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A THEOREM OF ŠARKOVSKII CHARACTERIZING CONTINUOUS MAPS OF ZERO TOPOLOGICAL ENTROPY

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1. Introduction

Throughout this paper f will be a continuous map of the compact real interval I to itself.

For any non-negative integer n let f^n be the nth *iterate* of f (i.e., $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for every x). A $p \in I$ is a *periodic point* of f of *period* n, if n is the least positive integer with $f^n(p) = p$. For any x, $\omega_f(x)$ is the limit set of the sequene $\{f^n(x)\}_{n=0}^{\infty}$, and we call it the ω-*limit set of* x.

In 1966 the following result was proved (cf. [8, p.71]).

1.1 Theorem (A. N. Šarkovskii). *The next two conditions are equivalent*:

C1: *f has a periodic point of period different from 2^n, for any n.*

C2: *For some x, $\omega_f(x)$ is infinite and contains a periodic point.*

This result is fundamental and very strong, and implies a number of important consequences (cf., e.g., [3], [10], [5] or (6]). However, the original proof is very long and even incomplete. The main aim of our paper is to give a simple, new proof.

1.2. Remark. C1 *is equivalent to each of the conditions*

C3: *f has a horseshoe, i.e., there are disjoint compact intervals U, V and positive integers m, n such that*

$$f^m(U) \cap f^n(V) \supset U \cup V$$

C4: *f has a homoclinic point, i.e., a point x such that there is a periodic point $p \neq x$ of f of period n with the following properties: $x \neq p$, $f^{kn}(x) = p$ for some positive integer k, and for any neighbourhood U of p there is some m with $x \in f^{mn}(U)$.*

This is also Šarkovskii's result [7] and [9]. A simple proof was given later by Block [2]. Note that this result is very strong, too (cf., e.g., [3], [4] or [10]) and we will use it in the sequel.

Recall that C1, and hence also the other conditions, are equivalent the statement that f has a positive topological entropy (Misiurewicz [4]).

2. Proof of Theorem 1.1

We begin with the following

2.1. Proposition. C2 $\Rightarrow$ C1

To prove this we use a sequence of lemmas. Till the end of the proof of 2.1, we assume that $\omega = \omega_f(x)$ is infinite and denote $a = \min \omega$, $b = \max \omega$ and $x_n = f^n(x)$ for every n.

2.2 Lemma. *There is a* $c \in (a, b)$ *with* $f(c) = c$.

Proof. It suffices to show that $f(u) > u$ and $f(v) < v$ for some $u, v \in (a, b)$. Assume that, e.g., $f(u) < u$ for every $u \in (a, b)$. Then $a < x_n < b$ implies $x_{n+1} < x_n$. Hence for any small $\varepsilon > 0$ there is a sequence $n(1) < n(2) < \ldots$ of integers with $x_{n(i)} \leqslant a$ and $x_{n(i)+1} \geqslant b - \varepsilon$, for every i. Since $\lim x_{n(i)} = a$ when $i \to \infty$, we have $f(a) \geqslant b - \varepsilon$. By the continuity of f, if $u > a$ is near to a, then $f(u) > u$ — a contradiction. □

2.3. Lemma. *If* $d < c$ *are fixed points of* f *contained in* (a, b) *and if there are* m, n *with* $d < x_m < x_n < c$, *then* C3 *is true.*

Proof. Choose positive integers k, s with $x_{m+k} > c$ and $x_{n+s} < d$. Then $f^k([d, x_m]) \cap f^s([x_n, c]) \supset [d, c]$. □

2.4 Lemma. *Let* $p \in \omega$ *with* $f(p) = p$. *If for some* $d \in [a, b]$, $d \neq p$, $f(d) = p$, *then* C4 *is true.*

Proof. If $d \in (a, b)$, the set $q = d$, otherwise let $q \in (a, b)$ be such that $f(q) = d$ or $f^2(q) = d$; this is possible since $f(\omega) = \omega$. Then for any neighbourhood U of p there is an n with $q \in f^n(U)$, i.e., q is a homoclinic point of f. □

2.5 Lemma. *If* $f(a) = a$ *and* a *is an isolated point of* ω *then* C4 *is true.*

Proof. Choose a neighbourhood U of a such that $\bar{U} \cap \omega = \{a\}$. Let $\{n(k)\}$ be the increasing sequence of all positive integers with $x_{n(k)} \in U$ and $x_{n(k)-1} \notin U$. Since a is isolated we have $\lim x_{n(k)} = a$ for $k \to \infty$. Let d be a limit point of $\{x_{n(k)-1}\}$. Then $d \in \omega$ and $f(d) = a$. Now **2.4** applies with $p = a$. □

2.6 Lemma. *If* $f(a) = a$ *or* $f(b) = b$, *then* C1 *is true.*

Proof. Let, e.g., $f(a) = a$. By **2.2** there is a fixed point $c \in (a, b)$. By **2.5** we may assume that there is some $p \in (a, c) \cap \omega$ with $(p, c) \cap \omega \neq \emptyset$. If $f(u) \leqslant u$ for all $u \in [a, p]$, then there is a sequence $\{x_{n(k)}\}$ converging from the left to a such that $x_{n(k)+1} > p$. Hence $f(a) \geqslant p$, which is imposible.

