and neither conf A_6 nor conf B_6 are formed. This is followed by an appropriate number of A-steps, B-step, or C-steps. Fig. 7 shows the initial positions for the insertion of a $(6t+1)$-gon and a $(6w+1)$-gon into conf A_6.



a) Fig. 7 b)

Lemma 3.α. *Let $f=(f_i\mid i\geqslant 7)$ be a sequence of nonnegative integers with a finite number of nonzero elements and let*

$$j=6+\sum_{i\geqslant 7}(i-6)f_i .$$

If there is a map $M=M(q,\ v,\ g,\ a,\ b,\ c)$ with $a+b+c\neq 0$, then there is a map $M'=M(q',\ v',\ g,\ a',\ b',\ c')$ with

$$q'=(q'_i\mid q'_3=q_3+s_3,\ q'_4=q_4+s_4,\ q'_5=q_5+s_5,\ q'_6=q_6+s_6,\ q'_i=q_i+f_i \text{ for all } i\geqslant 7),$$

$$v'=\left(v'_i\mid v'_i=v_i \text{ for all } i\neq 3;\ v'_3=\frac{1}{3}\left(\sum_{i\geqslant 3}iq'_i-\sum_{i\geqslant 4}iv'_i\right)\right);$$

for the values s_3, s_4, s_5, a', b', c' see Table 1, lines 1—9 if $a\neq 0$; lines 10—18 if $b\neq 0$, lines 12—27 if $c\neq 0$. The value s_6 is a constant depending on the sequence f.

Proof. There exists a sequence of maps $M_0=M$, M_1, ..., $M_h=M'$, $h=\sum_{i\geqslant 7}f_i-\left|\frac{\sum_{i\geqslant 3}f_{2i+1}}{2}\right|$ such that the existence of a map M_z follows from the existence of a map M_{z-1}, $z=1,\ 2,\ \ldots,\ h$, by some of Lemmas 1.α or 2.β for suitable α or β.

Inserting into the one from among the configurations A_6 (for $\alpha = 1$—9), or configurations B_6 ($\alpha = 10$—18), or configurations C_6 ($\alpha = 19$—27), respectively, of the map M_0 an even-gon, or a pair of odd-gons required we obtain a map M_1. We obtain the map M_z, $z = 2, \ldots, h$, from the map M_{z-1} by inserting and even-gon, or a pair of required odd-gons with $\geqslant 7$ edges (or a single odd-gon if $z = h$ and $\sum_{i>3} f_{2i+1} \equiv 1 \pmod 2$) into the new conf A_6, or the new conf B_6 of M_{z-1}. (A conf A_6, or conf B_6 is called a *new* conf A_6, or a *new* conf B_6, respectively, of M_z if it contains a face, which has not appeared in the map M_{z-1}. It should be remarked that at most one of new conf A_6 or new conf B_6 appears in the map M_z, — see Lemmas 1.α and 2.β).

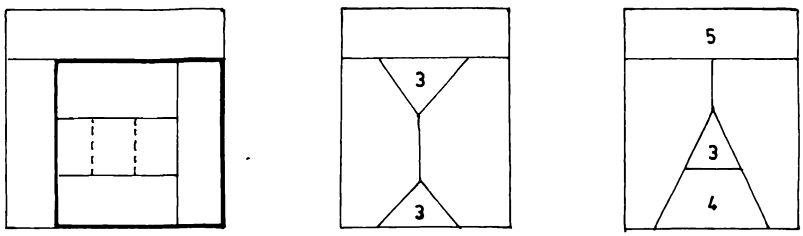
If neither new conf A_6, nor new conf B_6 appear in M_{z-1}, one from among the configurations C_6 is employed for creating an even-gon or a pair of odd-gons required.

