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CONCEPT LATTICES UNIFY THE BIRKHOFF REPRESENTATION THEOREMS

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ABSTRACT. A unified approach to the Birkhoff representation theorems is given by means of concept lattices.

1. Preliminaries

The Birkhoff representation of finite distributive lattices by the posets of their nonzero $\vee$-irreducible elements is frequently used in lattice theory. Another Birkhoff theorem gives as representation of the lattice of all equivalences on an n-element set by the subalgebra lattice of the Boolean algebra $\mathbf{2}^n$. The aim of this paper is to show that both the mentioned representations can be obtained in the same way by means of concept lattices introduced by R. Wille. Moreover a representation of the lattice of all quasiorders on a finite nonvoid set is given. A number of corollaries follows.

To make this paper selfcontained we recall some definitions and notations used in the sequel. By a *quasiorder* is meant a reflexive transitive binary relation, an *equivalence* is a symmetric quasiorder, and an *order* is an antisymmetric quasiorder. Let R be a binary relation on a set G. Then $\neg R$ is an abbreviation of $G \times G \backslash R$; the symbol R^{-1} denotes the relation $\{\langle x, y\rangle \in G \times G;\ \langle y, x\rangle \in R\}$.

Recall further from [5] that a *context* is a triple $\langle G, M, r\rangle$ where G, M are finite nonvoid sets and r is a correspondence from G to M, i.e. $r \subseteq G \times M$. Denote by $\mathbf{B}(X)$ the set of all subsets of a set X. One can easily verify that the pair of mappings $\mathbf{B}(G) \underset{t}{\overset{s}{\rightleftarrows}} \mathbf{B}(M)$ introduced by the rules

$$s(H) = \{m \in M;\ \langle g, m\rangle \in r \text{ for all } g \in H\},\ H \in \mathbf{B}(G), \text{ and}$$
$$t(N) = \{g \in G;\ \langle g, m\rangle \in r \text{ for all } m \in N\}, N \in \mathbf{B}(M),$$

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establishes a Galois connection between the posets $\langle \boldsymbol{B}(M), \subseteq\rangle$ and $\langle \boldsymbol{B}(G), \subseteq\rangle$. In virtue of this fact a *concept* of the context $\langle G, M, r\rangle$ is defined as a pair $\langle A, B\rangle$, $A \subseteq G$, $B \subseteq M$, with the property $A = t(B)$ and $s(A) = B$. A and B are called the extent and the intent of the concept $\langle A, B\rangle$, respectively. It is well known, see [5], that all concepts form the *concept lattice* $\mathfrak{B}(G, M, r)$ in which $\langle A, B\rangle \wedge \langle C, D\rangle = \langle A \cap C, s(A \cap C)\rangle$, and $\langle A, B\rangle \vee \langle C, D\rangle = \langle t(B \cap D), B \cap D\rangle$ hold for any concepts $\langle A, B\rangle, \langle C, D\rangle \in \mathfrak{B}(G, M, R)$. Particularly if S is a sublattice of (G, M, r) we can assign the set $ext_S(g) = \cap\{A \subseteq G; g \in A$ and $\langle A, s(A)\rangle \in S\}$ to any element $g \in G$.

2. Representation

Definition 1. Let Q be a quasiorder on a set G. A subset $\emptyset \subseteq A \subseteq G$ is called a quasiorder ideal of $\langle G, Q\rangle$ whenever $g \in G$, $a \in A$, and $\langle g, a\rangle \in Q$ imply $g \in A$.

Lemma 1. *Let Q be a quasiorder on a finite set G, s: $\boldsymbol{B}(G) \to \boldsymbol{B}(G)$ a mapping determinated by the context $\langle G, G, \neg Q^{-1}\rangle$. Then $s(A) = G\backslash A$ holds for any quasiorder ideal A of $\langle G, Q\rangle$.*

Proof. The equivalence "$g \in A$ iff $\langle g, a\rangle \in Q$ for some $a \in A$" is clear. Consequently "$g \in G\backslash A$ iff $\langle a, g\rangle \in \neg Q^{-1}$ for all $a \in A$", as required.

Our next lemma states that Proposition 1 from [6] remains true for quasiorders.

