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# ON THE DIRECT POWER OF RELATIONAL SYSTEMS

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In the paper relations are considered in a general sense, i.e. as sets of mappings. We introduce three direct binary operations of addition, multiplication and exponentiation for relational systems which generalize the three cardinal operations for ordered sets discussed by G. Birkhoff in [1] and [2]. The aim of this note is to find a sufficient condition for the direct power of relational systems to have a certain property that is characteristic for powers in cartesian closed topological categories.

Let $F$ and $I$ be non-empty sets. Then a set of mappings $R \subseteq F^I$ is called a *relation* and the ordered pair $\mathcal{F} = (F, R)$ is sasid to be a *relational system*. The set $F$ is called the *carrier* of $\mathcal{F}$ and the set $I$ is called the *domain* of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are relational systems of domains $I$ and $J$ respectively, then $\mathcal{F}$ and $\mathcal{G}$ are said to be *of the same type* iff there exists a bijection of $I$ onto $J$, i.e. iff $I$ and $J$ are equipotent.

Let $\mathcal{F} = (F, R)$ of domain $I$ and $\mathcal{G} = (G, S)$ of domain $J$ be two relational systems of the same type. Let $\alpha$: $I \to J$ be a bijection and let $\varphi$: $F \to G$ be a mapping. If the implication $f \in R \Rightarrow \varphi \circ f \circ \alpha^{-1} \in S$ holds, then $\varphi$ is called a *homomorphism of $\mathcal{F}$ into $\mathcal{G}$ with regard to $\alpha$*. By $\mathrm{Hom}_\alpha(\mathcal{F}, \mathcal{G})$ we denote the set of all homomorphisms of $\mathcal{F}$ into $\mathcal{G}$ with regard to $\alpha$. If $I = J$ and $\alpha = \mathrm{id}_I$ (by $\mathrm{id}_I$ we denote the identity mapping of $I$), then we write $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ instead of $\mathrm{Hom}_\alpha(\mathcal{F}, \mathcal{G})$.

If $\mathcal{F} = (F, R)$ and $\mathcal{G} = (F, S)$ are two relational systems of the same domain and with the same carrier, then we put $\mathcal{F} \leqq \mathcal{G}$ iff $R \subseteq S$, i.e. iff $\mathrm{id}_F \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. Clearly, $\leqq$ is an order on the set of all relational systems of the same domain and with the same carrier.

**1.** Example. Let $R$ be a ternary relation on a set $F$ or, in other words, let $\mathcal{F} = (F, R)$ be a relational system of the domain $\{1, 2, 3\}$. By $\hat{R}$ denote the cyclic closure of $R$, i.e. $\hat{R}$ is the least (with respect to the set inclusion) ternary relation on $F$ such that $R \subseteq \hat{R}$ and that the implication $f \in \hat{R} \Rightarrow g \in \hat{R}$ is valid whenever $g \in F^{\{1, 2, 3\}}$ is the mapping with $g(1) = f(2)$, $g(2) = f(3)$, $g(3) = f(1)$. Put $\hat{\mathcal{F}} = (F, \hat{R})$. It can be easily seen that the following assertion is true: The identity mapping of $F$ is a homomorphism of $\mathcal{F}$ into $\hat{\mathcal{F}}$ with regard to any even permutation of the set $\{1, 2, 3\}$.

**2. Definition.** *Let $\mathcal{F} = (F, R)$ of domain $I$ and $\mathcal{G} = (G, S)$ of domain $J$ be two relational systems of the same type. Let $F \cap G = \emptyset$ and let $\alpha: I \to J$ be a bijection. The direct sum $\mathcal{F} \overset{\alpha}{+} \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ with regard to a is the relational system $\mathcal{H} = (H, T)$ of domain $\alpha$, where $H = F \cup G$ and $T \subseteq H^\alpha$ is defined in the following way: $h \in H^\alpha$, $h \in T \Leftrightarrow$ there exists $f \in R$ such that $h(x, y) = f(x)$ for all $(x, y) \in \alpha$ or there exists $g \in S$ such that $h(x, y) = g(y)$ for all $(x, y) \in \alpha$.*

**3. Definition.** *Let $\mathcal{F} = (F, R)$ of domain $I$ and $\mathcal{G} = (G, S)$ of domain $J$ be two relational systems of the same type. Let $\alpha: I \to J$ be a bijection. The direct product $\mathcal{F} \overset{\alpha}{\cdot} \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ with regard to $\alpha$ is the relational system $\mathcal{H} = (H, T)$ of domain $\alpha$, where $H = F \times G$ and $T \subseteq H^\alpha$ is defined as follows: $h \in H^\alpha$, $h \in T \Leftrightarrow$ there exist $f \in R$ and $g \in S$ such that $h(x, y) = (f(x), g(y))$ for all $(x, y) \in \alpha$.*