Lemma 4.α. *If there is a map $M = M(q, v, g, a, b, c)$ with $c \neq 0$, then there is a map $M' = (q', v', g, a', b', c')$, where*

1. $q' = (q'_i \mid q'_i = q_i$ *tor all* $i \neq 6$, $p'_6 = p_6 + 2t)$,
   $v' = (v'_i \mid v'_i = v_i$ *for all* $i \neq 3$, $v'_3 = v_3 + 4t)$,

where t is a nonnegative integer and $a' = a$, $b'b$, $c' = c$, or

2. $q' = (q'_i \mid q'_3 = q_3 + 2$, $q'_4 = q_4 - 3$, $q'_i = q_i$ *for all* $i \geqslant 5)$,
   $v' = (v'_i \mid v'_3 = v_3 - 2$, $v'_i = v_i$ *for all* $i \neq 3)$
   and $a' = a$, $b' = b$, $c' = c - 1$, *or*
3. $q' = (q'_i \mid q'_3 = q_3 + 1$, $q'_4 = q_4 - 2$, $q'_5 = q_5 + 1$, $q'_6 = q_6 - 1$, $q'_i = q_i$ *for all* $i \geqslant 7)$,
   $v' = (v'_i \mid v'_3 = v_3 - 2$, $v'_i = v_i$ for all $i \neq 3)$
   and $a' = a$, $b' = b$, $c' = c - 1$.



a)   b)   c)

Fig. 8

Proof. Adding into the conf C_6 of the map M a pair of edges as shown in Fig. 8a (broken lines) we receive two new hexagons and a conf C_6. Repeating the above procedure t-times a map M' in the case $\alpha = 1$ is obtained.

The transformation of a conf C_6 of the map M as shown in Fig. 8b or 8c gives *a map* M' in the case $\alpha=2$, or $\alpha=3$, respectively.

Lemma 5. *If there is a map* $M=M(q, v, g, a, b, c)$ *with* $b\neq 0$, *then there is a map* $M'=M(q', v', g, a, b-1, c)$, *where*

$q'=(q'_i \mid q'_3=q_3+1,\ q'_4=q_4-2,\ q'_5=q_5+1,\ q'_6=q_6-1,\ q'_i=q_i$ *for all* $i\geq 7)$ *and* $v'=(v'_i \mid v'_3=v_3-2,\ v'_i=v_i$ *for all* $i\neq 3)$.

Proof. It is sufficient to transform a conf B_6 of the map M as shown in Fig. 9.

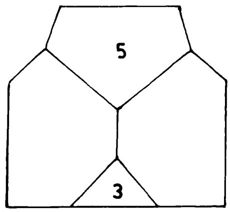


Fig. 9

Lemma 6. *Let* $M=M(q, v, g, a, b, c)$ *be a map and let* f_3, f_4, f_5 *be nonnegative integers satisfying the following conditions*

(i) $3f_3+2f_4+f_5=3q_3+2q_4+q_5$,
(ii) $f_3\geqslant q_3,\ q_5\leqslant f_5\leqslant q_5+1$,
(iii) $f_3\leqslant 2c+q_3$ *or* $f_3=2c+q_3+1$ *and* $b\neq 0$.

Then there is a map $M'=M(q', v', g, a', b', c')$ *with*

$q'=(q'_i \mid q'_3=f_3,\ q'_4=f_4,\ q'_5=f_5,\ q'_6=q_6-(f_5-q_5),\ q'_i=q_i$ *for all* $i\geqq 7)$,

$v'=\left(v'_i \mid v'_i=v_i \text{ for all } i\neq 3,\ v'_3=\frac{1}{3}\left(\sum_{i\geqslant 3} iq'_i-\sum_{i\geqslant 4} iv'_i\right)\right)$

and $a'=$, $b'\geqslant 0$, $c'\geqslant 0$.

Proof. According to the assumptions of the lemma, new triangles and (possibly) a pentagon of a map M' must be obtained from the quadrangles of the map M. As (iii) is valid, we can use Lemma 4.2 $\left(\left[\frac{f_3-q_3}{2}\right]\text{-times}\right)$ and, if $f_3-q_3\equiv 1 \pmod 2$ (and also $f_5-q_5=1$), Lemma 4.3 (if $f_3\leqslant 2c+q_3$), or Lemma 5 (if $f_3=2c+q_3+1$ and $b\neq 0$).

Lemma 7.α. 1. *There exists a trivalent map* $M(q, v, 0, 0, 3, 0)$ *with* $q=(q_i/q_i=0$ *for all* $i\neq 4,6$; $q_4=6,\ q_6=9$ *or* $12)$.

