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OSCILLATION CRITERIA FOR DIFFERENTIAL EQUATION $y^{(4)}+P(t)y''+R(t)y'+Q(t)y=0$

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1. Introduction

This paper is concerned with the oscillation of the differential equation

(R) $$(L[y]=)\,y^{(4)}+P(t)y''+R(t)y'+Q(t)y=0,$$

where $P(t)$, $R(t)$, $Q(t)$ are real-valued continuous functions on the interval $I=\langle a,\infty)$, $-\infty<a<\infty$.

We shall assume throughout that

(B) $$P(t)\leqq 0,\quad R(t)\leqq 0,\quad R^2(t)\leqq 2P(t)Q(t)$$

or

(C) $$P(t)\leqq R(t)\leqq 0,\quad 2Q(t)\leqq R(t)$$

for all $t\in I$ and $Q(t)$ not identically zero in any subinterval of I. One can verify easily that (C) imply (B).

A nontrivial solution of a differential equation of the n-th order is called *oscillatory* if its set of zeros is not bounded from above. Otherwise, it is called *nonoscillatory*. A differential equation of the n-th order will be called *nonoscillatory*, when all its solutions are nonoscillatory; *oscillatory*, when at least one of its solutions is oscillatory.

This paper is a continuation of [6]. So we shall use the results obtained earlier, without explaining them again here.

There are proved the oscillation theorems and criteria for equation (R). Oscillation criteria for equation (R) will be obtained by an application of the theory developed in [5] and by the oscillation of linear differential equation of the third order. For interesting results on the subject we refer to the papers [1, 2, 3, 4, 5, 6].

2. Oscillations theorems

We will state the relevant theorems which are needed in proving the main theorems in this paper.

Theorem 1 [5]. *Suppose that* (B) *holds. Then equation* (R) *is oscillatory if and only if for every nonoscillatory solution* $y(t)$ *of* (R) *there holds either*

$$\operatorname{sgn} y(t) = \operatorname{sgn} y^{(j)}(t), \qquad j = 1, 2, 3$$

on $\langle t_0, \infty)$ *for some* $t_0 \in I$, *or*

$$\operatorname{sgn} y(t) \neq \operatorname{sgn} y'(t) \quad \text{on } I.$$

Theorem 2 [5]. *Suppose that* (B) *holds. Then equation* (R) *is nonoscillatory on* I *if and only if there exists a number* $t_0 \in I$ *and a solution* $y(t)$ *of* (R) *such that either*

$$y(t) > 0, \quad y'(t) > 0, \quad y''(t) < 0,$$

or

$$y(t) > 0, \quad y'(t) > 0, \quad y''(t) > 0, \quad y'''(t) < 0$$

for all $t \geqq t_0$.

The following theorem is proved by the same technique as Theorem 1.3 [4]. The proof will be omitted.

Theorem 3. *Suppose that* (B) *holds and let* $-2t^{-2} \leqq P(t)$ *for* $t > t_0 \geqq \max\{a, 0\}$. *Then there does not exist a solution* $y(t)$ *of* (R) *such that* $y(t) > 0$, $y'(t) > 0$ *and* $y''(t) < 0$ *for* $t > t_1 \geqq t_0$.

It can be shown that in general the condition $-2t^{-2} \leqq P(t)$ cannot be replaced by $-\mathrm{D}t^{-2} \leqq P(t)$, $\mathrm{D} > 2$.

Theorem 4. *Suppose that* (B) *holds and let* $\int_{\tau_0}^{\infty} t^{2+\alpha} Q(t)\, dt = -\infty$, $\tau_0 \geqq \max\{a, 0\}$, $0 \leqq \alpha < 1$. *Then* (R) *is nonoscillatory if and only if there exists a solution* $y(t)$ *of* (R) *and a number* $t_0 \in I$, $t_0 > \tau_0$, *such that* $y(t) > 0$, $y'(t) > 0$ *and* $y''(t) < 0$ *for all* $t \geqq t_0$.

The proof is obtained similarly to that of Theorem 1.2 [4] by using Theorem 2 and Lemma 1.1 [4] and is omitted.

Theorem 5 [6]. *Suppose that* (B) *holds and let*

$$\int_{t_0}^{\infty} s^{2+\alpha} Q(s)\, ds = -\infty, \qquad \int_{t_0}^{\infty} s^{2+\alpha} R(s)\, ds > -\infty,$$

$t_0 \geqq \max\{a, 0\}$, $0 \leqq \alpha < 1$. *Then for every solution* $y(t)$ *of* (R) *such that* $y(t)y'(t) \leqq 0$, $y(t)y''(t) \geqq 0$, $y(t)y'''(t) \leqq 0$ *for* $t \geqq t_0$ *there holds*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0.$$

Now, oscillation theorem for equation (R) will be obtained by using preceding results.

