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Marián Slodička

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PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH MEMORY

MARIAN SLODIČKA

Introduction

In this paper we consider the first initial-boundary value problem for the system of quasilinear parabolic equations

$$\frac{\partial u^{(r)}}{\partial t} + \sum_{|i|,|j|\leq k} (-1)^{|i|} \mathrm{D}^i (a_{ij}^{(r)}(x)\, \mathrm{D}_u^{j(r)}) + \tag{1}$$

$$+ f^{(r)}\left(t, x, \mathrm{D}u, \int_0^t K_\alpha(t, \tau)\, \mathrm{D}u(\tau)\, \mathrm{d}\tau\right) = 0$$

for $r = 1, \ldots, N$

in the domain $Q = \Omega \times (0, T)$, where $t \in \langle 0, T\rangle$ $(T < \infty)$, is a bounded domain $x \in \Omega \subset E^M$ (M — dimensional Euclidean space) with a Lipschitzian boundary $\partial\Omega$.

$i = (i_1, \ldots, i_M)$ is a multi-index

$$\mathrm{D}^i = \frac{\partial^{|i|}}{x_1^{i_1} \ldots x_M^{i_M}}, \qquad |i| = \sum_{p-1}^{M} i_p$$

and $\mathrm{D}u$ is the vector function, $\mathrm{D}u \equiv (\mathrm{D}^i u^{(1)}, \ldots, \mathrm{D}^i u^{(N)};\ i \leq k)$.

Let $\int_0^t K_\alpha(t, \tau)\, \mathrm{D}u(\tau)\, \mathrm{d}\tau$ denote the vector function

$$\left(\int_0^t K_{\alpha_{i,1}}(t, \tau)\, \mathrm{D}^i u^{(1)}(\tau)\, \mathrm{d}\tau, \ldots, \int_0^t K_{\alpha_{i,N}}(t, \tau)\, \mathrm{D}^i u^{(N)}(\tau)\, \mathrm{d}\tau;\ i \leq k\right),$$

where $\alpha = (\alpha_{i,j};\ |i| \leq k,\ j = 1, \ldots, N)$, $\alpha_{i,j} \in E^d$, where $d = \mathrm{card}\,\{i, |i| \leq k\}$.

The function $K_\alpha(t, \tau)$ is absolutely integrable in $\langle 0, T\rangle$.

The integral in the system (1) is in the sense of Bochner integral (see [9]). All the functions $f^{(r)}(t, x, \xi)$, $\xi \in E^{2Nd}$, $r = 1, \ldots, N$ are Lipschitz continuous in t and ξ.

Initial-boundary conditions are of the form

$$u^{(r)}(x, 0) = u_0^{(r)}(x) \quad \mathrm{D}_\nu^l u^{(r)} / \partial\Omega \times (0, T) = 0 \tag{2}$$

for $l = 0, 1, \ldots, k-1$ and $r = 1, \ldots, N$

where D_ν^l is the outward normal derivate of order l with respect to $\partial\Omega$, $u_0^{(r)}(x) \in \mathring{W}_2^k(\Omega)$ (Sobolev space).

To solve our problem we use Rothe's method and some techniques recently developed in [1]—[7]. In the paper [10] a solution of this problem u ing othe method as in the present paper is given. These references con ern our problems. The present papers contribution is a proof of regularity, un quene s and approximation of a solution

An approximate solution $u^n(x, t) = (u^1\ (x, t), \ldots, u^N\ (x\ t))$ of he pr blem (1), (2) is constructed in terms of functions $u_s^{(r)}(x)$, $s = 1, \ldots, n$ and $r = 1, \ldots, N$ which are obtained in the following way:

Let $\{t_s\}_{s=1}^n$ be the uniform partition of $\langle 0, T\rangle$, $h = \frac{T}{n}$, $t_s = h$ Successively fo $s = 1, \ldots, n$ we solve the linear Dirichlet boundary value problems

$$\frac{u_s^{(r)} - u_{s-1}^{(r)}}{h} + \sum_{|i|, |j| \le k} (-1) \ D\,(a \ D\,u\ )\Big) + \tag{1 }$$

$$+ f^{(r)}\Big(t, x, Du_{s-1}, \sum_{i=0}^{s-1} \tilde{K}_\alpha(t, ih)\, Du\, h\Big) = 0 \quad \text{for} \quad r = 1, \ldots, N$$

and $\tilde{K}_\alpha(t, y) = \dfrac{1}{y - (i-1)h} \displaystyle\int_{(i-1)h}^{y} K_\alpha(t, \tau)\, d\tau$ in $\langle (i-1)h, ih\rangle$

$$u_0^{(r)} - u_0^{(r)}(x)\ D^l u_s^{(r)}(x)/\partial\Omega = 0 \tag{2 }$$

for $r = 1, \ldots, N$ and $l = 0, 1, \ldots, k-1$

where $u_0^{(r)} = u_0^{(r)}(x)$ is taken from (2). Then we construct Rothe's functions

$$u^{r,n}(x, t) = u_{s-1}^{(r)}(x) + (t - t_{s-1})h^{-1}(u_s^{(r)}(x) - u_{s-1}^{(r)}(x))$$

for $t_{s-1} \le t \le t_s$, $s = 1, \ldots, n$, $r = 1, \ldots, N$.

We prove that $u^{r,n}(x, t)$ converges for $n \to \infty$ to the weak solution $u^{(r)}(x, t)$ where $r = 1, \ldots, N$ (see Definition 3) in the norm of the space $W_2^{k+1}(\Omega') \cap W_2^k(\Omega)$ for all $t \in (0, T)$, where Ω' is an arbitrary subdomain of Ω with $\overline{\Omega'} \subset \Omega$. If (3) is satisfied for $l = k$, then our weak solution $u(x, t) = (u^{(1)}(x, t), \ldots u^{(N)}(\ , t))$ satisfies (1) for a.e. $(x, t) \in Q$ in the classical sense.

Notation and definitions

By C, K with index we always mean positive constants By $C^{0,1}(\Omega)$ we denote the space of Lipschitz continuous functions in Ω $C^{p,1}(\Omega)$ the subset of all $v \in C^{0,1}(\bar{\Omega})$ such that $D^i v \in C^{0,1}(\bar{\Omega})$ for all i with $|i| \le p$.

We shall assume.

$$a_{ij}^{(r)}(x) \in C^{p_{i,l}, 1}(\bar{\Omega})\, |i|,\, |j| \leq k \tag{3}$$

where $p_{i,l} = \max\{0, |i| + l - k - 1\}$ and l is an integer satisfying $1 \leq l \leq k$.
Ellipticity is assumed in the form

$$\sum_{|i|, |j| = k} a_{ij}^{(r)}(x) \xi_i \xi_j \geq C_1 \sum_{|i| = k} \xi_i^2 \tag{4}$$

where $C_1 > 0$ and $r = 1, \ldots, N$.

$f^{(r)}(t, x, \xi)$ is continuous in all variables and satisfies

$$|f^{(r)}(t, x, \xi) - f^{(r)}(t', x, \xi')| \leq C(|t - t'| + |t - t'|\ |\xi| + |\xi - \xi'|) \tag{5}$$

where $C > 0$ and $r = 1, \ldots, N$.

Let us consider the Sobolev space

$$W_2^k(\Omega) \equiv W = \{u \in L_2(\Omega);\, \mathrm{D}^i u \in L_2(\Omega) \text{ for all } |i| \leq k\}$$

with the norm $\|u\|_W = \sum_{|i| \leq k} \|\mathrm{D}^i u\|$, where $\|\cdot\|$ is the norm in $L_2(\Omega)$. The scalar product in $L_2(\Omega)$ is denoted by $(.,.)$.

Let $C_0^\infty(\Omega)$ be the set of all infinitely differentiable functions with support in Ω. We denote $\mathring{W}_2^k(\Omega) = \overline{C_0^\infty(\Omega)}$, where the closure is taken in the norm of the space $W_2^k(\Omega)$.

