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CODOMAIN OF THE TENSOR PRODUCT OF SEMIGROUPS

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In this paper we prove that every commutative semigroup is a c-tensor product of some commutative semigroups and there exist semigroups (finite and infinite) which are not tensor products of any semigroups. The last fact follows from the properties of indecomposable elements.

1. Introduction

Let $\mathcal{T}$ be the class of all semigroups and $\mathcal{T}_c$ be the class of all commutative semigroups. The tensor product $\otimes$ in the class $\mathcal{T}$ and the tensor product $\otimes^c$ in the class $\mathcal{T}_c$ have been defined in [1—3]. The tensor product $\otimes^c$ in the class $\mathcal{T}_c$ will be called a c-tensor product.

Definition 1. *Let A and B be semigroups and $A \times B$ be the Cartesian product of A and B. A semigroup S together with a bilinear map $\omega\colon A \times B \to S$ is called a tensor product of A and B if the following condition is satisfied:*

For any semigroup C and for any bilinear map $\beta\colon A \times B \to C$ there exists a homomorphism $\alpha\colon S \to C$ such that $\beta = \omega\alpha$.

The semigroup S will be denoted as $A \otimes B$.

The c-*tensor product* $\otimes^c$ is defined analogously as the tensor product $\otimes$, we require that the semigroups A, B, C, S be commutative semigroups.

It is proved in [1] that for any $A, B \in \mathcal{T}$ the semigroup $A \otimes B$ is isomorphic to $F_{A\times B}/\tau$, where $F_{A\times B}$ is the free semigroup on $A \times B$ and τ is the smallest congruence over the relation τ_0, which is defined on $A \times B$ in this way:

For any a, a_1, $a_2 \in A$ and b, b_1, $b_2 \in B$ the relations

$$(a, b_1b_2)\tau_0(a, b_1)(a, b_2)$$

$$(a_1a_2, b)\tau_0(a_1, b)(a_2, b) \text{ hold.}$$

The relation τ_0 will be called the tensor relation and τ will be called the tensor congruence (on $F_{A\times B}$). The class of the tensor congruence which contains the element $(a_1, b_1)(a_2, b_2) \ldots (a_n, b_n) \in F_{A\times B}$ will be denoted by $(a_1 \otimes b_1)(a_2 \otimes b_2) \ldots (a_n \otimes b_n)$. This is an element of $A \otimes B$.

The function $\omega: A \times B \to A \otimes B$ in the Definition 1 is given by $\omega = \iota\omega'$, where $\iota: A \times B \to F_{A\times B}$ is the embedding and $\omega': F_{A\times B} \to A \otimes B$ is the natural homomorphism. The function ω will be called the tensor function.

The c-tensor product is given analogously. For any $A, B \in \mathscr{T}_c$ we have $A \otimes^c B \cong F^c_{A\times B}/\tau$, where $F^c_{A\times B}$ is the free commutative semigroup on $A \times B$. The relations τ_0, τ and the element $(a_1 \otimes^c b_1) \ldots (a_n \otimes^c b_n) \in A \otimes^c B$ are defined in the same way.

Definition 2. *A semigroup $S \in \mathscr{T}$ will be called an N-semigroup, if for any $x, y \in S$ and any natural number n the condition $(xy)^n = x^n y^n$ holds.*

Let $A \in \mathscr{T}$. We will denote:

$C(A)$ — the greatest commutative homomorphic image of A.
$N(A)$ — the greatest homomorphic image of A which is an N-semigroup.
F — the free semigroup with one generator.
$|X|$ — the cardinality of a set X.

Grilet proved in [1] and [2] the following properties:

G1. *If* $A, B \in \mathscr{T}_c$, then $A \otimes^c B \cong C(A \otimes B)$.
G2. *If $A \in \mathscr{T}$, then $A \otimes F \cong N(A)$.*
G3. *For any $A, B \in \mathscr{T}$ is $A \otimes B \cong N(A) \otimes N(B)$.*

The following corollaries are consequences of G1—G3.

Corollary 1. *Any N-semigroup is the tensor product of some semigroups.*
Proof. If A is an N-semigroup, then $A \cong N(A) \cong A \otimes F$.

Corollary 2. *Any commutative semigroup is the c-tensor product of some commutative semigroups.*
Proof. Every commutative semigroup A is an N-semigroup and we have $A \otimes^c F \cong C(A \otimes F) \cong C(N(A)) \cong A$.

