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GRAPHS WITH PRESCRIBED NEIGHBOURHOOD GRAPHS

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Let G be an undirected graph without loops and multiple edges, let u be a vertex of G. By $N_G(u)$ we denote the subgraph of G induced by the set of all vertices which are adjacent to u in G; we call it the neighbourhood graph of u in G.

At the Symposium on Graph Theory in Smolenice in 1963 [1] A. A. Zykov has proposed the problem (by himself and B. A. Trachtenbrot) to characterize graphs H with the property that there exists a graph G such that $N_G(u) \cong H$ for each vertex u of G.

We shall study the graphs G with the property that for each vertex u of G the graph $N_G(u)$ is isomorphic to the complement of a path or of a circuit. Evidently a graph has the required property if and only if each of its connected components has this property and therefore we shall consider only connected graphs with such a property.

The symbol P_n usually denotes the (simple) path of the length n, i.e. with $n+1$ vertices. Its complement will be denoted by $\bar{P}_n$. Similarly C_n denotes the circuit of the length n and $\bar{C}_n$ denotes its complement.

Theorem 1. *Let G be a graph with the property that $N_G(u) \cong \bar{P}_n$ for each vertex u of G, where $n \geqq 4$. Then $G \cong \bar{C}_{n+4}$.*

Proof. Let u be a vertex of G. We have $N_G(u) \cong \bar{P}_n$. Let $v_0, v_1, \ldots, v_n$ be the vertices of $N_G(u)$ such that $\{v_{i-1}, v_i\}$ for $i = 1, \ldots, n$ are the unique non-adjacent pairs of its vertices. Consider $N_G(v_0)$. This graph contains the vertices $v_2, \ldots, v_n$, u and its subgraph induced by these vertices is the complement of the graph consisting of a path of the length $n-2$ and an isolated vertex. As $N_G(v_0) \cong \bar{P}_n$, there exists a vertex w in it which is non-adjacent to u and one of the vertices v_2, v_n and adjacent to all other vertices v_i. Suppose that w is non-adjacent to v_2. Then v_2 is adjacent to none of the vertices v_1, v_3, w. If $n = 4$, then the graph $N_G(v_2)$ contains a triangle with the vertices u, v_0, v_4. We have $N_G(v_2) \cong \bar{P}_4$; hence in the complement of $N_G(v_2)$ one of the vertices u, v_0, v_4 is the centre and the others are terminal vertices of the path of the length 4. Therefore there exists a vertex z of $N_G(v_2)$ which is adjacent to exactly one of the vertices u, v_0, v_4. This vertex z must be evidently distinct from the vertices u, v_0, v_1, v_2, v_3, v_4, w. But $N_G(u)$ contains the vertices v_0, v_1, v_2, v_3, v_4, the graph $N_G(v_0)$ contains u, v_2, v_3, v_4, w and $N_G(v_4)$

contains u, v_0, v_1, v_2, w. As z is adjacent to one of the vertices u, v_0, v_4, one of the graphs $N_G(u)$, $N_G(v_0)$, $N_G(v_4)$ contains at least six vertices and is not isomorphic to $\bar{P}_4$, which is a contradiction. Now suppose $n \geqq 5$. All vertices v_1, v_2, v_3, w are contained in $N_G(v_5)$. In the complement of $N_G(v_5)$ the vertex v_2 has the degree at least 3 and thus this graph is not a path, which is a contradiction. Hence w is non-adjacent to v_n and $N_G(v_0)$ is the complement of the path with the vertices $v_2, \ldots, v_n, w, u$ (in this order). Analogously taking $N_G(v_n)$ instead of $N_G(v_0)$, we prove that there exists a vertex x non-adjacent to v_0 and u and adjacent to $v_1, \ldots, v_n$. Using the graph $N_G(v_2)$, we prove that also x is adjacent to w. Therefore the set $\{v_0, \ldots, v_n, w, u, x\}$ induces a subgraph F of G such that $F \cong \bar{C}_{n+4}$. We have $N_F(y) \cong \bar{P}_n$ for each vertex y of F. As also $N_G(y) \cong \bar{P}_n$ for each vertex y of F, we have $N_G(y) = N_F(y)$ and, as G is connected, $G = F$ and $G \cong \bar{C}_{n+4}$. □