By means of the bilinear form

$$[Au, v]_{(r)} = \int_\Omega \sum_{|i|, |j| \leq k} a_{ij}^{(r)}(x)\, \mathrm{D}^j u\, \mathrm{D}^i v\, \mathrm{d}x$$

for $u, v \in \mathring{W}_2^k(\Omega)$ and $r = 1, \ldots, N$

we define a linear continuous operator A_r from $W_2^k(\Omega)$ into $W_2^{-k}(\Omega)$ ($W_2^{-k}(\Omega)$ is the dual space to $\mathring{W}_2^k(\Omega)$).

Let X be a Banach space with a norm $\|.\|_X$ and $v(t): \langle 0, T\rangle \to X$ be an abstract function.

Definition 1. *We denote by* $L_p(\langle 0, T\rangle, X)$ $(1 \leq p \leq \infty)$ *the set of all measurable abstract functions* $v(t): \langle 0, T\rangle \to X$ *such that*

$$\|v\|_{L_p(\langle 0, T\rangle, X)} = \int_0^T \|v(t)\|_X\, \mathrm{d}t < \infty \quad 1 \leq p < \infty$$

$$\|v\|_{L_\infty(\langle 0, T\rangle, X)} = \sup_{t \in \langle 0, T\rangle} \mathrm{ess}\, \|v(t)\|_X < \infty \quad p = \infty$$

where

$$\sup_{t\in\langle 0,T\rangle} \text{ess} = \inf_{\substack{A\subset\langle 0,T\rangle\\ \mu(A)=0}} \{ \sup_{\langle 0,T\rangle - A} \|v(t)\|_X \}.$$

Let $C(\langle 0, T\rangle, X)$ be the set of all continuous functions $v(t): \langle 0, T\rangle \to X$ with $\|v(t)\|_{C(\langle 0,T\rangle, X)} = \max_{t\in\langle 0,T\rangle} \|v(t)\|_X < \infty$

The set of all abstract functions $v(t): \langle 0, T\rangle \to X$ such that $x^*(v(t)) \in C(\langle 0, T\rangle)$ for all $x^* \in X^*$ (X^* is the dual space to X) is denoted by $C_w(\langle 0, T\rangle, X)$.

Definition 2. *$C_w^1((0, T), L_2(\Omega))$ is the set of all $v \in C(\langle 0, T\rangle, L_2(\Omega))$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(v(t), w) \in C((0, T)) \cap L_\cdot(\langle 0, T\rangle)$$

for all $w \in L_2(\Omega)$ and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(v(t), w)\right| \leq C\|w\|.$$

In this case there exists a uniquely determined

$$g(t) \in L_\infty(\langle 0, T\rangle, L_2(\Omega)) \cap C_w((0, T), L_2(\Omega))$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(v(t), w) = (g(t), w)$$

for all

$$w \in L_2(\Omega)$$

and we denote by $\dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = g(t)$. *the weak derivate of* $v(t)$.

$$\text{By } (F^{(r)}u)(t) = f^{(r)}(t, x, \mathrm{d}u, \int_0^t K_\alpha(t, \tau)\, Du(\tau)\, \mathrm{d}\tau),$$

$r = 1, \ldots, N$, we denote the operator from $\langle 0, T\rangle \times W_2^k(\Omega) \to L_2(\Omega)$.

We denote by $\mathring{W} = [\mathring{W}_2^k(\Omega)]^N$ and $L_2 = [L_2(\Omega)]^N$ the Cartesian products of $\mathring{W}_2^k(\Omega)$, $L_2(\Omega)$, respectively.

Definition 3. *$u(t) \in L_\infty(\langle 0, T\rangle, \mathring{W})$ is a weak solution of the problem* (1), (2) *if*

$$u(t) \in C_w^1((0, T, L_2) \quad u(0) = (u_0^{(1)}, \ldots, u_0^{(N)})$$

and

$$\left(\frac{\mathrm{d}u^{(r)}(t)}{\mathrm{d}t}, v\right) + [A_r u^{(r)}(t), v] + ((F^{(r)}u)(t), v) = 0$$

holds for all $v \in \mathring{W}_2^k(\Omega)$, $r = 1, \ldots, N$ and $t \in (0, T)$.

We shall assume

$$A_r u_0^{(r)} \in L_2(\Omega) \qquad r = 1, \dots, N. \tag{6}$$

A priori estimates

$u_s^{(1)}, \dots, u_s^{(N)} \in \mathring{W}_2^k(\Omega)$ are a solution of (1′), (2′) if

$$\left(\frac{u_s^{(r)} - u_{s-1}^{(r)}}{h}, v\right) + [A_r u_s^{(r)}, v] + ((F^{(r)} u_{s-1})(t), v) = 0 \tag{7}$$

holds for all $v \in \mathring{W}_2^k(\Omega)$, $r = 1, \dots, N$, where

$$(F^{(r)} u^{(n)})(t) = f^{(r)}\left(t_s, x, \mathrm{D}u_{s-1}, \sum_{i=0}^{s-1} K_\alpha(t, ih)\, \mathrm{D}u_i h\right)$$

$t_{s-1} \leqslant t < t_s$, $u^{(n)} = (u^{1,n}, \dots, u^{N,n})$, $u^{j,n}$ is a Rothe's function. By Ω' we denote an arbitrary subdomain of Ω with $\overline{\Omega'} \subset \Omega$.

Lemma 1. *If* (3)—(5) *are satisfied, then there exists a unique solution* $u_s^{(1)}, \dots, u_s^{(N)} \in \mathring{W}_2^k(\Omega) \cap W_2^{k+1}(\Omega')$ *of* (1′), (2′)

$$(s = 1, \dots, n).$$

Proof: Let $r \in \langle 1, \dots, N\}$. From (3), (4) and due to lemma of Lions (see [8]) and inequality of Garding we obtain

$$[A_r u^{(r)}, u^{(r)}] \geqslant C_1 \|u^{(r)}\|_w^2 - C_4 \|u^{(r)}\|^2 \tag{8}$$

The operator $A_r u^{(r)} + h^{-1} u^{(r)}$ is elliptic for $h \leqslant h_0 < C_4^{-1}$ (see [8]).

If $u_{s-1}^{(1)}, \dots, u_{s-1}^{(N)} \in W_2^k(\Omega)$, then $(F^{(r)} u_{s-1})(t) \in L_2(\Omega)$ because of (5). From the results on linear elliptic equations (see [8]) we conclude that there exists a unique solution $u_s^{(1)}, \dots, u_s^{(N)} \in \mathring{W}_2^k(\Omega)$.

If $u_s^{(1)}, \dots, u_s^{(N)} \in \mathring{W}_2^k(\Omega)$ are a solution of the equation

$$A_r u^{(r)} = \frac{u_s^{(r)} - u_{s-1}^{(r)}}{h} + (F^r u_{s-1})(t) \equiv f_{hs}^{(r)}$$

for $r = 1, \dots, N$

we have

$$\sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ \|\varphi\|_{W_2^{k-1}} \leqslant 1}} |(\mathrm{D}^\alpha f_{hs}^{(r)}, \varphi)| =$$

$$= \sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ \|\varphi\|_{W_2^{k-1}} \leqslant 1}} |(f_{hs}^{(r)}, \mathrm{D}^\alpha \varphi)| \leqslant \|f_{hs}^{(r)}\|$$

for $|\alpha| \leqslant l - 1$

Thus, from [8] we deduce $u_s^{(r)} \in W_2^{k+1}(\Omega')$ and

$$\|u_s^{(r)}\| W_2^{k+1}(\Omega') \leq C(\Omega')(\|u_s^{(r)}\|_W + \|f_{hs}^{(r)}\|) \tag{9}$$

holds for all $h \leq h_1$ and $s = 1, \ldots, n$, $r = 1, \ldots, N$.

In the sequel we shall assume that (3)—(6) are satisfied.

Lemma 2. *There exist C and $h_0 > 0$ such that the estimates*

$$\|u_s^{(r)}\| \leq C, \quad \sum_{s=1}^{n} h \|u_s^{(r)}\|_W^2 \leq C$$

take place for all $h < h_0$, $s = 1, \ldots, n$, $r = 1, \ldots, N$.

