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A NOTE ON INDECOMPOSABLE ELEMENTS IN THE TENSOR PRODUCT OF SEMIGROUPS

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Let $\mathcal{T}$ be the class of all semigroups. In [1] the tensor product $\otimes$ is defined and following property is proved:

For any $A, B \in \mathcal{T}$ the semigroup $A \otimes B$ is isomorphic to $F_{A\times B}/\tau$, where $F_{A\times B}$ is the free semigroup on the Cartesian product $A \times B$ and τ is the smallest congruence over the relation τ_0, which is defined on $A \times B$ in this way:

For any a, $a_1 a_2 \in A$ and b, b_1, $b_2 \in B$ the relations

$$(a, b_1 b_2)\ \tau_0\ (a, b_1)(a, b_2)$$
$$(a_1 a_2, b)\ \tau_0\ (a_1, b)(a_2, b)$$

hold.

The relations τ_0 will be called the tensor relation and τ will be called the tensor congruence (on $F_{A\times B}$). The class of the tensor congruence which contains the element $(a_1, b_1)\dots(a_n, b_n) \in F_{A\times B}$ will be denoted by $(a_1 \otimes b_1)\dots(a_n \otimes b_n)$. This is an element of $A \otimes B$.

The following properties of $A \otimes B$ are proved in [1]:

G1. *If E is a one-element semigroup, then $A \otimes E \cong E(A)$ holds for any $A \in \mathcal{T}$. $E(A)$ is the greatest idempotent homomorphic image of A.*

G2. *If $A, B \in \mathcal{T}$, $A_1 \subset A$, $B_1 \subset B$ and A_1 is a set of generators of A, B_1 a set of generators of B, then the set*

$$\otimes(A_1, B_1) = \{a \otimes b \in A \otimes B \colon a \in A_1,\ b \in B_1\}$$

is a set of generators of $A \otimes B$.

Definition 1. *Let $A \in \mathcal{T}$ and $a \in A$. Then the element a is called indecomposable (in A), if $a \in A - A^2$. If $a \in A^2$, then a is called decomposable (in A).*

The following properties are proved in [2]:

J1. *Let $A, B \in \mathcal{T}$, $a \in A$ and $b \in B$. Then $a \otimes b$ is indecomposable in $A \otimes B$ iff $a \in A - A^2$ and $b \in B - B^2$.*

J2. *Let $A, B \in \mathcal{T}$, $a_i \in A - A^2$, $b_i \in B - B^2$, $i = 1, \dots, n$ and $a_{j+1} \neq a_j$, $b_{j+1} \neq b_j$ for $j = 1, \dots, n-1$. Then the element $(a_1, b_1)\dots(a_n, b_n) \in F_{A\times B}$ is the only element of the class $(a_1 \otimes b_1)\dots(a_n \otimes b_n)$ of the tensor congruence on $F_{A\times B}$.*

In particular we have $(a_1, b_1) \neq (a_2, b_2)$ in $F_{A \times B}$ iff $a_1 \otimes b_1 \neq a_2 \otimes b_2$ in $A \otimes B$.

If I is an ideal in a semigroup A, then A/I denotes the Rees factor semigroup.

The cardinality of a set X will be denoted by $|X|$.

The purpose of this note is to prove the statements C1—C5 formulated below which clarify the influence of the indecomposable elements of A and B on the structure of $A \otimes B$.

Statement C1. *Let A, $B \in \mathcal{T}$. Then*

$$|(A \otimes B) - (A \otimes B)^2| = |(A - A^2) \times (B - B^2)|.$$

Proof. This follows from J1 and J2, since $a \otimes b \in [(A \otimes B) - (A \otimes B)^2]$ iff $(a, b) \in [(A - A^2) \times (B - B^2)]$.

Statement C2. *If $|A - A^2| > 1$ and $|B - B^2| > 1$, then the semigroup $A \otimes B$ is an infinite non-commutative semigroup.*

Proof. Let $a_1, a_2 \in A - A^2$, $b_1, b_2 \in B - B^2$ and $a_1 \neq a_2$, $b_1 \neq b_2$. Denote $s_1 = a_1 \otimes b_1$, $s_2 = a_2 \otimes b_2$. Since $(a_1, b_1)(a_2, b_2) \neq (a_2, b_2)(a_1, b_1)$ we have $s_1 s_2 \neq s_2 s_1$ by J2 and $A \otimes B$ is non-commutative semigroup.

The following elements are different (by J2):

$$s_1,\ s_2,\ s_1 s_2,\ s_1 s_2 s_1,\ s_1 s_2 s_1 s_2,\ s_1 s_2 s_1 s_2 s_1,\ \ldots.$$

Hence the semigroup $A \otimes B$ is infinite.