The natural question arises. Is any semigroup the tensor product of some semigroups? The answer is negative and this fact will be proved in the third part of this paper.

2. Indecomposable elements

Definition 3. *Let $A \in \mathscr{T}$. An element $a \in A$ is indecomposable (in A) iff $a \in A - A^2$. If $a \in A^2$, then a is called decomposable (in A).*

The connection between indecomposable elements in A, B and indecomposable elements in $A \otimes B$ and $A \otimes^c B$ is given in the following Theorems 1—3.

Theorem 1. *Let $A, B \in \mathcal{T}$. Then $a \otimes b$ is an indecomposable element in $A \otimes B$ iff a is an indecomposable element in A and b is an indecomposable element in B.*

Proof. Let $a \otimes b$ be an indecomposable element in $A \otimes B$. Suppose that the element a or b is decomposable, say $a = a_1a_2$. Then $a \otimes b = (a_1 \otimes b)(a_2 \otimes b)$ and this shows that $a \otimes b$ is decomposable in $A \otimes B$. This is a contradiction.

Conversely suppose that $a \in A$ and $b \in B$ are indecomposable elements. We have to show that $a \otimes b$ is an indecomposable element.

It is clear that $|A| > 1$ and $|B| > 1$. Let $T = \{x, 0\}$ be the zero semigroup and 0 the zero of T.

Let β: $A \times B \to T$ be the function defined by

$$\beta(a, b) = x$$
$$\beta(a', b') = 0, \text{ if } (a', b') \neq (a, b).$$

For any $a_1, a_2 \in A$ and $b_1, b_2 \in B$ we have $(a_1a_2, b_1) \neq (a, b)$ and $(a_1, b_1b_2) \neq (a, b)$. This implies

$$\beta(a_1a_2, b_1) = 0 = \beta(a_1, b_1)\beta(a_2, b_1),$$
$$\beta(a_1, b_1b_2) = 0 = \beta(a_1, b_1)\beta(a_1, b_2).$$

Hence β is a bilinear map $A \times B$ to T.

By Definition 1 there exists a homomorphism α: $A \otimes B \to T$ and $\beta = \omega\alpha$, where ω is the tensor function.

Let us remark that for any homomorphism f if $f(y) = z$ and z is indecomposable, so is y.

We have $\beta(a, b) = x$ and this implies $\alpha(a \otimes b) = \alpha(\omega(a, b)) = \beta(a, b) = x$. Since x is indecomposable element, so is $a \otimes b$.

This completes the proof of Theorem 1.

Since T is commutative, the same proof implies

Theorem 2. *Let $A, B \in \mathcal{T}_c$. Then $a \otimes^c b$ is an indecomposable element in $A \otimes^c B$ iff $a \in A - A^2$ and $b \in B - B^2$.*

Corollary 1.1. *Let $a, a_1 \in A - A^2$, $b, b_1 \in B - B^2$. Then $(a, b) \neq (a_1, b_1)$ iff $a \otimes b \neq a_1 \otimes b_1$.*

Proof. Let T, α be the same as in the proof of Theorem 1. If $(a, b) \neq (a_1, b_1)$ then $\alpha(a \otimes b) = x \neq 0 = \alpha(a_1 \otimes b_1)$ and it follows that $a \otimes b \neq a_1 \otimes b_1$.

The converse is obvious.

Theorem 3. *Let $A, B \in \mathcal{T}$ and $a_1, a_2, \ldots, a_n \in A - A^2$, $b_1, b_2, \ldots, b_n \in B - B^2$, where $a_i \neq a_{i+1}$, $b_i \neq b_{i+1}$ for $i = 1, \ldots, n-1$. The class $(a_1 \otimes b_1) \ldots (a_n \otimes b_n)$ of the* *tensor congruence on $F_{A\times B}$ contains a unique element, namely $(a_1, b_1) \dots (a_n, b_n)$*

Proof. Let $c_1, \dots, c_k$ be all different elements of $a_1, \dots, a_n$ and $d_1, \dots, d_s$ all different elements of $b_1, \dots, b_n$. By Corollary 1.1 the elements $c_i \otimes d_j$ for $i = 1, \dots, k$ and $j = 1, \dots, s$ are different indecomposable elements of $A \otimes B$.