Proof: Let us put $v = u_s^{(r)}$ into (7) and then sum up for $s = 1, \ldots, p$ where $1 < p \leq n$. We have

$$\sum_{s=1}^{p} (u_s^{(r)} - u_{s-1}^{(r)}, u_s^{(r)}) + h \sum_{s=1}^{p} [A_r u_s^{(r)}, u_s^{(r)}] + \tag{10}$$

$$+ \sum_{s=1}^{p} h((F^{(r)} u_{s-1})(t_s), u_s^{(r)}) = 0$$

From the identity

$$2 \sum_{s=1}^{p} (u_s^{(r)} - u_{s-1}^{(r)}, u_s^{(r)}) = \sum_{s=1}^{p} \|u_s^{(r)} - u_{s-1}^{(r)}\|^2 \|u_p^{(r)}\|^2 - \|u_0^{(r)}\|^2$$

and from (8) we deduce

$$\|u_p^{(r)}\|^2 + C_1 \sum_{s=1}^{p} h \|u_s^{(r)}\|_W^2 \leq \|u_0^{(r)}\|^2 + C_2 h \sum_{s=1}^{p} \|u_s^{(r)}\|^2 - \tag{11}$$

$$- 2h \sum_{s=1}^{p} ((F^{(r)} u_{s-1})(t_s), u_s^{(r)})$$

Applying Young's inequality

$$|ab| \leq \frac{a^2 \varepsilon^2}{2} + \frac{b^2}{2\varepsilon^2} \tag{12}$$

we estimate

$$\sum_{s=1}^{p} h |(F^{(r)} u_{s-1})(t_s), u_s^{(r)})| \leq \sum_{s=1}^{p} h \|(F^{(r)} u_{s-1})(t_s)\| \cdot \|u_s^{(r)}\| \leq \tag{13}$$

$$< \sum_{s=1}^{p} \frac{\varepsilon h}{2} \|(F^{(r)} u_{s-1})(t_s)\|^2 + \frac{h}{2\varepsilon} \sum_{s=1}^{p} \|u_s^{(r)}\|^2$$

Owing to (5) we have

$$|(F^{(r)} u_{s-1})(t_s)| \leq K_1 + K_2 \sum_{r=1}^{N} |Du_{s-1}^{(r)}| + K_3 \sum_{r=1}^{N} \sum_{i=0}^{s-1} h |Du_i^{(r)}|$$

Therefore we obtain

$$\|(F^{(r)}u_{s-1})(t_s)\|^2 \leqslant K_4 + K_5 \sum_{r=1}^{N} \|u_{s-1}^{(r)}\|_W^2 + \tag{14}$$

$$+ K_6 h \sum_{r=1}^{N} \sum_{i=0}^{s-1} \|u_i^{(r)}\|_W^2$$

and hence, due to (11), (13), (14) we conclude

$$\|u_p^{(r)}\|^2 + C_1 \sum_{s=1}^{p} h \|u_s^{(r)}\|_W^2 \leqslant \|u_0^{(r)}\|^2 + C_2 \sum_{ps=1} h \|u_s^{(r)}\|^2 +$$

$$+ \sum_{s=1}^{p} \frac{h}{2\varepsilon} \|u_s^{(r)}\|^2 + \sum_{s=1}^{p} \frac{\varepsilon h}{2} K_4 +$$

$$+ \sum_{s=1}^{p} \frac{\varepsilon h}{2} \sum_{j=1}^{N} [K_5 \|u_{s-1}^{(j)}\|_W^2 + K_6 h \sum_{i=0}^{s-1} \|u_i^{(j)}\|_W^2$$

Applying the identity

$$\sum_{s=1}^{p} \sum_{i=0}^{s-1} \|u_i^{(j)}\|_w^2 = \sum_{s=0}^{p} (p-s) \|u_s^{(j)}\|_w^2$$

we have

$$\|u_p^{(r)}\|^2 + C_1 h \sum_{s=1}^{p} \|u_s^{(r)}\|_W^2 \leqslant \|u_0^{(r)}\|^2 + \left(C_2 + \frac{1}{2\varepsilon}\right) h \sum_{s=1}^{p} \|u_s^{(r)}\|^2 +$$

$$+ \frac{\varepsilon T K_4}{2} + \varepsilon C(u_0^{(1)}, \ldots, u_0^{(N)}) + \frac{\varepsilon h K_5}{2} \sum_{j=1}^{N} \sum_{s=1}^{p} \|u_s^{(j)}\|_W^2 +$$

$$+ \frac{\varepsilon K_6 h^2}{2} \sum_{j=1}^{N} \sum_{s=0}^{p} (p-s) \|u_s^{(j)}\|_W^2$$

and hence summing up for $r = 1, \ldots, N$ we estimate

$$\sum_{r-1}^{N} \|u_p^{(r)}\|^2 + \left(C_1 - \frac{\varepsilon N (K_5 + K_6 T)}{2}\right) \sum_{s=1}^{p} \sum_{r=1}^{N} h \|u_s^{(r)}\|_W^2 \leqslant$$

$$\leqslant C(\varepsilon, u_0^{(1)}, \ldots, u_0^{(N)}) + \sum_{s=1}^{p} \left(C_2 + \frac{1}{2\varepsilon}\right) \sum_{r=1}^{N} h \|u_s^{(r)}\|^2$$

Let us choose $\varepsilon > 0$ so that

$$C_1 - \frac{\varepsilon K_5 N}{2} - \frac{\varepsilon K_6 T N}{2} = \frac{C_1}{2}.$$

The we obtain

$$\sum_{r=1}^{N}\|u_p^{(r)}\|^2+\frac{C_1}{2}\sum_{s=1}^{p}\sum_{r=1}^{N}h\|u_s^{(r)}\|_W^2\leq \tag{15}$$

$$\leq C(\varepsilon, u_0^{(1)}, \ldots, u_0^{(N)})+C_6\sum_{s=1}^{p}\sum_{r=1}^{N}h\|u_s^{(r)}\|^2$$

and in particular

$$\sum_{r=1}^{N}\|u_p^{(r)}\|^2\leq C_5+C_6\sum_{s=1}^{p}h\sum_{r=1}^{N}\|u_s^{(r)}\|^2.$$

We denote

$$a=\sum_{r=1}^{N}\|u_p^{(r)}\|^2$$

$$a_p\leq C_5+C_6\sum_{s=1}^{p}ha_s$$

Now $h\leq h_0<C_6^{-1}$ implies succesively

$$a_p\leq C_5(1-C_6h)^{-1}[1+C_6h(1-C_6h)^{-1}]^{p-1} \tag{16}$$

There exists C such that $a_p\leq C$ for all p and $h\leq h_0<C_6^{-1}$. The rest of the proo follows from (15).

Lemma 3. *There exist $C>0$ and $h_0>0$ such that the estimates*

$$\left\|\frac{u_s^{(r)}-u_{s-1}^{(r)}}{h}\right\|^2\leq C \quad h^{-1}\|u_s^{(r)}-u_{s-1}^{(r)}\|_W^2<C$$

take place for all $h\leq h$, $s=1, \ldots, n$, $r=1, \ldots, N$.

Proof: Let us consider (7) for $s=i$, $s=i-1$, putting $v=u^{(r)}-u^{(r)}_{-1}$. Subtracting these equalities we obtain

$$\left(\frac{u_i^{(r)}-u_{i-1}^{(r)}}{h}, u_i^{(r)}-u_{i-1}^{(r)}\right)+[A_r(u_i^{(r)}-u_{i-1}^{(r)}), u_i^{(r)}-u_{i-1}^{(r)}]+$$
$$+((F^{(r)}u_{i-1})(t_i)-F^{(r)}u_{i-2})(t_{i-1}), u_i^{(r)}-u_{i-1}^{(r)})$$
$$=\left(\frac{u_{i-1}^{(r)}-u_{i-2}^{(r)}}{h}, u_i^{(r)}-u_{i-1}^{(r)}\right)$$

from where, due to (8), we deduce

$$\left\|\frac{u_i^{(r)}-u_{i-1}^{(r)}}{h}\right\|^2+C_1h^{-1}\|u_i^{(r)}-u_{i-1}^{(r)}\|_W^2\leqq$$

$$\leqslant \left\| \frac{u_{i-1}^{(r)} - u_{i-2}^{(r)}}{h} \right\| \left\| \frac{u_i^{(r)} - u_{i-1}^{(r)}}{h} \right\| + C_4 h \left\| \frac{u_i^{(r)} - u_{i-1}^{(r)}}{h} \right\|^2 +$$

$$+ \left\| \frac{u_i^{(r)} - u_{i-1}^{(r)}}{h} \right\| \| (F^{(r)} u_{i-1})(t_i) - (F^{(r)} u_{i-2})(t_{i-1}) \|$$