**4. Definition.** *Let $\mathcal{F} = (F, R)$ of domain $I$ and $\mathcal{G} = (G, S)$ of domain $J$ be two relational systems of the same type. Let $\alpha: I \to J$ be a bijection. The direct power $\mathcal{F} \overset{\alpha}{\Delta} \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ with regard to $\alpha$ is the relational system $\mathcal{H} = (H, T)$ of domain $\alpha$ where $H = \mathrm{Hom}_{\alpha^{-1}}(\mathcal{G}, \mathcal{F})$ and $T \subseteq H^\alpha$ is defined in the following way: $h \in H^\alpha$, $h \in T \Leftrightarrow {}^t h \in R$ for all $t \in G$. Here, whenever $t \in G$* and *$h \in H^\alpha$, ${}^t h$ is the mapping ${}^t h$: $I \to F$ defined by ${}^t h(x) = h(x, \alpha(x))(t)$ for all $x \in I$.*

The following two assertions are proved in [6]:

**5. Proposition.** *Let $\mathcal{F} = (F, R)$ of domain $I$ and $\mathcal{G} = (G, S)$ of domain $J$ be two relational systems of the same type. Let $F \cap G = \emptyset$, let $\alpha: I \to J$ be a bijection and let $\mathcal{H} = (H, T) = \mathcal{F} \overset{\alpha}{+} \mathcal{G}$. Then $\mathcal{H}$ the least element (with respect to $\leqq$) in the set of all such relational systems $\mathcal{L}$ of the same domain $\alpha$ and with the same carrier $H$ for which the following two conditions are fulfilled:*

(i) $\mathrm{id}_F \in \mathrm{Hom}_\beta(\mathcal{F}, \mathcal{L})$, *where $\beta$: $I \to \alpha$ is the bijection defined by $\beta(x) = (x, \alpha(x))$ for all $x \in I$,*

(ii) $\mathrm{id}_G \in \mathrm{Hom}_\gamma(\mathcal{G}, \mathcal{L})$, *where $\gamma$: $J \to \alpha$ is the bijection defined by $\gamma(y) = (\alpha^{-1}(y), y)$ for all $y \in J$.*

**6. Proposition.** *Let $\mathcal{F} = (F, R)$ of domain $I$ and $\mathcal{G} = (G, S)$ of domain $J$ be two relational systems of the same type. Let $\alpha: I \to J$ be a bijection and let $\mathcal{H} = (H, T) = \mathcal{F} \overset{\alpha}{\cdot} \mathcal{G}$. Then $\mathcal{H}$ is the greatest element (with respect to $\leqq$) in the set of all such relational systems $\mathcal{L}$ of the same domain $\alpha$ and with the same carrier $H$ for which the following two conditions are fulfilled:*

(i) $\mathrm{pr}_F H \in \mathrm{Hom}_\beta(\mathcal{L}, \mathcal{F})$, *where $\beta$: $\alpha \to I$ is the bijection defined by $\beta(x, y) = x$ for all $(x, y) \in \alpha$,*

(ii) $\mathrm{pr}_G H \in \mathrm{Hom}_\gamma(\mathscr{L}, \mathscr{G})$, *where* $\gamma\colon \alpha \to J$ *is the bijection defined by* $\gamma(x, y) = y$ *for all* $(x, y) \in \alpha$.

Here $\mathrm{pr}_F H$ and $\mathrm{pr}_G H$ denote the projections of $H$ onto $F$ and $G$, respectively.

Now there arises the question whether the direct power of any two relational systems of the same domain can be described in a similar way as the direct sum and product of them are described in sections 5 and 6. The answer is negative in general. However, we discover such a condition for relational systems under which this answer is positive.

**7. Definition.** *Let* $\mathscr{F} = (F, R)$ *be a relational system of domain* $I$.

1. *The system* $\mathscr{F}$ *is called reflexive iff for any constant mapping* $c\colon I \to F$ *there holds* $c \in R$.
2. *Let* $J$ *be a set equipotent with* $I$ *and let* $\alpha\colon I \to J$ *be a bijection. The system* $\mathscr{F}$ *is called diagonal with regard to* $\alpha$ *iff the following condition is valid*:

*If* $\{f_y \mid y \in J\}$ *is a family with* $f_y \in R$ *for all* $y \in J$ *and if the family* $\{g_y \mid y \in J\}$ *of elements of* $F^I$, *defined by* $g_y(x) = f_{\alpha(x)}(\alpha^{-1}(y))$ *for all* $x \in I$ and $y \in J$, *has the property* $g_y \in R$ *for every* $y \in J$, *then putting* $h(x) = f_{\alpha(x)}(x)$ *for each* $x \in I$ *we get* $h \in R$.