Let $X = \{x_{i,j}: i = 1, \dots, k, \ j = 1, \dots, s\}$ be a set with $|X| = ks \geqq 1$. Let S be a semigroup with the generating set X and a zero $0 \notin X$. The operation on S for non-zero elements is defined in the following way:

For any $x_{i,j}$, $x_{k,l} \in X$ we have

$$x_{i,j} \cdot x_{k,l} = \begin{cases} 0 \text{ if } i = k \text{ or } j = l, \\ x_{i,j} x_{k,l} \text{ if } i \neq k \text{ and } j \neq l. \end{cases}$$

Let I be the ideal of S consisting of 0 and of all words from S of the lenght greater than n.

Let $T_1 = S/I$ be the Rees factor semigroup. Elements of T_1 will be denoted in the following way:

$\bar{0}$ denotes the zero of T_1. The class of the Rees congruence which contains the non-zero element $x \in S$ will be denoted by $\bar{x}$.

The set $T_1 - T_1^2 = \{\overline{x_{i,j}}: i = 1, \dots, k, \ j = 1, \dots, s\}$ is the set of indecomposable different generators of T_1.

Let the function β: $A \times B \to T_1$ be defined in the following way:

$$\beta(c_i, d_j) = \overline{x_{i,j}} \text{ for } i = 1, \dots, k \text{ and } j = 1, \dots, s,$$
$$\beta(c, d) = \bar{0} \text{ otherwise.}$$

For any $u, u_1, u_2 \in A$ and $v, v_1, v_2 \in B$ we have:

$$\beta(u_1 u_2, v) = \bar{0} \text{ since } u_1 u_2 \in A^2.$$

$\beta(u_1, v)\beta(u_2, v) = \bar{0}$ since one of $\beta(u_1, v)$, $\beta(u_2, v)$ is equal to $\bar{0}$ or $\beta(u_1, v) = \overline{x_{i,j}}$, $\beta(u_2, v) = \overline{x_{k,j}}$ and $\overline{x_{i,j}} \cdot \overline{x_{k,j}} = \bar{0}$.

In both cases we have $\beta(u_1 u_2, v) = \beta(u_1, v)\beta(u_2, v)$.

In the same way we prove $\beta(u, v_1 v_2) = \beta(u, v_1)\beta(u, v_2)$.

This means that the function β is bilinear function.

Let ω be the tensor map ω: $A \times B \to A \otimes B$ and $\acute{\omega}$, ι, τ be the same as in the Definition 1.

Let β': $F_{A\times B} \to T_1$ be the natural extension of β, $\beta = \iota\beta'$. According to the definition of the tensor congruence τ and the construction of the semigroup T_1 it is clear $\tau = \ker \omega' \subset \ker \beta'$. All classes of $\ker \beta'$ contain only one element except one class, namely the class whose image by β' is $\bar{0}$.

Since $a_i \neq a_{i+1}$, and $b_i \neq b_{i+1}$ for $i = 1, \dots, n-1$ we have $\beta'[(a_1, b_1) \dots (a_n, b_n)]$

$= \beta(a_1, b_1) \dots \beta(a_n, b_n) = \overline{x_{i_1, j_1}} \dots \overline{x_{i_n, j_n}} \neq \bar{0}$, where $\overline{x_{i_k, j_k}} \in T_1 - T_1^2$ for $k = 1, \dots, n$.

The class of ker β', which contains the element $(a_1, b_1) \dots (a_n, b_n)$ is a one-element class and this fact implies that the class $(a_1 \otimes b_1) \dots (a_n \otimes b_n)$ of the tensor congruence τ is the same element.

This proves Theorem 3.

3. Semigroups which are not tensor products

By G3 we have $N(A) \otimes N(B) \cong A \otimes B$ and denote by α this isomorphism. By the definition of the tensor product the tensor function ω: $N(A) \times N(B) \to N(A) \otimes N(B)$, for any fixed $b \in B$, determines a homomorphism σ_b: $N(A) \to A \otimes B$ defined by $\sigma_b(a) = a \otimes b$.

Let σ: $A \to N(A)$ and τ: $B \to N(B)$ be the natural homomorphism. We have

$$A \times B \xrightarrow{\sigma \times \tau} N(A) \times N(B) \xrightarrow{\omega} N(A) \otimes N(B) \xrightarrow{\alpha} A \otimes B.$$

The set $X = \{a \otimes b: a \in A, b \in B\}$ is the set of generators of $A \otimes B$. For any $a \in A$ we have $(\sigma \times \tau)(a, b) = (\sigma(a), \tau(b))$ and by the proof of G3 we have $a \otimes b = \sigma(a) \otimes \tau(b)$.

