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COMPACTIFICATIONS OF PARTIALLY ORDERED SETS

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In [1], the notions of a compact set and a compactification of a partially ordered set were introduced. It was proved that the ordinal sum of any two-element chain and $(P, \leqslant)$ is a compactification of $(P, \leqslant)$ provided that $(P, \leqslant)$ is a partially ordered set without the least element. Further, a necessary condition for the existence of a compactification of a partially ordered set with the least element was given. In this paper it is shown that this condition is also sufficient for the existence of a compactification.

Terminology left undefined here may be found in [2].

We suppose that $(P, \leqslant)$ is a partially ordered set (a poset for short) with the least element 0, which is called the zero.

1. Definition. *We say that a subset S of $(P, \leqslant)$ has the finite lower bound property (the f.l.b.p. for short) if every finite subset of S has a nonzero lower bound in P (i.e. a lower bound different from the zero of P). The poset $(P, \leqslant)$ is said to be compact provided that every its subset with the finite lower bound property has a nonzero lower bound.*

Remark. It is evident from the foregoing definition that the subset having the f.l.b.p. does not contain the zero.

2. Definition. *By a compactification of $(P, \leqslant)$, we mean a couple $((Q^*, \leqslant^*), \varphi)$, where $(Q^*, \leqslant^*)$ is a compact partially ordered set and φ is a mapping $P \to Q^*$ with the following properties:*

(1) *If $a, b \in P$, then $a \leqslant b \Leftrightarrow \varphi(a) \leqslant^* \varphi(b)$.*

(2) *$\varphi(0)$ is the zero of Q^*.*

(3) *φ preserves all existing suprema and infima of subsets of P, except the zero infima of infinite subsets of P with the finite lower bound property. If S is an infinite subset of P with the f.l.b.p. and $\inf S = 0$, then $\varphi(S)$ has in Q^* a nonzero infimum.*

Remark. The infimum of a subset S of a poset $(R, \leqslant)$ is mostly denoted by $\inf_R S$. But the denotation $\inf S$ is also used when no confusion is likely to arise.

Every finite poset is evidently compact, hence solving the problem of the existence of the compactification of $(P, \leqslant)$, we can confine to the case of P being infinite.

If S is a subset of P with the f.l.b.p., then by Zorn's Lemma there exists a subset of P maximal with respect to the f.l.b.p. and containing S. The following statement is proved in [1].

3. Theorem. *If M is a subset of P maximal with respect to the finite lower bound property, then* $\inf M$ *exists. If* $\inf M = p \neq 0$, *then p is an atom in* $(P, \leqslant)$ *and* $M = \{x \in P: x \geqslant p\}$.

Denote by $\mathcal{M}$ the system of all subsets of P maximal with respect to the finite lower bound property with zero infima. With respect to our object we can suppose that $\mathcal{M} \neq \emptyset$. In the opposite case P would be compact. Let $\mathcal{M} = \{M_h: h \in H\}$. Set $\mathcal{S}(H) = \{H_1 \subseteq H: H_1 \neq \emptyset, \inf(\cap\{M_h: h \in H_1\}) = 0\}$.

4. Definition. *A set H_1 from $\mathcal{S}(H)$ is called saturated if H_1 contains all $h_1 \in H$ such that* $M_{h_1} \supseteq \cap\{M_h: h \in H_1\}$.

Obviously the set $\{h\}$ belongs to $\mathcal{S}(H)$ and it is saturated for every $h \in H$. If $H_1 \in \mathcal{S}(H)$, then $H'_1 = \{h' \in H: \cap\{M_h: h \in H_1\} \subseteq M_{h'}\} (\supseteq H_1)$ is the unique saturated set from $\mathcal{S}(H)$ with the property $\cap\{M_h: h \in H'_1\} = \cap\{M_h: h \in H_1\}$.