Evidently also the converse assertion is true and even without the assumption that $n \geqq 4$. For each positive integer n the graph $G \cong \bar{C}_{n+4}$ has the property that $N_G(u) \cong \bar{P}_n$ for each vertex u of G. But for $n = 1$ not only the circuit $C_5 \cong \bar{C}_5$, but an arbitrary circuit of the length at least 4 has this property. For $n = 2$ the line graph of the graph obtained from an arbitrary regular graph of degree 3 by inserting one vertex onto each edge has the required property. For $n = 3$ we have $\bar{P}_3 \cong P_3$; in [2] it was proved that for any odd integer $n \geqq 7$ there exists a graph G with n vertices such that $N_G(u) \cong P_3$ for each u. This graph is constructed from a circuit C_n by joining any two vertices having the distance $\frac{1}{2}(n-1)$ in C_n by an edge. (For $n = 7$ such a graph is isomorphic to $\bar{C}_7$.)

Theorem 2. *A graph G with the property that $N_G(u) \cong \bar{C}_n$ for each vertex u of G exists if and only if $3 \leqq n \leqq 6$.*

Proof. The graph $\bar{C}_3$ consists of three isolated vertices. Every regular graph of degree 3 without triangles has the property that $N_G(u) \cong \bar{C}_3$ for each u. The graph $\bar{C}_4$ has two connected components which are both isomorphic to the complete graph K_2. If G is the line graph of a regular graph of degree 3 without triangles, then $N_G(u) \cong \bar{C}_4$ for each vertex u of G. The graph $\bar{C}_5 \cong C_5$; the graph of the regular icosahedron has the required property. The graph with the required property for $n = 6$ is the complement of the Petersen graph.

Now let $n \geqq 7$. Suppose that there exists a graph G such that $N_G(u) \cong \bar{C}_n$ for each vertex u of G. Let u be a vertex of G. Let the vertices of $N_G(u)$ be $v_1, \ldots, v_n$ such that $\{v_i, v_{i+1}\}$ for $i = 1, \ldots, n-1$ and $\{v_n, v_1\}$ are the unique pairs of non-adjacent vertices of $N_G(u)$. Consider $N_G(v_2)$. A subgraph of this graph is the graph consisting of the complement of the path with vertices $v_4, \ldots, v_n$ and of the vertex u adjacent to all of these vertices. As $N_G(v_2) \cong \bar{C}_n$, there exists a vertex w which is not adjacent to v_n and u and is adjacent to all the vertices $v_4, \ldots, v_{n-1}$. The graph $N_G(v_4)$ contains the vertices v_{n-1}, v_n, v_1, w. The subgraph of $N_G(v_4)$ induced by these vertices is the complement of the graph in which v_n has the degree 3; hence the complement of $N_G(v_4)$ is not a circuit, which is a contradiction. □

Quite analogously to the part of Theorem 2 for $n \geqq 7$ also the following two theorems can be proved.

Theorem 3. *There exists no graph G with the property that $N_G(u)$ for each vertex u of G is the complement of a one-way infinite path.*

Theorem 4. *There exists no graph G with the property that $N_G(u)$ for each vertex u of G is the complement of a two-way infinite path.*

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ГРАФЫ С ПРЕДПИСАННЫМИ ГРАФАМИ ОКРЕСТНОСТЕЙ

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Резюме

Пусть G — неориентированный граф, пусть u — его вершина. Символом $N_G(u)$ обозначаем подграф графа G, порожденный множеством вершин смежных с u. В статье изучаются графы G такие, что графы $N_G(u)$ для всех вершин u графа G изоморфны дополнению цепи или контура.