The set X contains the homomorphic image of $N(A)$, because $a \otimes b = \sigma_b(a)$ for $a \in N(A)$. Similarly for $N(B)$.

A homomorphic image of an N-semigroup is an N-semigroup and this implies the existence of an N-semigroup in the tensor product $A \otimes B$.

Every semigroup contains an N-semigroup, namely the subsemigroup generated by one element. These N-semigroups will be called trivial N-subsemigroups.

Theorem 4. *Let S be a semigroup and $|S - S^2| > 1$. Suppose that for every $x, y \in S - S^2$, $x \neq y$, there exists a natural number $n \in N$, $n > 1$, such that $(xy)^n \neq x^n y^n$. Then S is not a tensor product of semigroups.*

Proof. Let us suppose for an indirect proof that $S \cong A \otimes B$ for some $A, B \in \mathcal{T}$ and let α be this isomorphism. If $|S - S^2| > 1$, then there exist two elements $x_1, x_2 \in S - S^2$ and $x_1 \neq x_2$. Since x_1 and x_2 are indecomposable elements, $\alpha^{-1}(x_1)$ and $\alpha^{-1}(x_2)$ are indecomposable, too. Let $\alpha^{-1}(x_1) = a_1 \otimes b_1$ and $\alpha^{-1}(x_2) = a_2 \otimes b_2$, then by injectivity of α we have $a_1 \otimes b_1 \neq a_2 \otimes b_2$. By Theorem 1 we have $a_1, a_2 \in A - A^2$ and $b_1, b_2 \in B - B^2$ and by Theorem 3 we have $(a_1, b_1) \neq (a_2, b_2)$.

Let, e.q., $a_1 \neq a_2$. Then by Theorem 1 the elements $a_1 \otimes b_1$, $a_2 \otimes b_1$ are indecomposable and by Theorem 3 $a_1 \otimes b_1 \neq a_2 \otimes b_1$. Let $\alpha(a_2 \otimes b_1) = x_3$. Clearly we have $x_3 \in S - S^2$, $x_1 \neq x_3$ and for any natural number $n \geqq 1$ we have $(x_1 x_3)^n = [\alpha(a_1 \otimes b_1)\alpha(a_2 \otimes b_1)]^n = [\alpha(a_1 a_2 \otimes b_1)]^n = \alpha(a_1 a_2 \otimes b_1^n) = \alpha(a_1 \otimes b_1^n)\alpha(a_2 \otimes b_1^n) = [\alpha(a_1 \otimes b_1)]^n [\alpha(a_2 \otimes b_1)]^n = x_1^n x_3^n$.

This is a contradiction with the assumption and Theorem 4 is proved.

Corollary 4.1. *Let S be a semigroup having only trivial N-subsemigroups and $|S-S^2|>1$. Then S is not a tensor product of semigroups.*

Proof. This corollary follows immediately from Theorem 4 and the definition of an N-semigroup.

Corollary 4.2. *Let F_X be a free semigroup on a set X, where $|X|>1$. Then F_X is not a tensor product of some semigroups.*

Proof. This corollary follows immediately from Theorem 4 by putting $S=F_X$.

Corollary 4.3. *Let F_X be a free semigroup on a set X, $|X|>1$. Let J be the ideal of all words of F_X the lenght of which is greater then fixed natural number k, $k \geqq 4$. Then the Rees factor semigroup F_X/J is not a tensor product of semigroups.*

Proof. Corollary 4.3 follows from Theorem 4 for $n=2$ and $S=F_X/J$. It is $|S-S^2|=|X|>1$.

Corollary 4.4. *There exist infinitely many finite semigroups which are not tensor products of semigroups.*

Proof. We use the notations of Corollary 4.3. If the set X is a finite set, then F_X/J is finite. If the set X have different finite cardinalities or natural numbers k are different, we obtain different finite semigroups.

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КООБЛАСТЬ ТЕНЗОРНОГО ПРОИЗВЕДЕНИЯ ПОЛОГРУПП

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Резюме

В этой работе мы показали, что всякая коммутативная полугруппа является тензорным произведением некоторых коммутативных полугрупп (в классе всех коммутативных полугрупп).

Существуют полугруппы (конечные и бесконечные), которые не являются тензорным произведением никаких полугрупп. Такой является, например, свободная полугруппа F_X, $|X|>1$ и фактор-полугруппа Рисса F_X/J, где J-идеал всех слов из F_X длина которых больше, чем натуральное число k, $k \geqq 4$.