By virtue of (5) we estimate

$$\| (F^{(r)} u_{i-1})(t_i) - (F^{(r)} u_{i-2})(t_{i-1}) \|^2 \leqslant \tag{18}$$
$$\leqslant C \left[h^2 + h^2 \sum_{r=1}^{N} \| u_{i-1}^{(r)} \|_W^2 + \sum_{r=1}^{N} \| u_{i-1}^{(r)} - u_{i-2}^{(r)} \|_W^2 \right.$$

Applying (12) we estimate

$$\left\| \frac{u_i^{(r)} - u_{i-1}^{(r)}}{h} \right\|^2 \left[\frac{1}{2} - C_4 h - \frac{\varepsilon^2}{2} \right] + C_1 h^{-1} \| u_i^{(r)} - u_{i-1}^{(r)} \|_W^2 \leqslant$$
$$\leqslant \frac{Ch^2}{2\varepsilon^2} + \frac{1}{2} \left\| \frac{u_{i-1}^{(r)} - u_{i-2}^{(r)}}{h} \right\|^2 + \frac{Ch^2}{2\varepsilon^2} \sum_{r=1}^{N} \| u_{i-1}^{(r)} \|_W^2 +$$
$$+ \frac{C}{2\varepsilon^2} \sum_{r=1}^{N} \| u_{i-1}^{(r)} - u_{i-2}^{(r)} \|_W^2$$

let $\varepsilon = \sqrt{\dfrac{ChN}{2C_1}}$

$$\left\| \frac{u_i^{(r)} - u_{i-1}^{(r)}}{h} \right\|^2 \left[\frac{1}{2} - C_4 h - \frac{ChN}{4C_1} \right] + C_1 h^{-1} \| u_i^{(r)} - u_{i-1}^{(r)} \|_W^2 \leqslant$$
$$\leqslant \frac{1}{2} \left\| \frac{u_{i-1}^{(r)} - u_{i-2}^{(r)}}{2} \right\|^2 + \frac{C_1 h}{N} + \frac{C_1 h}{N} \sum_{r=1}^{N} \| u_{i-1}^{(r)} \|_W^2 +$$
$$+ \frac{C_1 h^{-1}}{N} \sum_{r=1}^{N} \| u_{i-1}^{(r)} - u_{i-2}^{(r)} \|_W^2$$

We denote $C_5 = \frac{1}{2}\left(C_4 + \frac{CN}{4C_1} \right) \quad 2C_1 = K$

Summing up for $r = 1, \ldots, N$ we obtain

$$(1 - C_5 h) \sum_{r=1}^{N} \left(\left\| \frac{u_i^{(r)} - u_{i-1}^{(r)}}{h} \right\|^2 + K h^{-1} \| u_i^{(r)} - u_{i-1}^{(r)} \|_W^2 \right) \leqslant \tag{19}$$
$$\leqslant Kh + Kh \sum_{r=1}^{N} \| u_{i-1}^{(r)} \|_W^2 + \sum_{r=1}^{N} \left(\left\| \frac{u_{i-1}^{(r)} - u_{i-2}^{(r)}}{h} \right\|^2 + \right.$$
$$\left. + K h^{-1} \| u_{i-1}^{(r)} - u_{i-2}^{(r)} \|_W^2 \right)$$

We denote

$$a_i = \sum_{r=1}^{N}\left(\left\|\frac{u_i^{(r)} - u_{i-1}^{(r)}}{h}\right\|^2 + Kh^{-1}\|u_i^{(r)} - u_{i-1}^{(r)}\|_W^2\right.$$

$$q = 1 - C_5 h \quad b_i = \sum_{r=1}^{N}\|u_i^{(r)}\|_W^2$$

From (19) we have

$$qa_i \leqslant a_{i-1} + Kh + Khb_{i-1}$$

From this recurrent relation we obtain successively

$$q^{i-1}a_i \leqslant Kh(1 + q + \ldots + q^{i-2}) + Kh(b_1 + qb_2 + \ldots + q^{i-2}b_{i-1}) + a_1$$

Since $1 \geqslant q^j = (1 - C_5 h)^j \geqslant e^{-C_5 T}$ for all j and $h \leqslant h_0 < C_5^{-1}$ we have

$$e^{-C_5 T}a_i \leqslant a_1 + KT + Kh\sum_{j=1}^{i-1} b_j \tag{20}$$

Applying Lemma 2 we obtain

$$h\sum_{j=1}^{i-1} b_j = h\sum_{j=1}^{i-1}\sum_{r=1}^{N}\|u_j^{(r)}\|_W^2 \leqslant CN$$

Now we must show that a_1 is bounded.

Putting $v = u_1^{(r)} - u_0^{(r)}$ we have from (7)

$$\left(\frac{u_1^{(r)} - u_0^{(r)}}{h}, u_1^{(r)} - u_0^{(r)}\right) + [A_r u_1^{(r)}, u_1^{(r)} - u_0^{(r)}] + ((F^{(r)}u_0)(t), u_1^{(r)} - u_0^{(r)}) = 0$$

Owing to (6) we estimate

$$\left|\left[A_r u_0^{(r)}, \frac{u_1^{(r)} - u_0^{(r)}}{h}\right]\right| \leqslant \|A_r u_0^{(r)}\| \left\|\frac{u_1^{(r)} - u_0^{(r)}}{h}\right\|$$

Then

$$\left\|\frac{u_1^{(r)} - u_0^{(r)}}{h}\right\|^2 + [A_r(u_1^{(r)} - u_0^{(r)}), u_1^{(r)} - u_0^{(r)}] \leqslant$$

$$\leqslant C(u_0^{(1)}, \ldots, u_0^{(N)})\left\|\frac{u_1^{(r)} - u_0^{(r)}}{h}\right\| + \|A_r u_0^{(r)}\| \left\|\frac{u_1^{(r)} - u_0^{(r)}}{h}\right\|$$

Applying (8), (12) we have

$$\left\|\frac{u_1^{(r)} - u_0^{(r)}}{h}\right\|^2\left(1 - C_4 h - \frac{\varepsilon^2}{2}\right) + h^{-1}C_1\|u_1^{(r)} - u_0^{(r)}\|_W^2 \leqslant$$

$$\leqslant C(u_0^{(1)}, \ldots, u_0^{(N)}, \varepsilon) \tag{21}$$

Since $h \leqslant h_0 < C_5^{-1}$ we can choose $\varepsilon > 0$ so that $1 - C_4 h - \frac{\varepsilon^2}{2} > \alpha > 0$. From (21) it follows that a_1 is bounded. The proof of Lemma 3 follows from (20).

Lemma 4. *There exists C and $h_0 > 0$ such that $\|u_i^{(r)}\|_w \leqslant C$ for all $h \leqslant h_0$ $i = 1, \ldots, n$, $r = 1, \ldots, N$.*

Proof: From (8) and (7) with $v = u_s^{(r)}$ we obtain

$$C_1 \|u_s^{(r)}\|_W^2 \leqslant \left\| \frac{u_s^{(r)} - u_{s-1}^{(r)}}{h} \right\| \|u_s^{(r)}\| + \|(F^{(r)} u_{s-1})(t_s)\| \, \|u_s^{(r)}\| + C_2 \|u_s^{(r)}\|^2$$

Owing to (5) we estimate

$$\|(F^{(r)} u_{s-1})(t_s)\| \leqslant K_1 + K_2 \sum_{r=1}^{N} \|u_{s-1}^{(r)}\|_w + K_3 \sum_{r=1}^{N} \sum_{i=0}^{s-1} h \|u_i^{(r)}\|_w$$

Since

$$h \|u_i^{(r)}\|_w \leqslant h \|u_i^{(r)}\|_W^2 + \frac{h}{4}$$

we have

$$\sum_{i=0}^{s-1} h \|u_i^{(r)}\|_w \leqslant \sum_{i=0}^{s-1} h \|u_i^{(r)}\|_W^2 + \frac{sh}{4}$$