2. *There exists a trivalent map* $M(q, v, 0, 6, 0, 0)$ *with* $q=(q_i \mid q_i=0$ *for all* $i \neq 4,6$; $q_4=6$, $q_6=t$ *for all integers* $t \geqslant 27)$.

3. *There exists a trivalent map* $M(q, v, 0, 3, 0,0)$ *with* $q=(q_i \mid q_i=0$ *for all* $i \neq 3, 4, 6$, $q_3=2$, $q_4=3$, $q_6=t$ *for all integers* $t \geqslant 18)$.

Proof. For $\alpha=1$ see Fig. 10. Let $\alpha=2$ or 3 (see Malkevitch [10]). We observe that if there exists a trivalent map L containing a circuit $\varkappa$ as drawn in Fig. 11a, then there is a trivalent map L_1 with $p_6(L_1)=p_6(L)+2t$, $t \geqslant 0$ and $p_j(L_1)=p_j(L)$ for all $j \neq 6$.

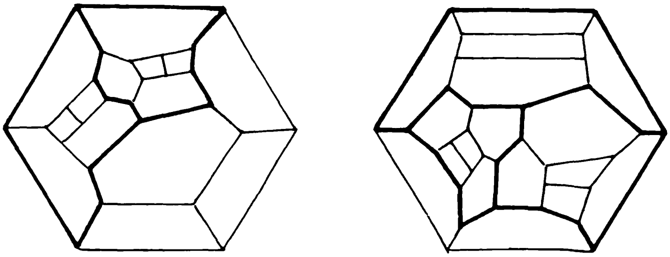


Fig. 10

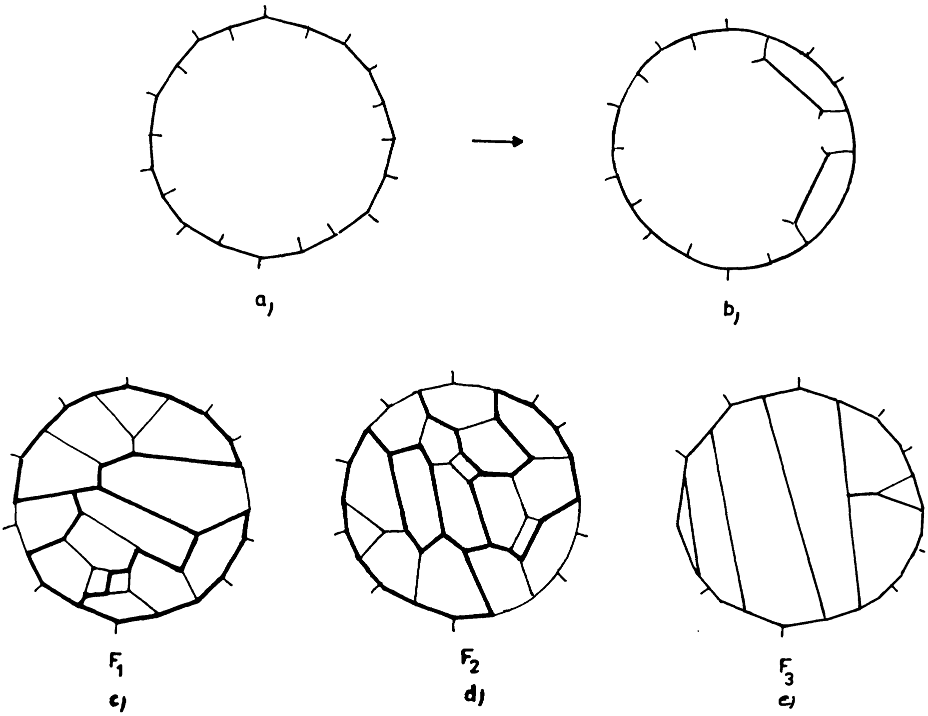


Fig. 11

The circuit $\varkappa$ separates two submaps P and Q of the map L. If we add two hexagons to the submap P as shown in Fig. 11b, we obtain a submap P_1 bounded by a circuit $\varkappa_1$ which has the same properties as $\varkappa$. This properties of $\varkappa_1$ allow to add two other hexagons to the submap P_1. After t-times repetition of the above procedure, we obtain a submap P_t bounded by a circuit $\varkappa_t$ with the same properties as $\varkappa$. To receive the need map L_1 we join suitably the submaps Q and P_t along their boundary circuits. To obtain the propositions of the lemmas, it is sufficient to take as the submaps P and Q the suitable ones among F_1, F_2, F_3 being drawn in Figures 11c—e.