Let $\mathcal{S}'(H) = \{H' \in \mathcal{S}(H): H'$ is saturated$\}$. Denote by Q the disjoint join of P and $\mathcal{S}'(H)$ and define a relation $\leqslant$ in Q as follows:

$\leqslant$ is the extension of the partial order given in P and of the set-inclusion in $\mathcal{S}'(H)$;

if $x \in P$, $H' \in \mathcal{S}'(H)$, then $x \leqslant H'$ if and only if $x = 0$, and $H' \leqslant x$ if and only if $x \in \cap\{M_h: h \in H'\}$.

5. Theorem. *The above defined relation $\leqslant$ in Q is a partial order. If $((Q^*, \leqslant^*), \varphi)$ is a compactification of $(P, \leqslant)$, then the mapping φ has an extension $\varphi^*: Q \to Q^*$ such that $a \leqslant b$ $(a, b \in Q)$ if and only if $\varphi^*(a) \leqslant^* \varphi^*(b)$.*

Proof. The first part of the statement is evident. Let $((Q^*, \leqslant^*), \varphi)$ be a compactification of $(P, \leqslant)$. Let φ^* be an extension of φ such that $\varphi^*(H') = \inf_{Q^*} \varphi(\cap\{M_h: h \in H'\})$ for $H' \in \mathcal{S}'(H)$. The property (3) of φ ensures that the last infimum exists and it is nonzero. We prove that $a \leqslant b$ $(a, b \in Q)$ if and only if $\varphi^*(a) \leqslant^* \varphi^*(b)$. If $a, b \in P$, the statement is evident. Suppose that $a = H' \in \mathcal{S}'(H)$, $b \in P$. We have to show that $H' \leqslant b$ if and only if $\inf_{Q^*} \varphi(\cap\{M_h: h \in H'\}) \leqslant^* \varphi(b)$. The implication $H' \leqslant b \Rightarrow \inf_{Q^*} \varphi(\cap\{M_h: h \in H'\}) \leqslant^* \varphi(b)$ is evident. Now, suppose that $\inf_{Q^*} \varphi(\cap\{M_h: h \in H'\}) \leqslant^* \varphi(b)$ and $H' \nleqslant b$. Then there exists $h_0 \in H'$ such that $b \notin M_{h_0}$. The maximality of M_{h_0} implies the existence of a finite subset K of M_{h_0} with $\inf_P (K \cup \{b\}) = 0$. Thus, $\inf_{Q^*} \varphi(K \cup \{b\}) = 0$. On the other hand $\inf_{Q^*} \varphi(M_{h_0}) \leqslant^* \inf_{Q^*} \varphi(\cap\{M_h: h \in H'\}) \leqslant^* \varphi(b)$ and $\inf_{Q^*} \varphi(M_{h_0}) \leqslant^* \varphi(k)$ for every $k \in K$, hence $\inf_{Q^*} \varphi(M_{h_0})$ is a nonzero lower bound of $\varphi(K \cup \{b\})$ in Q^*. We have a contradiction.

Let now $a \in P$, $b = H' \in \mathcal{S}'(H)$. If $a \leqslant H'$, then $a = 0$, hence evidently $\varphi^*(a) \leqslant^* \varphi^*(b)$. Assume $\varphi(a) \leqslant^* \inf_{Q^*} \varphi(\cap\{M_h: h \in H'\})$. Then a is a lower bound of the set $\cap\{M_h: h \in H'\}$ in P. Since $\inf_P(\cap\{M_h: h \in H'\}) = 0$, it must be $a = 0$, which follows $a \leqslant H'$.

Finally, let $a=H_1\in\mathcal{S}'(H)$, $b=H_2\in\mathcal{S}'(H)$. Suppose that $\varphi^*(a)\leqslant^*\varphi^*(b)$, i.e. $\inf_{Q^*}\varphi(\cap\{M_h\colon h\in H_1\})\leqslant^*\inf_{Q^*}\varphi(\cap\{M_h\colon h\in H_2\})$. Take $x\in\cap\{M_h\colon h\in H_2\}$. Then $\inf_{Q^*}\varphi(\cap\{M_h\colon h\in H_1\})\leqslant^*\varphi(x)$, which follows, with respect to the above proved, $H_1\leqslant x$, i.e. $x\in\cap\{M_h\colon h\in H_1\}$. We proved $\cap\{M_h\colon h\in H_2\}\subseteq\cap\{M_h\colon h\in H_1\}$. Considering that H_2 is saturated, we have $H_1\subseteq H_2$.