Due to Lemma 2 we obtain

$$\sum_{i=0}^{s-1} h \|u_i^{(r)}\|_w \leqslant K_4$$

It holds

$$\|(F^{(r)} u_{s-1})(t_s)\| \leqslant K\left(1 + \sum_{r=1}^{N} \|u_{s-1}^{(r)}\|_w\right) \tag{22}$$

Due to Lemma 3 we have

$$\|u_{s-1}^{(r)}\|_w \leqslant \|u_s^{(r)}\|_w + \|u_s^{(r)} - u_{s-1}^{(r)}\|_w \leqslant \|u_s^{(r)}\|_w + K_5$$

and thus the estimate

$$C_1 \|u_s^{(r)}\|_W^2 \leqslant K_6 + K_7 \sum_{r=1}^{N} \|u_s^{(r)}\|_w$$

Summing up for $r = 1, \ldots, N$ yields

$$\sum_{r=1}^{N} C_1 \|u_s^{(r)}\|_W^2 \leqslant C + C \sum_{r=1}^{N} \|u_s^{(r)}\|_w$$

$$\sum_{r=1}^{N}\left(\sqrt{C_1}\,\|u_s^{(r)}\|_w-\frac{C}{2}\right)^2\leqslant C+\frac{NC^2}{4}\quad\text{for all } h<h_1 \text{ and } s=1,\ \dots,n.$$

The proof of Lemma 4 follows from the last inequality.

Lemma 5. *There exist $C(\Omega')$ and $h_0>0$ such that*

$$\|u_s^{(r)}\|_{W_2^{k+l}(\Omega')}\leqslant C(\Omega')$$

for all $h\leqslant h_0$, $s=1,\ \dots,\ n$, $r=1,\ \dots,\ N$.

Proof: We denote

$$f_h^{(r)}=h^{-1}(u_s^{(r)}-u_{s-1}^{(r)})+(F^{(r)}u_{s-1})(t_s)$$

Due to Lemma 3, Lemma 4, (22) we have $\|f_{hs}^{(r)}\|\leqslant C$. Thus, the result follows from (9) and Lemma 4.

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Proof: *Let $r\in\{1,\dots,N\}$*. Lemma 4 implies the estimate

$$\|u^{r,n}(t)\|_{W_2^k(\Omega)}<C$$

for all n and $t\in\langle 0,\ T\rangle$.

From the compactne s of the imbedding $W_2^k(\Omega)\to L_2(\Omega)$ we conclude that for fixed $t\in\langle 0,\ T\rangle$ it is possible to choose a subsequence of $\{u^{r,n}(t)\}$ convergent in th norm of the space $L_2(\Omega)$. By the diagonal method we choo e a subsequence $\{u_{n_k}^{(r)}(t)\}$ such that $\{u_{n_k}^{(r)}(t)\}$ is convergent in $L_2(\Omega)$ for each rational point $t\in\langle 0,\ T\rangle$.

From Lemma 3 and the triangle inequality we deduce

$$\|u^{r,n}(t)-u^{r,n}(t')\|\leqslant C|t-t'| \tag{23}$$

for all n and $t,\ t'\in\langle 0,\ T\rangle$. For an arbitrary point $t'\in\langle 0,\ T\rangle$ it holds

$$\begin{aligned}&\|u_{n_k}^{(r)}(t')-u_{n_l}^{(r)}(t')\|\leqslant \\ &\leqslant\|u_{n_k}^{(r)}(t')-u_{n_k}^{(r)}(t)\|+\|u_{n_k}^{(r)}(t)-u_{n_l}^{(r)}(t)+\|u_{n_l}^{(r)}(t)-u_{n}^{(r)}(t')\|\end{aligned} \tag{24}$$

where t is a rational point.

Owing to (23) we have that $u_{n_k}^{(r)}\to u^{(r)}$ in $L_2(\Omega)$ for all $t\in\langle 0\ \ T\rangle$. It holds

$$\begin{aligned}\|u^{(r)}(t)-u^{(r)}(t')\|\leqslant\|u^{(r)}(t)-u^{r,n}(t)\|+\|u^{r,n}(t)-u^{r,n}(t')\|+\\+\|u^{r,n}(t')-u^{(r)}(t)\|\end{aligned}$$

where with (23) we can see that $u^{(r)}(t)\in C(\langle 0,\ T\rangle\ \ L_2(\Omega))$

From (24) passing to the limit for $l \to \infty$, we conclude that $u_{n_k}^{(r)}(t) \to u^{(r)}(t)$ locally uniformly, i.e.

$$\forall \varepsilon > 0 \; \exists K > 0 \; \delta_l(\varepsilon) > 0 \; \forall t' \; |t - t'| < \delta_l(\varepsilon): \|u_{n_k}^{(r)}(t) - u^{(r)}(t)\| < \varepsilon$$

Thus, the rest of the proof follows from the Borel covering theorem.

We denote the sequence $\{u_{n_k}^{(r)}(t)\}$ from Lemma 6 by $\{u_n^{(r)}(t)\}$ for $r = 1, \ldots, N$.

Lemma 7. *Let $r \in \{1, \ldots, N\}$, $u^{(r)}(t) \in C(\langle 0, T\rangle, L_2(\Omega))$ be thesame as in Lemma 6. The following assertions hold:*

a) *$u^{(r)}(t)$ is Lipschitz continuous from $\langle 0, T\rangle$ into $L_2(\Omega)$*

b) *$u^{(r)}(t) \in L_\infty(\langle \delta, T\rangle, \mathring{W}_2^k(\Omega) \cap W_2^{k+l}(\Omega'))$ for all $0 < \delta < T$ if $u_0^{(r)} \in W_2^{k+1}(\Omega)$, then $\delta = 0$.*

c) *$u_n^{(r)}(t) \to u^{(r)}(t)$ in the norm of the space $W_2^{k+l-1}(\Omega') \cap W_2^{k-1}(\Omega)$ for all $t \in \langle 0, T\rangle$*

d) *$u^{(r)}(t) \in C(\langle \delta, T\rangle, \mathring{W}_2^{k-1}(\Omega) \cap W_2^{k+l-1}(\Omega'))$ if $u_0^{(r)} \in W_2^{k+l}(\Omega)$, then $\delta = 0$.*

Proof: Assertion a) follows from Lemma 6 and (23).

b) The space $H = \mathring{W}_2^k(\Omega) \cap W_2^{k+l}(\Omega')$ is a separable Hilbert space with respect to the scalar product

$$(.,.)_H = (.,.)_{W_2^k(\Omega)} + (.,.)_{W_2^{k+l}(\Omega')}$$

From Lemma 4 and Lemma 5 we deduce that

$$\max_{t \in \langle \frac{T}{n} \cdot T\rangle} \|u_n^{(r)}(t)\|_H \leq C \quad \text{for all} \quad n \tag{25}$$

Thus, if $0 < \delta < T$ then there exists $w \in L_\infty(\langle \delta, T\rangle, H)$ and a subsequence $\{u_{n_k}^{(r)}\}$ of $\{u_n^{(r)}\}$ such that $u_{n_k}^{(r)} \rightharpoonup w$ (the symbol $\rightharpoonup$ denotes the weak convergence) in $L_\infty(\langle \delta, T\rangle, H)$. From this fact it follows also that $u_{n_k}^{(r)} \rightharpoonup w$ in $L_2(\langle \delta, T\rangle, L_2(\Omega)) \supset L_\infty(\langle \delta, T\rangle, H)$ and hence due to Lemma 6, we have $u^{(r)} = w$. If $u_0^{(r)} \in W_2^{k+l}(\Omega)$ then we can put $\delta = 0$.