3. Proof of Theorem 5

For the case $\varrho \leq 2$ the Theorem is a direct consequence of Theorems 2 and 4, for $p_5 \geq 2$ it is a consequence of Theorem 3. The case $\sum_{i \geq 7} p_i = 0$ has been treated in Jucovič [8] and Malkevitch [10] (cf. Grünbaum [5] or Jucovič [9, p. 60]). There remains to be proved Theorem 5 in the remaining cases, i. e. for all sequences $p = (p_i \mid 3 \leq i \neq 6)$ with

$$p_5 \leq 1, \quad \sum_{i \geq 7} p_n \neq 0, \qquad \varrho \geq 3. \tag{3}$$

We shall distinguish the following 16 cases:

1. $p_4 \geq 5$.
2. $p_4 = 2,3$ or 4, $p_3 \geq 2$.
3. $p_4 = 1$, $p_3 \geq 4$, $p_{6r+1} \geq 1$ for some $r \geq 1$.
4. $p_4 = 1$, $p_3 \geq 4$, $p_{6r+4} \geq 1$ for some $r \geq 1$.
5. $p_4 = 0$, $p_3 \geq 5$, $p_{6r+1} \geq 2$ for some $r \geq 1$.
6. $p_4 = 0$, $p_3 \geq 5$, $p_{6r+4} \geq 2$ for some $r \geq 1$.
7. $p_4 = 0$, $p_3 \geq 5$, $p_{6r+1} = p_{6s+1} = 1$ for some r, s, $r > s \geq 1$.
8. $p_4 = 0$, $p_3 \geq 5$, $p_{6r+1} = p_{6s+4} = 1$ for some r, s, $r \geq 1$, $s \geq 1$.
9. $p_4 = 0$, $p_3 \geq 5$, $p_{6r+4} = p_{6s+4} = 1$ for some r, s, $r > s \geq 1$.
10. $p_4 \leq 1$, $p_5 = 1$, $p_3 \geq 5$, $p_{6r+2} \geq 1$ for some $r \geq 1$.
11. $p_4 \leq 1$, $p_5 = 1$, $p_3 \geq 5$, $p_{6r+5} \geq 1$ for some $r \geq 1$.
12. $p_4 \leq 1$, $p_5 = 0$, $p_3 \geq 5$, $p_{6r+2} \geq 2$ for some $r \geq 1$.
13. $p_4 \leq 1$, $p_5 = 0$, $p_3 \geq 5$, $p_{6r+5} \geq 2$ for some $r \geq 1$.
14. $p_4 \leq 1$, $p_5 = 0$, $p_3 \geq 5$, $p_{6r+2} = p_{6s+2} = 1$ for some r, s, $r > s \geq 1$.
15. $p_4 \leq 1$, $p_5 = 0$, $p_3 \geq 5$, $p_{6r+2} = p_{6s+5} = 1$ for some $r \geq 1$, $s \geq 1$.
16. $p_4 \leq 1$, $p_5 = 0$, $p_3 \geq 5$, $p_{6r+5} = p_{6s+5} = 1$ for some $r > s \geq 1$.

The conditions (1) and (3) guarantee that at least one of these cases will always hold.

Let $\sum_{i\geq 7}(i-6)p_i=6(k-1)+z$ for some integers k, z, $k\geq 1$, $0\leq z\leq 5$. We shall now prove Theorem 5 in the cases 2 and 14.

Case 2. By Lemma 7.3 there exists a trivalent map $M_0=M(q', v', 0, 3, 0, 0)$ with $q'=(q'_i\,|\,q'_i=0$ for all $i\neq 3, 4, 6$; $q'_3=2$, $q'_4=3$, $q'_6=t$ for any integer $t\geq 18)$. Now we shall apply Lemma 3.α for suitable α. Consider a sequence $f=(f_i\,|\,f_i=p_i$ for all $i\geq 7)$. Then $j=6+\sum_{i\geq 7}(i-6)f_i=6+\sum_{i\geq 7}(i-6)p_i=6k+z$. If $z=0$, 2 or 4 we use Lemma 3.1, 3.4 or 3.7, respectively, if $z=1$, 3 or 5 and $p_5=0$ $(p_5=1)$, we shall continue by Lemma 3.2, 3.5 or 3.8 (3.3, 3.6 or 3.9), respectively.