Lemma 2. *Let Q be a quasiorder on a finite set G and let $A, B, C, D \subseteq G$. Then*

(i) *$\langle A, B\rangle$ is a concept of the context $\langle G, G, \neg Q^{-1}\rangle$ iff A is a quasiorder ideal of $\langle G, Q\rangle$ and $B = G\backslash A$.*

(ii) *$\langle A, B\rangle \wedge \langle C, D\rangle = \langle A \cap C, B \cup D\rangle$, and*

$\langle A, B\rangle \vee \langle C, D\rangle = \langle A \cup C, B \cap D\rangle$ hold for any concepts $\langle A, B\rangle, \langle C, D\rangle \in \mathfrak{B}(G, G, \neg Q^{-1})$.

Proof. (i) Let $\langle A, B\rangle$ be a concept of the context $\langle G, G, \neg Q^{-1}\rangle$. Then

$$A = t(B) = \bigcap_{b \in B} t(\{b\}) = \bigcap_{b \in B} \{g \in G;\ \langle g, b\rangle \in \neg Q^{-1}\} = \bigcap_{b \in B} \{g \in G;\ \langle b, g\rangle \notin Q\}.$$

Since any subset $\{g \in G; \langle b, g\rangle \notin Q\}$ is a quasiorder ideal of $\langle G, Q\rangle$ the set A has the same property. Thus $s(A) = G\backslash A$, by Lemma 1. Combining this equality with the hypothesis $s(A) = B$ the required conclusion $B = G\backslash A$ follows.

Conversely suppose that A is a quasiorder ideal of $\langle G, Q\rangle$ and $B = G\backslash A$. Then $s(A) = B$, by Lemma 1. Since B is an quasiorder ideal of $\langle G, Q^{-1}\rangle$ we have $t(B) = A$, by Lemma 1 again.

(ii) is an immediate corollary of part (i) of this Lemma.

Quasiorder ideals of $\langle G, Q\rangle$ evidently form a closure system on G. This fact ensures the existence of the least quasiorder ideal $(g]\,Q$ of $\langle G, Q\rangle$ containing an element $g \in G$.

Lemma 3. *Let Q_1, Q_2 be quasiorders on a finite set G. Then $Q_1 \supseteq Q_2$ iff $\mathfrak{B}(G, G, \neg Q_1^{-1})$ is a* (0, 1)*-sublattice of* $\mathfrak{B}(G, G, \neg Q_2^{-1})$.

Proof. First suppose that $Q_1 \supseteq Q_2$ holds. Let $\langle A, B\rangle$ be an arbitrary concept of the context $\langle G, G, \neg Q_1^{-1}\rangle$. By Lemma 2 (i), A is a quasiorder ideal of $\langle G, Q_1\rangle$ and $B = G\backslash A$. A is also a quasiorder ideal of $\langle G, Q_2\rangle$ since $Q_1 \supseteq Q_2$. Thus $\langle A, B\rangle$ is a concept of the context $\langle G, G, \neg Q_2^{-1}\rangle$, see Lemma 2 (i) again. This fact together with Lemma 2 (ii) establishes that $\mathfrak{B}(G, G, \neg Q_1^{-1})$ is a (0, 1)-sublattice of $\mathfrak{B}(G, G, \neg Q_2^{-1})$.

Conversely let $\mathfrak{B}(G, G, \neg Q_1^{-1})$ be a (0, 1)-sublattice of $\mathfrak{B}(G, G, \neg Q_2^{-1})$. Assume that $\langle x, y\rangle \in Q_2$. Then $x \in (y]Q_2$. In virtue of the hypothesis $(y]Q_2 \subseteq \subseteq (y]Q_1$ holds. Consequently $x \in (y]Q_1$, i.e. $\langle x, y\rangle \in Q_1$. Hence $Q_1 \supseteq Q_2$ and the proof of Lemma 3 is complete.