Since $\|u^{(r)}(t)\| \leq \underline{\lim} \|u_n^{(r)}(t)\|$ and due to (25) we obtain

$$\sup_{t \in \langle \delta, T\rangle} \|u^{(r)}(t)\|_H \leq C \quad \text{for all} \quad \delta > 0 \tag{26}$$

c) From (25) and the compactness of the imbedding $W_2^k(\Omega) \to W_2^{k-1}(\Omega)$ we obtain that $u_{n_k}^{(r)} \to w$ in $W_2^{k-1}(\Omega)$. Due to Lemma 6 we have $u_n^{(r)} \rightharpoonup u^{(r)}$ in $L_2(\Omega)$, i.e. $u^{(r)} = w$ since $L_2(\Omega) \supset W_2^{k-1}(\Omega)$. Similarly we can show that $u_n^{(r)} \to u^{(r)}$ in $W_2^{k+l-1}(\Omega')$.

d) From (26) and the compactness of the imbedding $W_2^k(\Omega)\to w_2^{k-1}(\Omega)$ and $W_2^{k+l}(\Omega')\to W_2^{k+l-1}(\Omega')$ for $t_n\to t$ ($t_n\in\langle\delta, T\rangle$) we obtain that $u^{(r)}(t_{n_k})\rightharpoonup w_t$ in $W_2^{k-1}(\Omega)\cap W_2^{k+l-1}(\Omega')$. From assertion a) we have $u^{(r)}(t_n)\to u^{(r)}(t)$ in $L_2(\Omega)$. Since $L_2(\Omega)\supset W_2^{k-1}(\Omega)\cap W_2^{k+l-1}(\Omega')$ we have $u^{(r)}(t_n)\to u^{(r)}(t)$ in the norm of the space $W_2^{k-1}(\Omega)\cap W_2^{k+l-1}(\Omega')$. If $u_0^{(r)}\in W_2^{k+l}(\Omega)$, then we can put $\delta=0$.

Existence and uniqueness of the solution

Let us define step functions $w_n^{(r)}(t)\colon\langle 0, T\rangle\to H$ for $r=1,\dots,N$ by

$$w_n^{(r)}(t)=u_s^{(r)}\qquad t_{s-1}<t\leq t_s$$
$$w_n^{(r)}(t)=u_0^{(r)}\qquad t\in\left\langle -\frac{T}{n}, 0\right\rangle$$

where $H=W_2^{k+l}(\Omega')\cap\mathring{W}_2^k(\Omega)$.

The identity (7) can be rewritten into the form

$$\left(\frac{d^-u^{(r)}(t)}{dt}, v\right)+[A_r w_n^{(r)}(t), v]+((F^{(r)}u_n)(t), v)\quad 0 \tag{27}$$

for all $v\in\mathring{W}_2^k(\Omega)$, $t\in(0, T)$, $r=1,\dots,N$

where $\frac{d^-}{dt}$ is the lefthand derivative. By integrating (27) over $\langle 0, t\rangle$ we obta n

$$(u_n^{(r)}(t), v)+\int_0^t [A_r w_n^{(r)}(\tau), v]\,d\tau+\int_0^t ((F^{(r)}u_n)(\tau), v)\,d\tau- \tag{28}$$
$$-(u_0^{(r)}, v)=0$$

for all $v\in\mathring{W}_2^k(\Omega)$, $r=1,\dots,N$.

Lemma 3 implies

$$\|w_n^{(r)}(t)-u_n^{(r)}(t)\|\leq\frac{C}{n}\quad\text{for all}\quad t\in\langle 0, T\rangle. \tag{29}$$

From Lemma 4 and Lemma 5 we have

$$\|u_s^{(r)}\|_H\leq K$$

and due to (25) we obtain

$$\|w_n^{(r)}\|_h\leq C\quad\text{for all}\quad r=1,\dots,N. \tag{30}$$

From (29), (30) we deduce that

$$w_n^{(r)}(t)\to u^{(r)}(t) \tag{31}$$

in the norm of space $W_2^{k+l-1}(\Omega')\cap W_2^{k-1}(\Omega)$ for all $t\in(0, T)$.

Lemma 8. a) *if* $v \in \mathring{W}_2^k(\Omega)$, *then*

$$((F^{(r)}u_n)(t), v) \to ((F^{(r)}u)(t), v) \quad \textit{for } n \to \infty \textit{ and } r = 1, \dots, N$$

b)
$$((F^{(r)}u)(t), v) \in C(\langle 0, T\rangle) \cap L_\infty(\langle 0, T\rangle)$$

and the estimate

$$|((F^{(r)}u)(t), v)| \leq C \|v\|$$

takes place for all $t \in \langle 0, T\rangle$ *and* $v \in L_2(\Omega)$, $r = 1, \dots, N$.

Proof: a) First we prove

$$\sum_{i=0}^{s-1} \tilde{K}_\alpha(t, ih)h \, \mathrm{D}u_i \to \int_0^t K_\alpha(t, \tau) \, \mathrm{D}u(\tau) \, \mathrm{d}\tau \tag{32}$$

in the norm of the space $L_2(\Omega)$, where

$$\tilde{K}_\alpha(t, y) = \frac{1}{y - (i-1)h} \int_{(i-1)h}^{y} K_\alpha(t, \tau) \, \mathrm{d}\tau \quad \text{in} \quad \langle (i-1)h, ih\rangle$$

$$\|w_n^{(r)}(t) - u^{(r)}(t)\|_{W_2^{k+l-1}(\Omega')\cap W_2^k(\Omega)} \geq \|w_n^{(r)}(t) - u^{(r)}(t)\|_{W_2^{k-1}(\Omega)} =$$

$$= \sum_{i=0}^{k-1} \|\mathrm{D}^i(w_n^{(r)}(t) - u^{(r)}(t)\|$$

From (31) we have that

$$\mathrm{D}^i w_n^{(r)}(t) \to \mathrm{D}^i u^{(r)}(t) \quad \text{in} \quad L_2(\Omega') \quad \text{for} \quad |i| \leq k-1$$

$$\left\|\sum_{i=0}^{s-1} \tilde{K}_\alpha(t, ih)h \, \mathrm{D}u_i - \int_0^t K_\alpha(t, \tau) \, \mathrm{D}u(\tau) \, \mathrm{d}\tau\right\| =$$

$$= \sum_{i=0}^{s-1} \int_{(i-1)h}^{ih} K_\alpha(t, \tau)[\mathrm{D}u_i - \mathrm{D}u(\tau)] \, \mathrm{d}\tau\Big\| =$$

$$= \left\|\int_0^t K_\alpha(t, \tau)[\mathrm{D}u_i - \mathrm{D}u(\tau)] \, \mathrm{d}\tau\right\| \leq$$

$$\leq \int_0^t |K_\alpha(t, \tau))| \, \|\mathrm{D}u_i - \mathrm{D}u(\tau)\| \, \mathrm{d}\tau$$

From the properties of the Bochner's integral and applying the Bochner's theorem we conclude that (32) holds. (32) is true for function $K_\alpha(t, \tau)$ as well. The rest of the proof follows from (5).

b) From Lemma 7 and (5) we have $((F^{(r)}u)(t), v) \in C(\langle 0, T\rangle)$. (26) and (5) imply

$$|((F^{(r)}u)(t), v)| \leq \|(F^{(r)}u)(t)\| \, \|v\| \leq C \|v\|$$

Lemma 9. a) $A_r u^{(r)}(t) \in L_2(\Omega)$ *for all* $t \in \langle 0, T\rangle$ *and* $r = 1, \ldots, N$ *with* $\|A_r u^{(r)}(t)\| \leq C$

b) $[A_r u^{(r)}(t), v] \in C(\langle 0, T\rangle)$ *for all* $v \in \mathring{W}_2^k(\Omega)$ *and* $r = 1, \ldots, N$.

Proof: Due to Lemma 3, Lemma 4 and (5) we deduce

$$\left\| \frac{\mathrm{d}\ u_n^{(r)}(t)}{\mathrm{d}t} \right\| \leq C \| F^{(r)} u_n)(t) \| \leq C$$

Since

$$[A_r w_n^{(r)}(t), v] \to [A_r u^{(r)}(t), v],$$

we obtain from (27)

$$|[A_r u^{(r)}(t), v]| \leq C \| v \| \quad \text{for all} \quad v \in C_0^\infty(\Omega) r = 1, \ldots, N. \tag{33}$$

The rest of the proof follows from the density of $C_0^\infty(\Omega)$ in $L_2(\Omega)$.

b) The required result follows from the continuity of the operator A_r and from Lemma 7.

Remark: In virtue of (26) and Lemma 7a we prove easily that $u^{(r)}(t) \in C_w(\langle \delta, T\rangle, H)$. If $u_0^{(r)} \in W_2^{k+1}(\Omega)$, then $\delta = 0$. Indeed, if $t \to t_0 (t_n, t_0 \in \langle \delta, T\rangle)$ then $u_{n_k}^{(r)} \rightharpoonup w$ in the reflexive space H, because of (26). But, owing to Lemma 7a we have $w = u^{(r)}(t_0)$. Thus, $u^{(r)}(t_n) \rightharpoonup u^{(r)}(t_0)$ in H.