The proof of the following Theorem resembles that of the compactness of the β-cover of a completely regular topological space (cf. [3]).

6. Theorem. *The above defined poset* $(Q, \leqslant)$ *is compact.*

Proof. Let $S\subseteq Q$ have the f.l.b.p. If $S\subseteq P$, then S has the f.l.b.p. in P. Suppose $\inf_P S=0$. Then there exists $h_0\in H$ with $S\subseteq M_{h_0}$. Evidently $\{h_0\}$ is a nonzero lower bound of S in Q.

Let $S\cap\mathcal{S}'(H)=\{H_i\colon i\in I\}$, $I\neq\emptyset$. Set $T=(S\cap P)\cup(\cup\{\cap\{M_h\colon h\in H_i\}\colon i\in I\})$. To show that T has the f.l.b.p. in Q, let K be a finite subset of T. If $K\subseteq S\cap P$, then K has a nonzero lower bound in Q, by the assumption. Let $K\cap(\cup\{\cap\{M_h\colon h\in H_i\}\colon i\in I\})=\{y_1, \ldots, y_l\}$, $l\geqslant 1$. Then for every $j\in\{1, \ldots, l\}$ there exists $i_j\in I$ such that $y_j\in\cap\{M_h\colon h\in H_{i_j}\}$, i.e. $H_{i_j}\leqslant y_j$. By the assumption, the set $(K\cap S\cap P)\cup\{H_{i_1}, \ldots, H_{i_l}\}$ has a nonzero lower bound in Q, and this is a nonzero lower bound of K, too. Since $T\subseteq P$, T has the f.l.b.p. also in P. Let M be a subset of P containing T and maximal with respect to the f.l.b.p. By 3, $\inf_P M$ exists. Set $p=\inf_P M$. p is a lower bound of $\cap\{M_h\colon h\in H_i\}$ for every $i\in I$ and since $\inf_P(\cap\{M_h\colon h\in H_i\})=0$, we have $p=0$. Therefore $M=M_{h_0}$ for some $h_0\in H$. Since $T\subseteq M_{h_0}$ and $\{h_0\}\in\mathcal{S}'(H)$, $\{h_0\}$ is a lower bound of T in Q. To show that $\{h_0\}$ is also a lower bound of S in Q, it is sufficient to prove $\{h_0\}\subseteq H_i$ for every $i\in I$. Take any $i\in I$. If $x\in\cap\{M_h\colon h\in H_i\}$, then $\{h_0\}\leqslant x$, which follows $x\in M_{h_0}$. Therefore $\cap\{M_h\colon h\in H_i\}\subseteq M_{h_0}$. As H_i is saturated, we have $h_0\in H_i$.

Consider the following condition for $(P, \leqslant)$:

(a) *If M is a subset of P maximal with respect to the finite lower bound property and* $\inf_P M=0$, *then M is closed under the existing nonzero infima of its subsets.*

By 3, this condition is equivalent to the following one:

(α) *If* $A(\subseteq P)$ has the finite lower bound property and $\inf_P N=p\neq 0$ *for some* $N\subseteq A$, *then* $A\cup\{p\}$ *has also the finite lower bound property.*

The following Theorem is proved in [1].

7. Theorem. *If* $(P, \leqslant)$ *has a compactification, then* $(P, \leqslant)$ *satisfies* (a).

We prove the converse.