**8.** Remark. Let $I$ be a set with card $I = n < \aleph_0$. Then the relations of domain $I$ coincide with the $n$-ary relations, obviously. The homomorphism and the direct operations of addition, multiplication and exponentiation for sets equipped with the $n$-ary relations introduced in [5] correspond to those for relational system with regard to the identity mapping of the set $I$. Also the diagonality of sets with the $n$-ary relations defined in [5] is equivalent with the diagonality of them with regard to the identity mapping of the set $I$. The statements of sections 5 and 6 generalize the corresponding statements of [5].

**9. Lemma.** *Let* $\mathscr{F} = (F, R)$ *of domain* $I$ *and* $\mathscr{G} = (G, S)$ *of domain* $J$ *be relational systems of the same type. Let* $\alpha\colon I \to J$ *be a bijection and let* $\mathscr{H} = (H, T) = \mathscr{F} \overset{\alpha}{\Delta} \mathscr{G}$. *Let* $\beta\colon J \to \alpha$ and $\gamma\colon \beta \to I$ be bijections defined by $\beta(y) = (\alpha^{-1}(y), y)$ *for all* $y \in J$ *and* $\gamma(y, x, y) = x$ *for all* $(y, x, y) \in \beta$. *If* $\mathscr{F}$ *is diagonal with regard to* $\alpha$, *then the mapping* $e\colon G \times H \to F$, *defined by* $e(t, \varphi) = \varphi(t)$ *whenever* $(t, \varphi) \in G \times H$, *fulfils* $e \in \mathrm{Hom}_\gamma(\mathscr{G} \overset{\alpha}{\cdot} \mathscr{H}, \mathscr{F})$.

Proof. Denote $(M, U) = \mathscr{G} \overset{\alpha}{\cdot} \mathscr{H}$ and let $r \in U$. Then there exist $p \in S$ and $q \in T$ such that $r(y, x, y) = (p(y), q(x, y))$ holds for all $(y, x, y) \in \beta$. For each $y \in J$ put $f_y = {}^{p(y)}q$. Since ${}^t q \in R$ for all $t \in G$, we have $f_y \in R$ for all $y \in J$. For each $x \in I$ and $y \in J$ put $g_y(x) = f_{\alpha(x)}(\alpha^{-1}(y))$. There holds $g_y(x) = f_{\alpha(x)}(\alpha^{-1}(y)) = {}^{p(\alpha(x))}q(\alpha^{-1}(y)) = q(\alpha^{-1}(y), y)(p(\alpha(x))) = ((q(\alpha^{-1}(y), y)) \circ p \circ \alpha)(x)$ for all $x \in I$ and $y \in J$, hence $g_y = (q(\alpha^{-1}(y), y)) \circ p \circ \alpha$ for each $y \in J$. As $q(\alpha^{-1}(y), y) \in H = \mathrm{Hom}_{\alpha^{-1}}(\mathscr{G}, \mathscr{F})$, there holds $g_y \in R$ for each $y \in J$. For any $x \in I$ we have

$e(r(\gamma^{-1}(x))) = e(r(\alpha(x), x, \alpha(x))) = e(p(\alpha(x)), q(x, \alpha(x))) = q(x, \alpha(x))(p(\alpha(x))) = {}^{p(\alpha(x))}q(x) = f_{\alpha(x)}(x)$. Now, putting $h(x) = f_{\alpha(x)}(x)$ for each $x \in I$ we get $e(r(\gamma^{-1}(x))) = h(x)$ for each $x \in I$, thus $e \circ r \times \gamma^{-1} = h$. If $\mathscr{F}$ is diagonal with regard to $\alpha$, then $e \circ r \circ \gamma^{-1} = h \in R$ and consequently $e \in \mathrm{Hom}_\gamma(\mathscr{G} \overset{\beta}{\cdot} \mathscr{H}, \mathscr{F})$.

**10. Theorem.** *Let $\mathscr{F} = (F, R)$ of domain $I$ and $\mathscr{G} = (G, S)$ of domain $J$ be two relational systems of the same type. Let $\alpha$: $I \to J$ be a bijection. Let $\beta$: $J \to \alpha$ and $\gamma$: $\beta \to I$ be the bijections defined by $\beta(y) = (\alpha^{-1}(y), y)$ for all $y \in J$ and $\gamma(y, x, y) = x$ for all $(y, x, y) \in \beta$. If $\mathscr{F}$ is diagonal with regard to $\alpha$ and $\mathscr{G}$ is reflexive, then the direct power $\mathscr{H} = (H, T) = \mathscr{F} \overset{\alpha}{\Delta} \mathscr{G}$ is the least element (with respect to $\leqq$) in the set of all such relational systems $\mathscr{L}$ of the same domain $\alpha$ and with the same carrier $H$ which have the following property:*