Let e. g. $z=3$ and $p_5=1$. By Lemma 3.6 there is a trivalent map $M_1=M(q, v, 0, 2, 1, k-1)$ with $q=(q_i\,|\,q_i=p_i$ for all $i\geq 7$, $q_3=2$, $q_4=3k+1$, $q_5=1$, $q_6=r_6+t$, $t\geq 18)$. Consider $M=M_1$ and $f_3=p_3$, $f_4=p_4$, $f_5=p_5$ in Lemma 6. Since the terms of the face-vector q or M_1 and the terms of the sequence p satisfy condition (1), we have fulfilled condition (i) of Lemma 6. Conditions (ii) and (iii) are evidently satisfied. Thus there exists a trivalent map $M_2=M(p^*, v^*, 0, a^*, b^*, c^*)$ with $p^*=(p^*_i\,|\,p^*_i=p_i$ for all $i\neq 6$, $p^*_6=t$ for any t, $t\geq d=r_6+18)$, $a^*\geq 0$, $b^*\geq 0$, $c^*\geq 0$. This map satisfies the assertion of Theorem 5. In the remaining subcases of this case we continue analogically.

Case 14. By Lemma 7.1 and a repeated application of Lemma 1.13 we get a trivalent map $M_1=M(q', v', 0, 0, 1, r+s)$ with $q'=(q'_i\,|\,q'_i=0$ for all $i\neq 4,6$, $6r+2$, $6s+2$; $q'_4=3(r+s)+2$, $q'_6=10(r+s)-5$ or $q'_6=10(r+s)-2$, $q'_{6r+2}=q'_{6s+2}=1)$. An application of Lemma 4.1 implies the existence of a trivalent map $M_2=M(q, v, 0, 0, 1, r+s)$ with $q=(q_i\,|\,q_i=q'_i$ for any $i\neq 6$, $q_6=10(r+s)-2+t$ for any nonnegative integer $t)$. Consider a sequence $f=(f_i\,|\,f_i=p_i$ for all $i\geq 7$, $i\neq 6r+2$, $6s+2$, $f_{6r+2}=p_{6r+2}-1$, $f_{6s+2}=p_{6s+2}-1)$. Then $j=6+\sum_{i\geq 7}(i-6)f_i=6+\sum_{i\geq 7}(i-6)p_i-(6r-4)-(6s-4)=6(k-r-s+1)+z+2$. It follows from (1) for this case that $z=0$, 1, 3 or 4 only. If $z=0$ (i. e. $j\equiv 2 \pmod 6$) we use Lemma 3.13 in the sequel; if $z=1$, 3, or 5 we shall proceed by Lemmas 3.14, 3.17, or 3.10, respectively. To finish the proof of Theorem 5 in this case we use Lemma 6 as above.

To prove Theorem 5 in the remaining cases we proceed similarly as above. In the following there are given Lemmas in the order in which they have to be used in order to prove the theorem.

Case 1. 7.2 – 3.1, 3.4, or 3.7 if $z=0$, 2, or 4, respectively, 3.2, 3.5, or 3.8 (3.3, 3.6 or 3.9) if $z=1$, 3, or 5 and $p_5=0$ $(p_5=1)$, respectively – 6.

Case 3. 7.3 – 1.2 – 3.4 or 3.7 if $z=3$, or 5, respectively, 3.2, or 3.8 (3.3 or 3.9) if $z=2$ or 0 and $p_5=0$ $(p_5=1)$, respectively – 6.

Case 4. 7.3 – 1.7 – 3.4, or 3.7 if $z=0$, or 2, respectively, 3.2, or 3.8 (3.3), or 3.9) if $z=5$ or 3 and $p_5=0$ $(p_5=1)$, respectively – 6.