Theorem 1. *The lattice $\mathbf{Q}(n)$, $n \geqslant 1$, of all quasiorders on an n-element set G is dually isomorphic to the lattice of all* (0, 1)*-sublattices of the Boolean algebra* $\mathfrak{B}(G, G, \neg =) \cong \mathbf{2}^n$. *The dual isomorphism is given by $Q \mapsto \mathfrak{B}(G, G, \neg Q^{-1})$ for any quasiorder Q on G.*

Proof. It follows directly from Lemma 3 that $\mathfrak{B}(G, G, \neg Q^{-1})$ is a (0, 1)-sublattice of $\mathfrak{B}(G, G, \neg =)$ for any quasiorder Q on G. The isomorphism $\mathfrak{B}(G, G, \neg =) \cong \mathbf{2}^n$ is evident, see [5].

Conversely let S be an arbitrary sublattice of the Boolean algebra $\mathfrak{B}(G, G, \neg =)$. Introduce the binary relation Q on G via $\langle x, y\rangle \in Q$ whenever $x \in ext_S(y)$. Apparently Q is a quasiorder on G. Now it is a routine to verify that $\mathfrak{B}(G, G, \neg Q^{-1}) = S$.

Lemma 3 completes the proof.

Further we restrict out attention to equivalence relations. In this way the Birkhoff representation of the lattice of all equivalences on a finite set is obtained. First we reformulate Lemma 2(i) for equivalence relations.

Lemma 4. *Let Θ be an equivalence relation on a finite set G and let $A, B \subseteq G$. Then $\langle A, B\rangle$ is a concept of the context $\langle G, G, \neg\Theta\rangle$ iff $A = \bigcup_{a \in A} [a]\Theta$ and $B = G\backslash A$.*

Proof. Apply Lemma 2 (i).

Theorem 2. *The lattice $\mathbf{E}(n)$, $n \geqslant 1$, of all equivalences on an n-element set G is dually isomorphic to the lattice of all Boolean subalgebras of $\mathfrak{B}(G, G, \neg =)$. The dual isomorphism is given by $\Theta \mapsto \mathfrak{B}(G, G, \neg\Theta)$ for any equivalence Θ on G.*

Proof. It follows directly from Lemma 4 that $\mathfrak{B}(G, G, \neg\Theta)$ is a Boolean subalgebra of $\mathfrak{B}(G, G, \neg =)$ for any equivalence Θ on G.

Conversely let S be an arbitrary Boolean subalgebra of $\mathfrak{B}(G, G, \neg =)$. Define a binary relation R on G via $\langle x, y\rangle \in R$ whenever $x \in ext_S(y)$. Clearly R is a quasiorder on G. Suppose that $\langle x, y\rangle \in R$ and $\langle y, x\rangle \notin R$. Then $ext_S(x) \subset$

$\subset ext_S(y)$ and $y \in G \backslash ext_S(x)$. Since $G \backslash ext_S(x)$ is an extent of some element of S we find that $ext_S(y) \subseteq G \backslash ext_S(x)$. Altogether $x \in ext_S(x) \subset ext_S(y) \subseteq G \backslash ext_S(x)$, a contradiction. This establishes the symmetry of R. Apparently $\mathfrak{B}(G, G, \neg R) = S$.

The rest of the proof follows from Lemma 3.

It remains to restrict Theorem 1 on orders. From [2] we quote

Definition 2. A $\vee$*-nearlattice* is a lower semilattice in which any two elements have a supremum whenever they are bounded above. The concept of a $\wedge$*-nearlattice* is introduced dually.

Theorem 3. *The* $\vee$*-nearlattice* $\boldsymbol{O}(n)$, $n \geqslant 1$, *of all orders on an n-elements set G is dually isomorphic to the* $\wedge$*-nearlattice of all sublattices of* $\mathfrak{B}(G, G, \neg =)$ *having the length n. The dual isomorphism is given by* $O \mapsto \mathfrak{B}(G, G, \neg O^{-1})$ *for any order O on G.*

Proof. By Lemma 3, $\mathfrak{B}(G, G, \neg O^{-1})$ is a sublattice of $\mathfrak{B}(G, G, \neg)$. As claimed in Lemma 2, $\mathfrak{B}(G, G, \neg O^{-1})$ is isomorphic to the lattice of all order ideals of $\langle G, O\rangle$. Finally it is wel known, see [1], that the lattice of all order ideals of $\langle G, O\rangle$ has the length n.