*For any relational system $\mathscr{M} = (M, U)$ of domain $\alpha$ and for any homomorphism $\psi \in \mathrm{Hom}_\gamma(\mathscr{G} \overset{\beta}{\cdot} \mathscr{M}, \mathscr{F})$ the mapping $\psi^*$: $M \to H$, defined by $\psi^*(u)(t) = \psi(t, u)$ whenever $u \in M$* and *$t \in G$, fulfils $\psi^* \in \mathrm{Hom}(\mathscr{M}, \mathscr{L})$.*

Proof. Let $r \in U$ be a mapping and denote $(N, V) = \mathscr{G} \overset{\beta}{\cdot} \mathscr{M}$. For any $t \in G$ and any $(y, x, y) \in \beta$ put $s_t(y, x, y) = (t, r(x, y))$. Then the reflexivity of $\mathscr{G}$ implies $s_t \in V$ for each $t \in G$. Thus, for each $t \in G$ there holds $\psi \circ s_t \circ \gamma^{-1} \in R$. For any elements $r \in U$, $t \in G$ and $x \in I$ we have ${}^t(\psi^* \circ r)(x) = (\psi^* \circ r)(x, \alpha(x))(t) = \psi^*(r(x, \alpha(x)))(t) = \psi(t, r(x, \alpha(x))) = \psi(s_t(\alpha(x), x, \alpha(x))) = \psi(s_t(\gamma^{-1}(x)))$. Hence ${}^t(\psi^* \circ r) = \psi \circ s_t \circ \gamma^{-1} \in R$ for each $t \in G$. Consequently $\psi^* \circ r \in T$, which yields $\psi^* \in \mathrm{Hom}(\mathscr{M}, \mathscr{H})$. Thus $\mathscr{H}$ has the property of the theorem.

Let $\mathscr{L} = (H, W)$ be a relational system of domain $\alpha$ fulfilling the property of the theorem. Put $e(t, \varphi) = \varphi(t)$ for each $(t, \varphi) \in G \times H$. Since $e \in \mathrm{Hom}_\gamma(\mathscr{G} \overset{\beta}{\cdot} \mathscr{H}, \mathscr{F})$ by Lemma, we have $e^* \in \mathrm{Hom}(\mathscr{H}, \mathscr{L})$. But $e^*(\varphi)(t) = e(t, \varphi) = \varphi(t)$ for all $t \in G$ and $\varphi \in H$. Therefore $e^* = \mathrm{id}_H$ and hence $\mathrm{id}_H \in \mathrm{Hom}(\mathscr{H}, \mathscr{L})$. Thus $\mathscr{H} \leqq \mathscr{L}$ and the proof is complete.

It $\mathscr{F} = (F, R)$ and $\mathscr{G} = (G, S)$ are sets $F$ and $G$ equipped with binary relations $R$ and $S$, respectively, then by $\mathscr{F} \cdot \mathscr{G}$ and $\mathscr{F}^{\mathscr{G}}$ we denote their usual direct product and power ([5]). It is clear what is meant by a homomorphism of $\mathscr{F}$ into $\mathscr{G}$. We obtain:

**11. Corollary.** *Let $\mathscr{F} = (F, R)$ and $\mathscr{G} = (G, S)$ be sets $F$ and $G$ equipped with binary relations $R$ and $S$, respectively, and let $(H, T) = \mathscr{F}^{\mathscr{G}}$. If $R$ is transitive and $S$ is reflexive, then $T$ is the least binary relation (with respect to the set inclusion) in the set of all such binary relations $U$ on $H$ which have the following property:*

*For any set equipped with a binary relation $\mathscr{M} = (M, V)$ and for any homomorphism $\psi$ of $\mathscr{G} \cdot \mathscr{M}$ into $\mathscr{F}$ the mapping $\psi^*$: $M \to H$, defined by $\psi^*(m)(h) = \psi(h, m)$ whenever $m \in M$ and $h \in H$, is a homomorphism of $\mathscr{M}$ into $(H, U)$.*

Proof. In [5] it is shown that a set equipped with a binary relation $(F, R)$ is diagonal iff $R$ is transitive. Regarding the section 8, the Corollary follows from the Theorem.

**12.** Remark. For preordered sets $\mathscr{F}$ and $\mathscr{G}$ the statement of Corollary follows also from the fact that the category of preordered sets is a cartesian closed topological category — see [4].

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## О ПРЯМОЙ СТЕПЕНИ РЕЛЯЦИОННЫХ СИСТЕМ

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### Резюме

Реляционная система — это упорядоченная пара $(F, R)$, где $F$ — непустое множество и $R$ — множество отображений какого-нибудь непустого множества в $F$. Для этих реляционных систем определяются три прямые бинарные операции суммы, произведения и возведения в степень. В статье надо достаточное условие для того, чтобы возведение в степень для реляционных систем обладало определенным свойством, которое является характеристическим для возведения в степень в декартовско замкнутых топологических категориях.