8. Theorem. *Let* $(Q, \leqslant)$ *be the poset constructed above,* ι *the identical mapping* $P\to Q$. *If* $(P, \leqslant)$ *satisfies* (a), *then* $((Q, \leqslant), \iota)$ *is the minimal compactification of* $(P, \leqslant)$.

Proof. It is evident that the mapping ι has the properties (1), (2) from Definition 2 and it is suprema-preserving. Let $A\subseteq P$, $\inf_P A=p\neq 0$. We show that if $H'\in\mathcal{S}'(H)$ is a lower bound of A in Q, then $H'\leqslant p$. Let $H'\leqslant a$ for every $a\in A$. Then $A\subseteq M_h$ for every $h\in H'$, whence, by (a), $p\in\cap\{M_h\colon h\in H'\}$, i. e. $H'\leqslant p$. Let now $A \subseteq P$, $\inf_P A = 0$. If 0 is the unique lower bound of A in Q, too, then $\inf_Q A = 0$. Assume that $H' \in \mathcal{S}'(H)$ is a lower bound of A. Then $A \subseteq M_h$ for every $h \in H'$, hence A is infinite and has the f.l.b.p. Let $H_1 = \{h \in H\colon A \subseteq M_h\}$. Then $H_1 \in \mathcal{S}'(H)$ and evidently $H_1 = \inf_Q A$. We proved that $((Q, \leqslant), \iota)$ is a compactification of $(P, \leqslant)$. The minimality of $((Q, \leqslant), \iota)$ follows from Theorem 5.

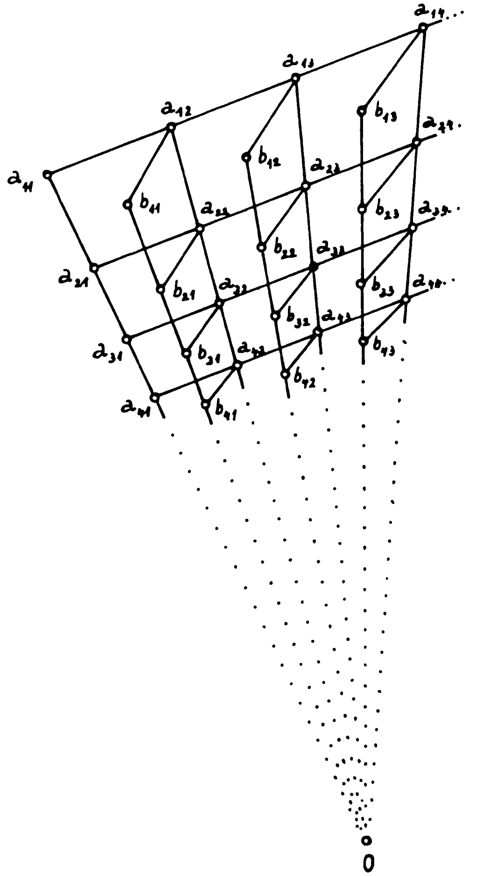

Fig. 1

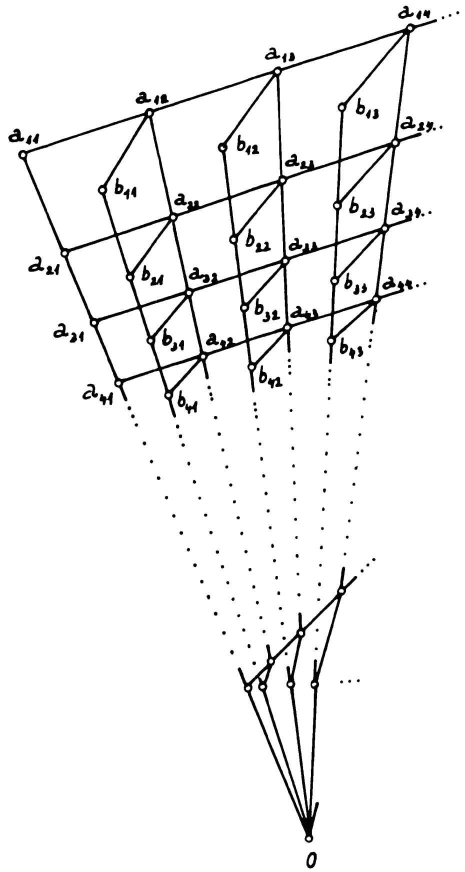

Fig. 2

9. Example. Let $P = \{a_{ij}\colon i, j \in N\} \cup \{b_{ij}\colon i, j \in N\} \cup \{0\}$ (N is the set of all positive integers). Define the relation $\leqslant$ on P as follows: if $i, j, k, l \in N$, then