Cases 5 and 7. 7.3 − 1.2 − 1.2 − 3.4, or 3.7 if $z=4$, or 0, respectively. 3.2, or 3.8 (3.3 or 3.9) if $z=3$ or 1 and $p_5=0$ ($p_5=1$) − 6.

Cases 6 and 9. 7.3 − 1.7 − 1.7 — as in the case 5.

Case 8. 7.3 − 1.2 − 1.7 − 3.4 or 3.7 if $z=1$, or 3, respectively, 3.2, or 3.8 (3.3 or 3.9) if $z=0$, or 4 and $p_5=0$ ($p_5=1$).

Case 10. 7.1 − 5 − 1.13 − 4.1 − 3.16, 3.17, 3.11, or 3.13 if $z=0$, 1, 3, or 4, respectively.

Case 11. 7.1 − 5 − 1.17 − 4.1 − 3.11, 3.13, 3.16, or 1.17 if $z=0$, 1, 3, or 4, respectively.

Case 12. 7.1 − 1.13 − 1.13 − 4.1 − 3.13, 3.14, 3.17, or 3.10 if $z=0$, 1, 3, or 5, respectively.

Cases 13 and 16. 7.1 − 1.17 − 1.17 − 4.1 − 3.10, 3.13, 3.16, or 3.17 if $z=4$, 0, 2, or 3, respectively.

Case 15. 7.1 − 1.13 − 1.17 − 4.1 − 3.17, 3.11, 3.13, or 3.14 if $z=0$, 2, 3, or 4, respectively.

Note. The values z, which had been considered in the Table, cannot occur in the corresponding cases. This follows from (1).

This completes the proof of Theorem 5.

4. Remarks

1. The main result of this paper — Theorem 5 — is mentioned (without a proof) in the book of Jucovič [9, p. 92].

2. A minor modification of the construction presented in this paper and a detailed analysis of the number of hexagons formed (we omitted this because of the great number of possibilities) shows that

$$d \leqslant \Sigma\, ip_i \text{ for } 3 \leqslant i \neq 6.$$

Fisher's results (cf. [2] and also Theorems 2 and 3) give a substance to the conjecture that

$$d \leqslant 3\sigma.$$

For the sake of completeness Grünbaum's result [5] must be mentioned: If $p_3=p_4=0$, then $d \leqslant 8$.

3. To prove the existence part of Theorem 5 in the cases (i) and (ii) it is sufficient to start with a map in Fig. 12 and then proceed by Lemma 4.1, Lemma 3.α for suitable α from among $\alpha = 19$—27, and Lemma 6.

We omit the proof for $p_5 \geqslant 2$ because the paper in the present form is already rather extensive.

4. In this connection it is to be said that in the Theorem of Fisher [2, Theorem 3.4] there is a mistake. I am indebted to professor Grünbaum for his pointing out this mistake. By [3] the revised version of this result of Fisher is as in Theorem 2 of the present paper.

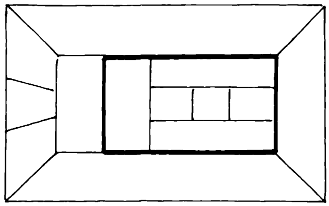


Fig. 12

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Received January 27, 1981

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О ГРАНЕВЫХ ВЕКТОРАХ ВЫПУКЛЫХ МНОГОГРАННИКОВ С РЕГУЛЯРНЫМ ГРАФОМ ТРЕТЬЕЙ СТЕПЕНИ

Stanislav Jendrol

Резюме

Граневим вектором выпуклого многогранника с регулярным графом третьей степени называется последовательность $(p_i(S))$, где $p_i(S)$ — число граней ограниченных i ребрами.

Каждой последовательности неотрицательных целых чисел p $(p \mid 3 \leq i \neq 6)$ удовлетворяющей следствию теоремы Эйлера (1), ставится в соответствие множество $P(p)$, где для любого $p_6 \in P(p)$ последовательность p дополненая p_6 является граневым вектором некоторого выпуклого многогранника с регулярным графом третьей степени. Задача характеризации всех возможных граневых векторов сводится к описанию множеств $P(p)$. В работе характеризуются множества $P(p)$ для всех последовательностей p за исключением конечного количества чисел p_6.