Theorem 6. *Suppose that*

$$(1) \qquad \int_{\tau_0}^{\infty} t^{2+\alpha} Q(t)\,dt = -\infty, \quad \tau_0 > \max\{a, 0\} \text{ for some } 0 \leqq \alpha < 1$$

and that (B) *holds and* $-2t^{-2} \leqq P(t)$ *for* $t \geqq \tau_0$ *or* (1) *and* (C) *hold and* $-2t^{-2} \leqq P(t)$ *for* $t \geqq \tau_0$. *Then* (R) *is oscillatory and there exists a fundamental system of solutions of* (R) *such that two solution of this system are oscillatory, other solutions of this system are nonoscillatory and one of them tends monotonically to* ∞ *as* $t \to \infty$ *and the other of them tends to zero if* $\int_{\tau_0}^{\infty} t^{2+\alpha} R(t)\,dt > -\infty$.

Proof. Theorems 3, 4 and 1 imply that (R) is oscillatory. Let z_0, z_1, z_2 and z_3 denote solutions of (R) defined on I by the initial conditions

$$z_i^{(j)}(a) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \text{for } i, j = 0, 1, 2, 3.$$

Then (R) has oscillatory solutions

$$u(t) = b_0 z_0(t) + b_3 z_3(t)$$

$$v(t) = c_2 z_2(t) + c_3 z_3(t),$$

Whose construction has already been shown in the paper [6].

Note that z_3 has no zero to the right of a by Lemma 2[5] and $\lim_{t\to\infty} z_3(t) = \infty$.

It follows from Theorem 2[5] that there exists a solution z with the properties $z > 0$, $z' < 0$, $z'' > 0$ and $z''' \leqq 0$ for $t \in I$. By theorem 5 there is $\lim_{t\to\infty} z(t) = 0$.

The solutions $z(t)$, $u(t)$, $v(t)$ and $z_3(t)$ form the fundamental system of (R). In fact, their Wronskian

$$W[z(t), u(t), v(t), z_3(t)]_{t=a} = -b_0 c_2 z'(a) \neq 0,$$

since $z'(a) < 0$ and $b_0 \neq 0$, otherwise it would be $u(t) = b_3 z_3(t)$, which would contradict the fact that $u(t)$ is oscillatory and $z_3(t)$ has no zeros to the right of a. By the same argument $c_2 \neq 0$. The proof of Theorem is complete.

Remark 1. Theorem 6 is a generalization of Theorem 1.7 [4]. If $R(t) \equiv 0$, $P(t) \equiv 0$ for $t \in I$ we obtain well-known results for equation $y^{(4)} + Q(t)y = 0$ [1, 2].

3. Oscillation criteria

Now Theorem 2.3 [4] will be extended for equation (R).

Theorem 7. *Suppose that* (B) *holds. Let* $\mu(t)$ *be a positive and continuous function in* (T, ∞), $T \geqq a$, *such that*

$$\liminf_{t\to\infty} \frac{t-t_0}{\mu(t)} \geqq 2 \tag{2}$$

for arbitrary $t_0 \geqq a$ *and let the differential equation of the third order*

$$x''' + P(t)x' + [R(t) + \Theta\mu(t)Q(t)]x = 0 \tag{3}$$

for some $\Theta \in (0, 1)$ *be oscillatory. Then* (R) *also is oscillatory.*

Remark. Examples of functions $\mu(t)$ for which the condition (2) is satisfied are as follows: $v(t-a)$, $v \in \left(0, \frac{1}{2}\right)$; $k(t-a)^{\varepsilon}$, $k>0$, $\varepsilon<1$; $\ln(t-a)$, $t>T$, $T \geqq a+1$.

Proof. The proof is similar to that done in [4]. For the sake of completeness we are going to give it here. Suppose on the contrary that (R) is nonoscillatory. Then by Theorem 2, there exists a number $t_0 \in I$ ($t_0 \geqq T$) and a solution y of (R) such that either $y>0$, $y'>0$, $y''>0$ and $y'''<0$ or $y>0$, $y'>0$ and $y''<0$ for $t \geqq t_0$. Applying Lemma 1.1 [4] to the solution y we obtain

$$y > \frac{t-t_0}{2} y' \quad \text{for } t \geqq t_0.$$

Then it follows from (R) that

$$y^{(4)} + P(t)y'' + R(t)y' + \frac{t-t_0}{2} Q(t)y' \geqq 0 \quad \text{for } t \geqq t_0. \tag{4}$$

If $\liminf_{t\to\infty} \frac{t-t_0}{\mu(t)} \geqq 2$, for $\Theta \in (0, 1)$ then there exists a number $\tau > t_0$ ($\tau > T$) such that

$$\frac{t-t_0}{\mu(t)} > 2\Theta \qquad \text{for } t > \tau.$$

and hence

$$\frac{t-t_0}{2} Q(t)y' \leqq \Theta\mu(t)Q(t)y' \quad \text{for } t > \tau. \tag{5}$$

It follows from (4) and (5) that

$$y^{(4)} + P(t)y'' + [R(t) + \Theta\mu(t)Q(t)]y' \geqq 0 \quad \text{for } t > \tau.$$

Setting $y' = u$, we obtain

$$(6) \qquad u''' + P(t)u' + [R(t) + \Theta\mu(t)Q(t)]u \geqq 0$$

and $u>0$, $u'>0$, $u''<0$ or $u>0$ and $u'<0$ on (τ, ∞). The above inequalities and Theorem 6.24 [7, Gera] imply that the linear differential equation

$$x''' + P(t)x' + [R(t) + \Theta\mu(t)Q(t)]x = 0$$

is nonoscillatory on $\langle \tau, \infty)$. This contradicts the hypothesis of the Theorem.