$a_{ij} \leqslant a_{kl} \Leftrightarrow i \geqslant k,\ j \leqslant l,$

$b_{ij} \leqslant b_{kl} \Leftrightarrow i \geqslant k,\ j = l,$

$b_{ij} \leqslant a_{kl} \Leftrightarrow i \geqslant k,\ j < l,$

$a_{ij} \nleqslant b_{kl},$

$0 \leqslant a_{ij},\ b_{ij}$

(cf. Figure 1). It is easy to verify that $(P, \leqslant)$ is a lattice. Maximal with respect to the f.l.b.p. are the sets $M_1=\{a_{ij}: i, j\in N\}$, $M_k=\{a_{ij}: i, j\in N, \ j\geqslant k\} \cup \{b_{i,k-1}: i\in N\}$ for $k\geqslant 2$. We have $\inf M_i=0$ for every $i\in N$, hence $H=N$. Obviously $\mathscr{S}(H)=\{H_1\subseteq H: H_1\neq\emptyset, \text{card } H_1<\aleph_0\}$, $\mathscr{S}'(H)=\{\{h\}: h\in H\}\cup\{\{1, 2, \ldots, k\}: k\in N, k\geqslant 2\}$. The diagram of the above constructed $(Q, \leqslant)$ is shown in Figure 2.

In what follows we suppose that $(P, \leqslant)$ fulfils (a). We shall investigate some properties of the above-mentioned compactification $((Q, \leqslant), \iota)$ of $(P, \leqslant)$.

10. Theorem. *Every non-void subsystem of the system $\mathscr{S}'(H)$ has an infimum in $(Q, \leqslant)$.*

Proof. Let $\emptyset\neq\{H_i: i\in I\}\subseteq\mathscr{S}'(H)$. First, assume that $\cap\{H_i: i\in I\}=\emptyset$. Then evidently $\inf_Q\{H_i: i\in I\}=0$. Now, let $H_0=\cap\{H_i: i\in I\}\neq\emptyset$. Take any $i_0\in I$. The relation $H_0\subseteq H_{i_0}$ implies $\cap\{M_h: h\in H_{i_0}\}\subseteq\cap\{M_h: h\in H_0\}$, whence $\inf_P(\cap\{M_h: h\in H_0\})=0$. Hence $H_0\in\mathscr{S}(H)$. Suppose that $\cap\{M_h: h\in H_0\}\subseteq M_{h_0}$ for some $h_0\in H$. Then also $\cap\{M_h: h\in H_i\}\subseteq M_{h_0}$ for every $i\in I$ and since every H_i is saturated, we have $h_0\in H_0$. We proved that $H_0\in\mathscr{S}'(H)$. Then evidently $H_0=\inf_Q\{H_i: i\in I\}$.

The following example shows that not even finite subsystems of $\mathscr{S}'(H)$ have a supremum in Q, in general.

11. Example. Let $(P, \leqslant)$ be a poset of Figure 3. Obviously $(P, \leqslant)$ is a lattice. There are three subsets of P maximal with respect to the f.l.b.p., $M_1=\{a_{-i}: i\in N\}\cup\{b_{-i}: i\in N\}$, $M_2=\{a_{-i}: i\in N\} \cup \{c_{-i}: i\in N\}$, $M_3=\{a_{-i}: i\in N\} \cup \{a_j: j\in N\}$. There is $\inf_P M_1 = \inf_P M_2=0$, $\inf_P M_3=a_1$. Hence $H=\{1, 2\}$, $\mathscr{S}'(H)=\{\{1\}, \{2\}\}$. Upper bounds of the set $\{\{1\}, \{2\}\}$ in Q are just the elements of the set $M_1\cap M_2=\{a_{-i}: i\in N\}$. Since the set $\{a_{-i}: i\in N\}$ has not the least element, $\sup_Q\{\{1\}, \{2\}\}$ does not exist.