Theorem 1. *There exists a solution* $u(t) = (u^{(1)}(t), \ldots, u^{(N)}(t))$ *of the problem* (1), (2) *with the following properties: (let* $r \in \{1, \ldots, N\}$*)*

a) $u^{(r)}(t) : \langle 0, T\rangle \to L_2(\Omega)$ *is Lipschitz continuous*

b) $u^{(r)}(t) \in C_w^{(1)}(\langle 0, T), L_2(\Omega))$ *and there exists* $u_t^{(r)}(t)$ *(strong derivative) for a.e.* $t \in (0, T)$ *with* $u_t^{(r)} = \dfrac{\mathrm{d}u^{(r)}(t)}{\mathrm{d}t}$ *and* $u_t^{(r)} \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$

c) $u^{(r)}(t) \in L_\infty(\langle 0, T\rangle, W_2^{k+1}(\Omega') \cap W_2^k(\Omega)) \cap C_w(\langle 0, T\rangle, W_2^{k+1}(\Omega') \cap W_2^k(\Omega))$ *if* $u_0^{(r)} \in W_2^{k+1}(\Omega)$ *then we can put* $\langle 0, T\rangle$ *instead of* $(0, T\rangle$

d) $u^{(r)}(t) \in C(\langle \delta, T\rangle, W_2^{k+1-1}(\Omega') \cap \mathring{W}_2^k(\Omega))$ *for* $0 < \delta < T$ *if* $u_0^{(r)} \in W_2^{k+1}(\Omega)$, *then* $\delta = 0$

e) $A_r u^{(r)}(t) \in L_\infty(\langle 0, T\rangle, L_2(\Omega)) \cap C_w(\langle 0, T\rangle, L_2(\Omega))$.

Proof: b) Let $v \in C_0^\infty(\Omega)$ in (28) Owing to Lemma 6, Lemma 8, (33) and Lebesque's theorem, by the limiting process $n \to \infty$ in (28) we deduce

$$(u^{(r)}(t), v) + \int_0^t [A_r u^{(r)}(\tau), v]\,\mathrm{d}\tau + \int_0^t ((F^{(r)}u)(\tau), v)\,\mathrm{d}\tau - (u_0^{(r)}, v) = 0 \tag{34}$$

for all $t \in (0, T\rangle$, $v \in C_0^\infty(\Omega)$ and hence also for $v \in \mathring{W}_2^k(\Omega)$. From (34), Lemma 8, Lemma 9 we conclude $u^{(r)}(t) \in C_w^1(\langle 0, T), L_2(\Omega))$. Differentiating in (34) we obtain that $u^{(r)}(t)$ is a solution of (1), (2) $r = 1, \ldots, N$.

The identity

$$\frac{d(u^{(r)}(t), w)}{dt} = (u_t^{(r)}(t), w)$$

for all $t \in (0, T)$ and $w \in L_2(\Omega)$ implies

$$\int_0^T (u^{(r)}(t), \psi'(t)w)\, dt = -\int_0^T \left(\frac{du^{(r)}(t)}{dt}, \psi(t)w\right) dt$$

for all $w \in L_2(\Omega)$ and $\psi(t) \in C_0^\infty(\langle 0, T\rangle)$. Thus (see [5]) $u^{(r)}(t) \in W_2^1(\langle 0, T\rangle, L_2(\Omega))$ and there exists the strong derivative $\frac{du^{(r)}(t)}{dt}$ and the equality $u_t^{(r)}(t) = \frac{du^{(r)}(t)}{dt}$ is true for $t \in (0, T)$. From (34), Lemma 7, Lemma 8 we deduce

$$\left|\left(\frac{du^{(r)}(t)}{dt}, v\right)\right| \leqq C \|v\|$$

for all $t \in (0, T)$ and $v \in L_2(\Omega)$.

The continuity of $\left(\frac{du^{(r)}(t)}{dt}, w\right)$ implies the measurability of the abstract function $\frac{du^{(r)}(t)}{dt}$. Thus, assertion b) is proved. The rest of the proof is contained in the previous lemmas and remark.

Now, we shall be concerned with the dependence of the solution (1), (2) on $u_0 = u_0^{(r)}$, $r = 1, \dots, N$ and on the operators $(F^{(r)}u)(t)$, $r = 1, \dots, N$.

Let $u_1(t) = (u_1^{(r)}(t);\ r = 1, \dots, N)$ and $u_2(t) = (u_2^{(r)}(t);\ r = 1, \dots, N)$ be the solutions of (1), (2) corresponding to $u_{01} = (u_{01}^{(r)};\ r = 1, \dots, N)$, $u_{02} = (u_{02}^{(r)},\ r = 1, \dots, N)$ and $((F_1^{(r)}u)(t);\ r = 1, \dots, N)$, $((F_2^{(r)}u)(t);\ r = 1, \dots, N)$, respectively.

Theorem 2. *If*

$$\|(F_1^{(r)}u)(t) - (F_2^{(r)}u)(t)\| \leq b_r(t) + \sum_{i=1}^{N} a_{ir}(t) \|u^i(t)\|_w \tag{35}$$

for all $u \in W_2^k(\Omega)$ and $r = 1, \dots, N$, where $a_{ir}(t)$, $b_r(t)$ are continuous nonnegative functions in $\langle 0, T\rangle$, then the estimate

$$\sum_{r=1}^{N} \|u_1^{(r)}(t) - u_2^{(r)}(t)\| \leq e^{Kt}\Big(\sum_{r=1}^{N} \|u_{01}^{(r)} - u_{02}^{(r)}\|^2 + \tag{36}$$
$$+ \sum_{r=1}^{N} \int_0^t b_r^2(\tau)\, d\tau + \sum_{i=1}^{N} \max_{t \in \langle 0, T\rangle} \|u_1^{(i)}(t)\|_w^2 \sum_{r=1}^{N} \int_0^t a_{ir}^2(\tau)\, d\tau\Big)$$

takes place for all $t \in \langle 0, T\rangle$.

Proof: From Definition 3 we deduce

$$\left(\frac{\mathrm{d}(u_1^{(r)}(t)-u_2^{(r)}(t))}{\mathrm{d}t},\ u_1^{(r)}(t)-u_2^{(r)}(t)\right)+$$

$$+[A_r(u_1^{(r)}(t)-u_2^{(r)}(t)),\ u_1^{(r)}(t)-u_2^{(r)}(t)]+$$

$$+((F_1^{(r)}u_1)(t)-(F_2^{(r)}u_2)(t),\ u_1^{(r)}(t)-u_2^{(r)}(t))=0$$

for all $t\in(0,\ T)$. Hence, integrating this equality over $\langle 0,\ t)$ and using (8), (12) we obtain

$$2^{-1}\|u_1^{(r)}(t)-u_2^{(r)}(t)\|^2+C_1\int_0^t\|u_1^{(r)}(\tau)-u_2^{(r)}(\tau)\|_w^2\,\mathrm{d}\tau\leqslant$$

$$\leqslant C_4\int_0^t\|u_1^{(r)}(\tau)-u_2^{(r)}(\tau)\|^2\,\mathrm{d}\tau+\frac{\varepsilon^2}{2}\int_0^t\|u_1^{(r)}(\tau)-u_2^{(r)}(\tau)\|^2\,\mathrm{d}\tau+$$

$$+\frac{1}{2\varepsilon^2}\int_0^t\|(F_1^{(r)}u_1)(\tau)-(F_2^{(r)}u_2)(\tau)\|^2\,\mathrm{d}\tau+2^{-1}\|u_{01}^{(r)}-u_{02}^{(r)}\|^2$$