If A, B satisfy the conditions of the Statement C2 then $A \otimes B$ contains indecomposable elements and it is infinite. The question arises: If $A \otimes B$ is finite are there indecomposable elements in $A \otimes B$. The answer is given in the Statement C3.

Statement C3. *let S be a finite semigroup, $S - S^2 \neq \emptyset$ and S be isomorphic to a tensor product $A \otimes B$ ($A, B \in \mathcal{T}$). If we denote $|A - A^2| = \alpha$ and $|B - B^2| = \beta$, then $\alpha = 1$ and β is a non-zero natural number or $\beta = 1$ and α is a non-zero natural number.*

Proof. By C1 we have $\alpha \neq 0$, $\beta \neq 0$, α and β finite. By C2 we have $\alpha \leqq 1$ or $\beta \leqq 1$.

Lemma. *Let $S = \{s, 0\}$ be a zero semigroup with zero 0 and T a zero semigroup. Then $S \otimes T$ is isomorphic to T.*

Proof. Let $0'$ be the zero of T. Then the set $T_1 = T - \{0'\}$ is the set of all indecomposable elements of T. Further $\otimes(\{s\}, T_1) = \{s \otimes t \colon t \in T_1\}$ is a set of generators of $S \otimes T$ by G2. By J1 we have $\otimes(\{s\}, T_1) = (S \otimes T) - (S \otimes T)^2$.

Let $\delta \colon T \to S \otimes T$ be a mapping defined by $\delta(t) = s \otimes t$ for any $t \in T$. We shall show that δ is an isomorphism:

By J2, δ is injective function, since $t \neq t_1$, $t \in T$, $t_1 \in T_1$ imply $\delta(t) = s \otimes t \neq s \otimes t_1 = \delta(t_1)$.

The function δ is surjective: The elements of $S \otimes T$ have the form $(s \otimes t_1) \dots (s \otimes t_n)$, where $t_1, \dots, t_n \in T_1$ and n is a natural number. We have $(s \otimes t_1) \dots (s \otimes t_n) = s \otimes t_1 \dots t_n = s \otimes 0'$ for $n > 1$. The elements of $S \otimes T$ are exactly the elements $s \otimes t = \delta(t)$, $t \in T$.

The function δ is a homomorphism: $\delta(t_1 t_2) = s \otimes (t_1 t_2) = (s \otimes t_1)(s \otimes t_2) = \delta(t_1)\delta(t_2)$ for any $t_1, t_2 \in T$.

This proves our Lemma.

Statement C4. *Let A, B be semigroups. Then $(A/A^2) \otimes (B/B^2)$ is isomorphic to $A \otimes B/(A \otimes B)^2$ iff $|A - A^2| \in \{0, 1\}$ or $|B - B^2| \in \{0, 1\}$.*

Proof. Let us remark that $A \otimes B \cong B \otimes A$.

If both $|A - A^2| > 1$ and $|B - B^2| > 1$, then, by C2, $A/A^2 \otimes B/B^2$ is non-commutative, while $(A \otimes B)/(A \otimes B)^2$ is commutative.

If $|A - A^2| = 1$, i.e. $|A/A^2| = 2$, then by the Lemma we have $A/A^2 \otimes B/B^2 \cong B/B^2$. Using C1, we have $|B/B^2| = |B - B^2| + 1 = |A - A^2|\,|B - B^2| + 1 = |(A \otimes B)/(A \otimes B)^2|$, whence $B/B^2 \cong (A \otimes B)/(A \otimes B)^2$.

If $|A - A^2| = 0$, i.e. $|A/A^2| = 1$, then $A/A^2 \otimes B/B^2$ is by G1 a one-point semigroup, and so is $(A \otimes B)/(A \otimes B)^2$, by C1.

This proves Statement C4.

Definition. *A semigroup S is called globally idempotent if $S = S^2$.*

Statement C5. *The semigroup $A \otimes B$ is globally idempotent iff A is globally idempotent or B is globally idempotent.*

Proof. The semigroup $A \otimes B$ is globally idempotent iff $|(A \otimes B) - (A \otimes B)^2| = 0$. We have $|A - A^2|\,|B - B^2| = 0$ by C1 and that means $A = A^2$ or $B = B^2$.

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ПРИМЕЧАНИЕ К НЕРОЗЛОЖНЫМ ЭЛЕМЕНТАМ ТЕНЗОРНОГО ПРОИЗВЕДЕНИЯ ПОЛУГРУПП

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Резюме

Пусть A, B — полугруппы и $\otimes$ — тензорное произведение в классе всех полугрупп. Если в A и в B существует более одного неразложимого элемента, то $A \otimes B$ — бесконечная некоммутативная полугруппа.

Фактор-полугруппа Рисса $A \otimes B/(A \otimes B)^2$ изоморфна $(A/A^2) \otimes (B/B^2)$ тогда и толькотогда, когда в A или в B существует не более одного неразложимого элемента.