Conversely, let S be a sublattice of $\mathfrak{B}(G, G, \neg =)$ having the length n. Define a binary relation R on G via $\langle x, y\rangle \in R$ whenever $x \in ext_S(y)$. Again by [1], the lattice formed by extents of all concepts from S has exactly n nonzero $\vee$-irreducible elements. Since $\{ext_S(g);\ g \in G\}$ is clearly the set of all nonzero $\vee$-irreducible elements of this lattice the assumptions $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$ imply $ext_S(x) = ext_S(y)$ from which the desired equality $x = y$ follows.

As usual, Lemma 3 completes the proof.

3. Corollaries

(1) Let $\leqslant$ be an order on an n-element set G, $n \geqslant 1$, and let $\geqslant\ = \ \leqslant^{-1}$. Consider an arbitrary element $\langle A, B\rangle \in \mathfrak{B}(G, G, \neg \leqslant) \cap \mathfrak{B}(G, G, \neg \geqslant)$. Lemma 2 (i) applied to $\langle A, B\rangle \in \mathfrak{B}(G, G, \neg \leqslant)$ yields that A is an order ideal of $\langle G, \geqslant\rangle$ and $B = G\ A$, whence B is an order ideal of $\langle G, \leqslant\rangle$. Analogously $\langle A, B\rangle \in \mathfrak{B}(G, G, \neg \geqslant)$ implies that A is an order ideal of $\langle G, \leqslant\rangle$ and B is an order ideal of $\langle G, \geqslant\rangle$. Altogether we find that $\langle B, A\rangle$, the complement of $\langle A, B\rangle$ in $\mathfrak{B}(G, G, \neg =)$, belongs to $\mathfrak{B}(G, G, \neg \leqslant) \cap \mathfrak{B}(G, G, \neg \geqslant)$, which proves that $\mathfrak{B}(G, G, \neg \leqslant) \cap \mathfrak{B}(G, G, \neg \geqslant)$ is a Boolean subalgebra of $\mathfrak{B}(G, G, \neg =)$. By Theorem 2 there is an equivalence Θ on G such that $\mathfrak{B}(G, G, \neg \Theta) = \mathfrak{B}(G, G, \neg \leqslant) \cap \mathfrak{B}(G, G, \neg \geqslant)$ or, dually, $\Theta = \ \leqslant \vee \geqslant$ in the lattice $\boldsymbol{Q}(n)$. This means that $\Theta = \Theta(\leqslant)$, the least equivalence on G containing $\leqslant$, and so $\mathfrak{B}(G, G, \neg \Theta(\leqslant))$ is the greatest Boolean subalgebra in $\mathfrak{B}(G, G, \neg \leqslant)$, i.e. $\mathfrak{B}(G, G, \neg \Theta(\leqslant))$ is the centre of $\mathfrak{B}(G, G, \neg \leqslant)$, see [1] for this concept.

Conversely, let L be a nontrivial finite distributive lattice and let $\langle J(L), \leqslant\rangle$ denote the poset of nonzero $\vee$-irreducible elements of L with order $\leqslant$ induced from L. As shown, e.g., in the proof of Theorem 3 there is an isomorphism $\alpha: \mathfrak{B}(J(L), J(L), \neg \geqslant) \mapsto L$. Then the centre $Cen\, L$ of L can be obtained by formula $Cen\, L = \alpha(\mathfrak{B}(J(L), J(L), \neg\, \Theta(\leqslant))$, where $\Theta(\leqslant)$ denotes the equivalence on $J(L)$ generated by $\leqslant$.

(2) Consider the Boolean algebra $\boldsymbol{2}^n$, $n > 1$, and a (0, 1)-sublattice S of $\boldsymbol{2}^n$. If G denotes an n-element set, then there is an isomorphism $\beta: \mathfrak{B}(G, G, \neg =) \mapsto \boldsymbol{2}^n$. The (0, 1)-sublattice $\beta^{-1}(S)$ of $\mathfrak{B}(G, G, \neg =)$ corresponds to some quasiorder Q on G, see our Theorem 1. Then the Boolean subalgebra of $\mathfrak{B}(G, G, \neg =)$ generated by $\beta^{-1}(S)$ corresponds to the greatest equivalence Θ on G with the property $\Theta \subseteq Q$. Evidently $\Theta = Q \cap Q^{-1}$. In this way the Boolean subalgebra generated by a (0, 1)-sublattice is the other side of the well-known construction of the greatest equivalence in a given quasiorder.