Owing to (5) and (35) we conclude

$$\|(F_1^{(r)}u_1)(\tau)-(F_2^{(r)}u_2)(\tau)\|\leqslant\|(F_1^{(r)}u_1)(\tau)-(F_2^{(r)}u_1)(\tau)\|+$$

$$+\|(F_2^{(r)}u_1)(\tau)-(F_2^{(r)}u_2)(\tau)\|\leqslant b_r(\tau)+\sum_{i=1}^N a_{ir}(\tau)\,\|u_1^i(\tau)\|_w+$$

$$+C\sum_{r=1}^N\|u_1^{(r)}(\tau)-u_2^{(r)}(\tau)\|_w+C\sum_{r=1}^N\int_0^\tau\|u_1^{(r)}(\xi)-u_2^{(r)}(\xi)\|_w\,\mathrm{d}\xi$$

and hence

$$\|(F_1^{(r)}u_1)(\tau)-(F_2^{(r)}u_2)(\tau)\|^2\leqslant K_1b_r^2(\tau)+K_1\sum_{i=1}^N a_{ir}^2(\tau)\,\|u_1^i(\tau)\|_w^2+$$

$$+CK_1\sum_{r=1}^N\|u_1^{(r)}(\tau)-u_2^{(r)}(\tau)\|_w^2+CK_1T\sum_{r=1}^N\int_0^\tau\|u_1^{(r)}(\xi)-u_2^{(r)}(\xi)\|_w^2\,\mathrm{d}\xi$$

owing to (37) and summing up for $r=1,\ \ldots,\ N$ we have

$$\sum_{r=1}^N 2^{-1}\|u_1^{(r)}(t)-u_2^{(r)}(t)\|^2+$$

$$+\left(C_1-\frac{CK_1N}{2\varepsilon^2}-\frac{CK_1T^2N}{2\varepsilon^2}\right)\int_0^t\sum_{r=1}^N\|u_1^{(r)}(\tau)-u_2^{(r)}(\tau)\|_w^2\,\mathrm{d}\tau\leqslant$$

$$\leqslant\frac{K_1}{2\varepsilon^2}\sum_{r=1}^N\sum_{i=1}^N\int_0^t a_{ir}^2(\tau)\|u_1^i(\tau)\|_w^2\,\mathrm{d}\tau+\frac{K_1}{2\varepsilon^2}\sum_{r=1}^N\int_0^t b_r^2(\tau)\,\mathrm{d}\tau+$$

$$\sum_{r=1}^{N} \|u_{01}^{(r)} - u_{02}^{(r)}\|^2 + 2^{-1}\left(C_4 + \frac{\varepsilon^2}{2}\right)\int_{r=1}^{t} \|u_1^{(r)}(\tau) - u_2^{(r)}(\tau)\|^2 \, d\tau$$

Let us choose $\varepsilon > 0$, so that

$$C_1 - \frac{CK_1N}{2\varepsilon^2} - \frac{CK_1T^2N}{2\varepsilon^2} \geq 0 \quad \frac{K_1}{2\varepsilon^2} \leq 2^{-1}$$

We denote by $K = C_4N + \frac{\varepsilon^2}{2}$, then

$$\sum_{r=1}^{N} \|u_1^{(r)}(t) - u_2^{(r)}(t)\|^2 \leq \sum_{r=1}^{N} \int_0^t b_r^2(\tau) \, d\tau +$$

$$+ \sum_{r=1}^{N} \|u_{01}^{(r)} - u_{02}^{(r)}\|^2 + \sum_{i=1}^{N} \max_{t \in \langle 0, T\rangle} \|u_1^i(t)\|_w^2 \sum_{r=1}^{N} \int_0^t a_{ir}^2(\tau) \, d\tau +$$

$$+ K \int_0^t \sum_{r=1}^{N} \|u_1^{(r)}(\tau) - u_2^{(r)}(\tau)\|^2 \, d\tau \tag{38}$$

Thus, (36) is a consequence of Gronwall's lemma.

Consequence: *The solution of* (1), (2) *is unique.*

Proof: In the proof of Theorem 2 we can put

$$(F_1^{(r)}u)(t) = (F_2^{(r)})(t), \; b_r(t) = 0, \; a_{ir}(t) = 0, \; u_{01}^{(r)} = u_{02}^{(r)}$$

for all $r = 1, \ldots, N$, $i = 1, \ldots$, $t \in \langle 0, T\rangle$.

Let $u_n = (u_n^{(r)}, r = 1, \ldots, N)$ be a solution of (1), (2) correspoding to $u_{0n} = (u_{0n}^{(r)}; r = 1, \ldots, N)$ and to the operators $(F_n^{(r)}u_n)(t) = f_n^{(r)}(t, x, du_n, \int_0^t K_\alpha(t, \tau) \, Du_n(\tau) \, d\tau)$ for $r = 1, \ldots, N$, $n = 1, \ldots$.

We shall assume that

$$|f_n^{(r)}(t, x, \xi) - f_n^{(r)}(t', x, \xi')| \leq C(|t - t'| + |t - t'| \, |\xi| + |\xi - \xi'|)$$

holds for all $t, t' \in \langle 0, T\rangle$; $\xi, \xi' \in E^{2Nd}$ $n = 1, 2, \ldots$, $r = 1, \ldots, N$ and

$$\|(F_n^{(r)}u)(t) - (F^{(r)}u)(t)\| \leq b_{rn}(t) + \sum_{i=1}^{N} a_{irn}(t) \|u^i\|_w$$

As a consequence of Theorem 2 we obtain:

Theorem 3. *If* $\|u_{0n}^{(r)} - u_0^{(r)}\| \to 0$, $\int_0^t a_{irn}^2(\tau) \, d\tau \to 0$ $\int_0^t b_{rn}^2(\tau) \, d\tau \to 0$ *for* $n \to \infty$, $r = 1, \ldots, N$ *and* $i = 1, \ldots, N$, *then* $u_n(t) \to u(t)$ *in the norm of the space* $C(\langle 0, T\rangle, L_2)$, *where* $L_2 = (L_2(\Omega))^N$.

Lemma 6 and Lemma 7 imply

Theorem 4. *For all* $r=1, .., N$ *it takes place* $u_n^{(r)}(t) \to u^{(r)}(t)$ *in* $C(\langle 0, T\rangle, L_2(\Omega))$ $u_n^{(r)}(t) \to u^{(r)}(t)$ *in* $W_2^{k+l}(\Omega) \cap W_2^{k+l}(\Omega')$ *for all* $t \in (0, T)$.

Theorem 5. *If the assumption* (3) *is satisfied for* $l=k$, *then the solution* $u^{(1)}(t)=u^1(x, t), \dots, u^{(N)}(t) = u^{(N)}(x, t)$ *of* (1), (2) *satisfies* (1) *in the classical sense for a.e.* $(x, t) \in Q$, *where* $Q=\Omega \times (0, T)$.

Proof: Owing to Theorem 1c. it suffices to prove that there exists the distribution derivate $\frac{\partial u^{(r)}(x, t)}{\partial t} \in L_2(Q)$ (see [8]) for all $r-1, .., N$.

Let $\psi(t) \in C_0^\infty(\langle 0, T\rangle)$ and $\varphi(x) \in C_0^\infty(\Omega)$, we have

$$\int_0^T \int_\Omega u^{(r)}(x, t)\psi'(t)\varphi(x)\, dx\, dt = -\int_0^T \int_\Omega q(x, t)\psi(t)\varphi(x)\, dx\, dt$$

where $q(x, t) \in L_2(Q)$ is generated by the abstract function $\frac{du^{(r)}(t)}{dt} \in L_2(\langle 0, T\rangle, L_2(\Omega)) \subset L_2(Q)$. Since linear combinations of all $\psi(t)\ \varphi(x)$ are dence in $C^\infty(Q)$, Theorem 5 is proved.

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Vypočtove stredisko SAV
Dubravska 9
842 35 Bratislava

ПАРАБОЛИЧЕСКИЕ ЧАСТНЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ С ПАМЯТЬЮ

Marian Slodička

Резюме

В работе рассматривается система параболических уравнений

$$\frac{\partial u^{(r)}}{\partial t} + A_r u^{(r)} = f^{(r)}\left(t, x, Du, \int_0^t K_\alpha(t, \tau)\, Du(\tau)\, d\tau\right)$$

для $r = 1, \ldots, N$ в области $\Omega \times (0, T)$, где A_r обозначает эллиптический оператор. Доказывается существование, единственность и некоторые свойства решения. Построено приближенное решение задачи и исследована его сходимость в отвечающих функциональных пространствах.