Based on Theorem 10, we have:

12. Corollary. *If $(P, \leqslant)$ is a lower semilattice, then $(Q, \leqslant)$ is also a lower semilattice.*

13. Corollary. *If $(P, \leqslant)$ is a complete lattice, then $(Q, \leqslant)$ is also a complete lattice.*

Proof of Corollary 12. It is sufficient to show that any two elements from Q, at least one of which belongs to P, have an infimum in Q. If $x, y\in P$, then there exists $\inf_P\{x, y\}$ by the assumption and since $((Q, \leqslant), \iota)$ is a compactification of $(P, \leqslant)$ we have $\inf_P\{x, y\} = \inf_Q\{x, y\}$. Let $x\in P$, $H'\in\mathscr{S}'(H)$. If $\{x, H'\}$ has no lower bound in $\mathscr{S}'(H)$, then evidently $0=\inf_Q\{x, H'\}$. Suppose that $H_1'\leqslant x$, $H_1'\leqslant H'$ for some $H_1'\in\mathscr{S}'(H)$. Set $H_0'=\{h\in H': x\in M_h\}$. It is easy to verify that $H_0'\in\mathscr{S}'(H)$ and $H_0'=\inf_Q\{x, H'\}$.

Proof of Corollary 13. Since the greatest element of P is also the greatest element of Q, it is sufficient to show that every non-void subset of Q, which is not disjoint from P, has an infimum in Q. If $\emptyset\neq X=\{x_i: i\in I\}\subseteq P$, then, by the assumption, there exists $x\in P$ with $x=\inf_P X$. If X is an infinite set with the f.l.b.p.

and $x=0$, then $H'=\{h\in H: X\subseteq M_h\}$ evidently belongs to $\mathscr{S}'(H)$ and $H'=\inf_Q X$. In the opposite case (i.e. if it does not hold that X is an infinite set with the f.l.b.p. and $x=0$), there is $\inf_P X=\inf_Q X$. Take $\emptyset\neq\{x_i: i\in I\}\subseteq P$, $\emptyset\neq\{H'_j: j\in J\}\subseteq \mathscr{S}'(H)$ and set $Y=\{x_i: i\in I\}\cup\{H'_j: j\in J\}$. By what we have already proved, there

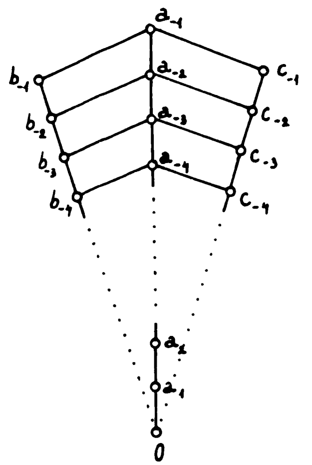

Fig. 3

exist elements $u, v\in Q$ such that $u=\inf_Q\{x_i: i\in I\}$, $v=\inf_Q\{H'_j: j\in J\}$. Using Corollary 12 we obtain that there exists $w\in Q$ with $w=\inf_Q\{u, v\}$. Then evidently $w=\inf_Q Y$.

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КОМПАКТИФИКАЦИИ ЧАСТИЧНО УПОРЯДОЧЕННЫХ МНОЖЕСТВ

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Резюме

В работе продолжается изучение компактификаций частично упорядоченных множеств. Показывается, что необходимое условие для существования компактификации частично упорядоченного множества, данное в работе [1], является также достаточным.