(3) For an integer $n \geqslant 1$, $\boldsymbol{E}(n)$ is a sublattice of $\boldsymbol{Q}(n)$ and the joins of $\boldsymbol{O}(n)$ (whenever they exist) coincide with the joins in the lattice $\boldsymbol{Q}(n)$. In fact meets in $\boldsymbol{Q}(n)$, $\boldsymbol{E}(n)$, and $\boldsymbol{O}(n)$ are given by set intersections. Joins in $\boldsymbol{Q}(n)$, $\boldsymbol{E}(n)$, and $\boldsymbol{O}(n)$ correspond to the intersections of (0, 1)-sublattices, Boolean subalgebras, and sublattices of the length n, respectively.

(4) It is easily seen that theorems from Section 2 can be reformulated as follows:

Theorem 1′. $\boldsymbol{Q}(n)$, $n \geqslant 1$, *is dually isomorphic to the lattice of all topologies on an n-element set.*

Theorem 2′. $\boldsymbol{E}(n)$, $n \geqslant 1$, *is dually isomorphic to the lattice of all topologies on an n-element set having clopen sets only.*

Theorem 3′. $\boldsymbol{O}(n)$, $n \geqslant 1$, *is dually isomorphic to the* $\wedge$*-nearlattice of all* T_0 *topologies on an n-element set.*

Consequently, using results of J. Hartmanis [3; Thm 3 and Corollary 1, p. 550] we immediately get that $\boldsymbol{Q}(n)$, $n \geqslant 1$, is a complemented lattice, moreover a quasiorder $Q \in \boldsymbol{Q}(n)$ has the unique complement in $\boldsymbol{Q}(n)$ iff $Q = \omega$ *or* $Q = \iota$.

(5) There are exactly $2^n - 2$ minimal (0, 1)-sublattices of the Boolean algebra 2^n, $n \geqslant 1$, i.e. there are exactly $2^n - 2$ maximal quasiorders on the set $\{1, \ldots, n\}$. Any quasiorder is an intersection of some maximal quasiorders.

On the other hand one can easily verify that any maximal sublattice of $\boldsymbol{2}^n$, $n > 1$, is a (0, 1)-sublattice. Hence the set of maximal sublattices in $\boldsymbol{2}^n$ is determined by the set of minimal quasiorders on $\{1, \ldots, n\}$, see our Theorem 1. These minimal quasiorders, denotes by $O(1, 2), \ldots, O(n, n-1)$, are defined by the incidence matrices in Tab. 1.

It is evident that $O(1, 2), \ldots, O(n, n-1)$ are orders. Since any quasiorder on an n-element set is generated by a suitable subset of $O(1, 2), \ldots, O(n, n-1)$ we

Tab. 1

$O(1,2)$	1	2	...	n
1	1	1	...	0
2	0	1	...	0
$\vdots$	$\vdots$			$\vdots$
n	0	0	...	1

, ...,

$O(n, n-1)$	1	2	...	n
1	1	0	...	0
2	0	1	...	0
$\vdots$	$\vdots$			$\vdots$
n	0	...	1	1

get that any (0, 1)-sublattice of $\mathbf{2}^n$ is an intersection of some maximal sublattices of $\mathbf{2}^n$.

Example. For $n = 3$, the Boolean algebra $\mathbf{2}^3$

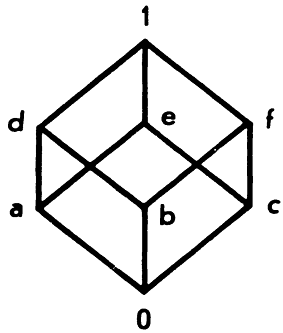


Fig. 1.

has $n^2 - n = 6$ maximal sublattices depicted in Fig. 2.