The proof of Theorem 7 implies the following corollary.

Corollary 1. *Suppose that* (B) *or* (C) *and* (2) *hold.*
If

$$x''' + P(t)x' + \Theta\mu(t)Q(t)x = 0$$

or

$$x''' + P(t)x' + R(t)x = 0,$$

where $R(t) \not\equiv 0$ *in any subinterval, is oscillatory, then* (R) *also is oscillatory.*

Throughout this paper we assume that coefficients of (R) satisfy conditions (B) or (C). If, in addition, we suppose that

$$(7) \qquad -2t^{-2} \leqq P(t) \quad \text{or}$$

$$(8) \qquad \int_T^\infty tP(t)\,dt > -\infty \quad \text{and} \quad Q(t) \leqq R(t)$$

(In the case that (B) holds) for $t \geqq T > \max\{a, 0\}$, then by Theorem 3 and 4 [6], respectively, the equation (R) has no solution $y(t)$ with the properties $y(t)>0$, $y'(t)>0$ and $y''(t)<0$ in $\langle t_0, \infty)$, $t_0 \geqq T$.

Using this fact, we can prove the following modifications of the oscillation Theorem 7.

Theorem 8. *Let* $\mu(t)$ *be a positive and continuous function in* (T, ∞), $T \geqq a$, *such that* (2) *holds. Suppose that* (B) *or* (C) *and* (7) *or* (8) *hold. If*

$$x''' + \Theta\mu(t)Q(t)x = 0$$

for some $\Theta \in (0, 1)$ *is oscillatory, then equation* (R) *also is oscillatory.*

The proof is similar to that of Theorem 7 and is omitted.

Theorem 9. *Suppose that* (B) *or* (C) *and* (7) *or* (8) *hold and that* $R(t)$ *not identically zero in any subinterval of* (T, ∞), $T > \max\{a, 0\}$. *If*

$$x''' + R(t)x = 0$$

is oscillatory, then equation (R) *also is oscillatory.*

Proof. Suppose that (R) is nonoscillatory. Then Theorem 2 implies the existence of a number $t_0 \in I$ ($t_0 \geqq T$) and of a solution y of (R) with $y>0$, $y'>0$, $y''>0$ and $y'''<0$ for $t \geqq t_0$. Since $P(t)y'' \leqq 0$ and $Q(t)y \leqq 0$ we obtain from (R)

$$y^{(4)} + R(t)y' \geqq 0 \quad \text{for} \quad t \geqq t_0.$$

Further the proof follows along the lines of the proof of the Theorem 7.

By combining Theorems 7, 8 and 9 with the known oscillation criteria for the third order equation we obtain oscillation criteria for (R).

Corollary 2. *Let $\mu(t)$ be a positive and continuous function in (T, ∞), $T > \max\{1, a\}$ such that* (2) *holds for arbitrary $t_0 \geqq a$. Suppose that* (B) *or* (C) *and* (7) *or* (8) *hold and that*

$$\int_T^\infty \mu(t) Q(t) f(t) \, dt = -\infty,$$

where $f(t)$ is one of the functions

$$t^{1+\alpha}, \quad t^2(\ln t)^{\alpha-2}, \quad t^2(\ln t)^{-1}(\ln(\ln t))^{\alpha-2}, \tag{9}$$

$0 < \alpha < 1$. *Then* (R) *is oscillatory.*

Proof. We note that under these assumptions the differential equation $x''' + \Theta\mu(t)Q(t)x = 0$ is oscillatory as follows from Theorem 2.6 [4]. Then the assertion of the Corollary 2 follows from Theorem 8.

Remark 2. The oscillation criterion which was proved in Theorem 7 [6] is a special case of Corollary 2 for $\mu(t) = vt$, $v \in \left(0, \frac{1}{2}\right)$ and $f(t) = t^{1+\alpha}$, $0 < \alpha < 1$.

On the bases of Theorem 9 and Theorem 2.6 [4] we obtain the following oscillation criterion of somewhat different character from that in Corollary 2.

Corollary 3. *Suppose that* (B) *or* (C) *and* (7) *or* (8) *hold and $R(t)$ not identically zero in any subinterval of $\langle T, \infty)$. Suppose that*

$$\int_T^\infty R(t) f(t) \, dt = -\infty,$$

where $f(t)$ is one of the functions (9). *Then* (R) *is oscillatory.*

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ОСЦИЛЛЯЦИОННЫЕ КРИТЕРИИ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y^{(4)}+P(t)y''+R(t)y'+Q(t)y=0$

Ján Regenda

Резюме

В работе приведены критерии для осцилляции уравнения (R). Некоторые из этих критериев являются обобщением некоторых известных результатов.