If $f(u) \geqslant u$ for all $u \in [a, p]$, then there is a sequence $\{x_{m(k)}\}$ converging to a with $x_{m(k)-1} > p$ for any k. Let d be a limit point of $\{x_{m(k)-1}\}$. Then $f(d) = a$, and by **2.4** and **1.2**, C1 is true.

Finally, if $f(u) > u$ and $f(v) < v$ for some $u, v \in (a, p)$ then there is a fixed point $d \in (a, p)$. Now since $p \in \omega$, we can find m, n with $d < x_m < x_n < c$ and **2.3** and **1.2** applies. □

2.7 Proof of 2.1. Let $p \in \omega$ be a periodic point of f. We may assume that $f(p) = p$ (otherwise replace f by a suitable f^m). By **2.6**, $p \in (a, b)$. By **2.4**, $f(a) \neq p$.

If $f(a) < p$, then by **2.4** and the continuity of f there is some $\delta > 0$ such that $f(y) \leqslant p$ for every $y \in (a - \delta, p]$. Consequently, $p = b$ must be the endpoint of ω — a contradiction.

Thus $f(a) > p$, and by **2.4** and the continuity of f we have $f(y) \geqslant p$ for any $y \in (a - \delta, p]$ if $\delta > 0$ is small. Repeating this argument (and using the symmetry) we can easily see that for every n sufficiently large, $f^2(x_n) < p$ iff $x_n < p$. Now let $g = f^2$. Then each of the sets $\omega_g(x)$, $\omega_g(f(x))$ is infinite and p is an endpoint of at least one of them. By **2.6** applied to g, g has a periodic point of period $\neq 2^n$ for any n. Clearly the same is true for f. □

Now it remains to prove the second part of Theorem 1.1. (Note that in Šarkovskii's original paper [8] this proof is omitted.) In view of **1.2** it suffices to prove the following

2.8 proposition. C3 ⇒ C2

Proof. Let C3 be true. Let $g = f^{mn}$. Since g is continuous there is a sequence

(1) $$U = U_0 \supset U_1 \supset U_2 \supset \dots, \qquad U_0 \neq U_1 \neq U_2 \neq \dots$$

of minimal closed intervals such that

(2) $$g(U_{k+1}) = U_k \qquad \text{for every } k.$$

Denote by $\{J_k\}_{k=0}^{\infty}$ the sequence

(3) $$U_0 V U_1 \, U_0 V U_2 U_1 \; U_0 V U_3 \dots U_0 V U_k \dots U_0 V \dots$$

Since $g(J_k) \supset J_{k+1}$ for any k, there is clearly a point $x \in U_0$ such that $g^n(x) \in J_n$ for every $n \geqslant 0$. Choose $y \in \omega_g(x) \cap V \neq \emptyset$. By (3) y cannot be periodic. Since every finite ω-limit set contains only periodic points (cf. [1]; however, this result is elementary and easily provable), $\omega_g(x)$ must be infinite.

It remains to prove that $\omega_g(x)$ contains a periodic point (of g, and hence also of f) since $\omega_f(x) \supset \omega_g(x)$. Put

(4) $$[p, q] = \bigcap_{k=0}^{\infty} U_k .$$

By (1) and (2),

(5) $$[p, q] = g([p, q])$$

is invariant. Hence $g^k(x) \notin [p, q]$ for any k. On the other hand, for every k there is some $n(k)$ with $g^{n(k)}(x) \in U_k$. This along with (5) implies that $p \in \omega_g(x)$ or $q \in \omega_g(x)$. Now the result follows from the next lemma, since $\omega_g(x)$ is invariant. □

2.9 Lemma. $g(\{p, q\}) \subset \{p, q\}$.

Proof. Assume that, e.g., $g(p) \notin \{p, q\}$. Then by (5) there is a neighbourhood $O(p)$ of p such that

$$g(O(p)) \subset (p, q). \tag{6}$$

Consider the following two cases A and B.

A. $g(q) \neq q$. Since by (5), $g(q) \in [p, q)$, there is a neighbourhood $O(q)$ of f with $g(O(q)) \subset O(p) \cup [p, q]$. Take a k such that $U_{k+2} \subset O(p) \cup [p, q] \cup O(q)$. Then by (6), $U_k = g^2(U_{k+2}) \subset g(O(p) \cup [p, q]) \subset [p, q]$, contrary to (1) and (4).

B. $g(q) = q$. Set $U_k = [a_k, b_k]$ and take a k with $a_k \in O(p)$. Let $y \in O(p)$, $y > a_k$. Then $g([a_k, y]) \subset (p, q)$, hence by (2) and (4), $g([y, b_k]) = U_{k-1}$, contrary to the minimality of U_k. □

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ОДНА ТЕОРЕМА ШАРКОВСКОГО, ХАРАКТЕРИЗУЮЩАЯ НЕПРЕРЫВНЫЕ ОТОБРАЖЕНИЯ С НУЛЕВОЙ ТОПОЛОГИЧЕСКОЙ ЭНТРОПИЕЙ

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Резюме

Статья содержит новое, краткое доказательство следующего утверждения А. Н. Шарковского из 1966 г.: Произвольное непрерывное отображение отрезка обладает периодической точкой, период которой не является степенью 2 тогра и только тогда, когда оно обладает бесконечным ω-предельным множеством, содержащим периодическию точку.