Our Example shows that any maximal sublattice of the Boolean algebra $\mathbf{2}^3$ is isomorphic to the lattice $\mathbf{3} \times \mathbf{2}$. In general any maximal sublattice of $\mathbf{2}^n$, $n > 1$, is isomorphic to the lattice $\mathbf{3} \times \mathbf{2}^{n-2}$. This is an immediate consequence of the fact that any maximal sublattice of $\mathbf{2}^n$ is isomorphic to the lattice of order ideals of $\langle\{1, \ldots, n\}, O(i, j)\rangle$ for some $i \neq j \in \{1, \ldots, n\}$. For illustration the poset $\langle\{1, \ldots, n\}, O(1, 2)\rangle$ has the following lattice of order ideals

Now the formula $M \cong \mathbf{3} \times \mathbf{2}^{n-2}$ for any maximal sublattice M of $\mathbf{2}^n$, $n > 1$, improves the following results of H. Sharp and D. Steven from [4]:

(i) $|M| = |\mathbf{3} \times \mathbf{2}^{n-2}| = |\mathbf{3}| . |\mathbf{2}^{n-2}| = 3 . 2^{n-2} = (3/4) . 2^n$;

(ii) $l(M) = l(\mathbf{3} \times \mathbf{2}^{n-2}) = l(\mathbf{3}) + l(\mathbf{2}^{n-2}) = 2 + (n-2) = n$.

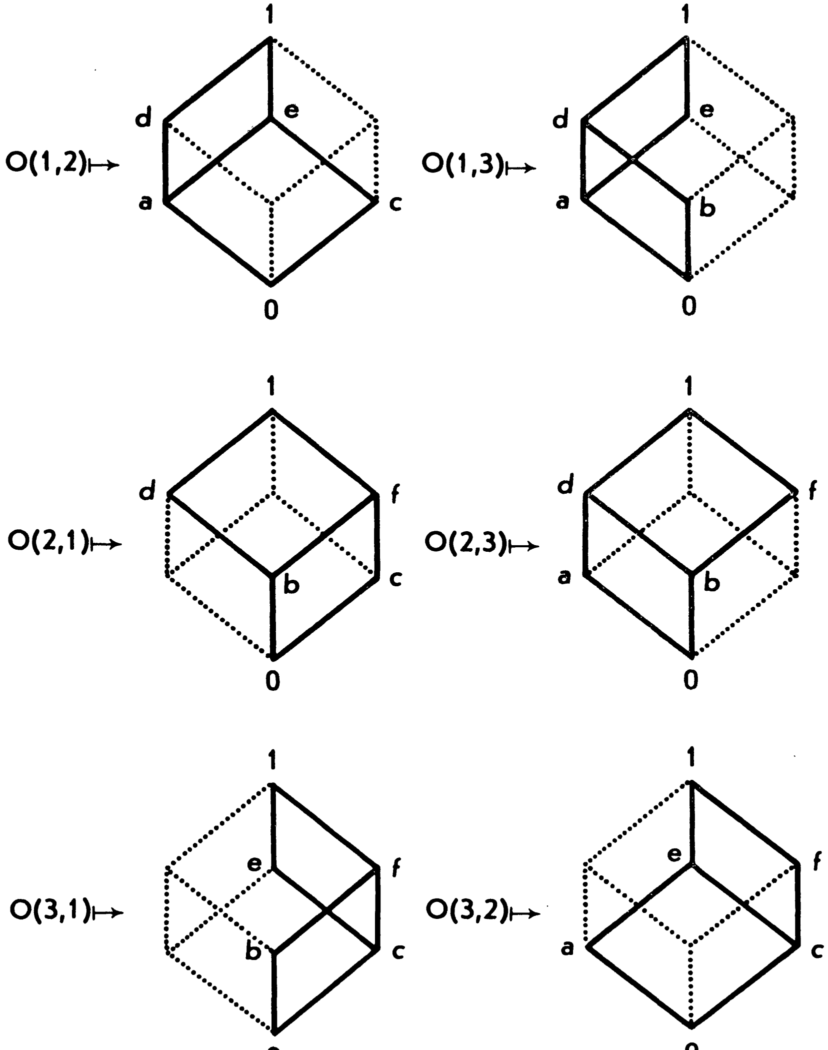
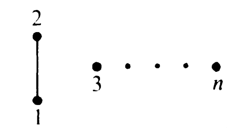


Fig. 2

Tab. 2

poset $\langle\{1, \ldots, n\}, O(1, 2)\rangle$	
lattices of order ideals	3 ; 2^{n-2}
the whole lattice of order ideals	$3 \times 2^{n-2}$

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