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LOCALLY–CYCLIC GRAPHS COVERING COMPLETE TRIPARTITE GRAPHS

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ABSTRACT. A new construction of the so-called locally-C_n graphs, for n even, based on the technique of voltage graphs is presented.

Let G be a graph and u a vertex. Denote by $G(u)$ the subgraph of G induced by the set of vertices adjacent to u. The graph G is called *locally* H if $G(u) \cong H$ for each vertex u of G. Further we shall be interested only in the case $H \cong C_n$, where $n \geq 3$ is fixed and C_n is a cycle of length n. The existence of finite locally-C_n graphs for each $n \geq 3$ was established in [1] and also in [2]. Later Ronan in [7] showed that there are infinitely many such graphs for each $n \geq 6$. A characterization of locally-C_n graphs is geometrical terms given by Vince [9] shows how to obtain locally-C_n graphs from groups. This was done in [8]. The relationship between locally-C_n graphs and 3-valent polygonal graphs is studied in [6]. In this note we present a way of constructing locally-C_{2n} graphs using voltage graphs.

An important and interesting property of locally C_n graphs is that each of them gives rise to a uniquely determined triangulation of a closed surface. In fact, denote for a given graph G by $K(G)$ the simplicial complex the simplices of which are the cliques of G and the incidence relation is given by subgraph inclusion. Then we have

THEOREM 1. ([5]) *A graph G is locally C_n if and only if $K(G)$ is an n-valent triangulation of a closed surface in which each cycle of length* 3 *forms a face-boundary.*

We obtain a class of locally-C_{2n} graphs as covering triangulations of the well-known triangular embedding of complete tripartite graphs $K_{n,n,n}$, $n \geq 2$ even, described in [10].

Further it is assumed that the reader is familiar with the terminology and the basic concepts of the topological graph theory, namely with the theory of

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2-cell embeddings of graphs into closed surfaces and with the theory of voltage graphs (see [3, 11]).

First we recall some definitions. For a given graph G choose for each edge of a graph G one of the two possible orientations. Then to each edge e of G we associate two arcs e, e^{-1} with the chosen and the opposite orientation, respectively. Denote by $D(G)$ the set of all arcs of G. Clearly, $|D(G)| = 2|E(G)|$. A ***voltage graph*** is a triple (G, φ, Γ), where G is a graph and φ is a mapping (sometimes called a *voltage assignment*) from $D(G)$ to a group Γ with a unique restriction $\varphi(e)^{-1} = \varphi(e^{-1})$. For the given voltage graph (G, φ, Γ) the ***derived covering graph*** $G \times_\varphi \Gamma$ is defined as follows: its vertex set is $V(G) \times \Gamma$ and each edge $e = uv$ of G generates the edges $(e, g) = (u, g)(v, g\varphi(e))$ of $G \times_\varphi \Gamma$, where g ranges over all the elements of the group Γ. It is easy to see that the natural projection, mapping an edge (e, g) of $G \times_\varphi \Gamma$ to e of G, is a covering mapping. If the original graph G is embedded into some surface, then this embedding may be lifted in a natural way into the ***derived embedding*** of the graph $G \times_\varphi \Gamma$. The set of cycles forming face-boundaries of faces of the derived embedding consists of the cycles of $G \times_\varphi \Gamma$ covering the boundaries of faces of the embedding of G in the natural projection sending an edge (e, g) of $G \times_\varphi \Gamma$ to e in G. It is not difficult to see that this new embedding forms a (branched) covering embedding of the original embedding. It is also known that the derived embedding is unbranched if and only if the product of voltages on a boundary of each face of the original embedding is the unit element of Γ. In the latter case a covering over a triangulation is again a triangulation.

Construction. We start with a triangular embedding j of $K_{n,n,n}$ into an orientable surface S described in [10]. Let $V(K_{n,n,n}) = A \cup B \cup C$ be the tri-partition of $K_{n,n,n}$. Since j is the triangulation, then its restriction $r = j\big|_{\langle A \cup B\rangle}$ is an embedding of an induced subgraph $\langle A \cup B\rangle \cong K_{n,n}$ of $K_{n,n,n}$ into S. Clearly, the boundary of each face of r forms a Hamiltonian cycle in $K_{n,n}$. We claim that if n is even, then the edges of $K_{n,n}$ can be oriented in such a way that arcs lying on the boundary of each face of r create a directed cycle. This follows from the fact that the dual embedding r^* of r is an embedding of a bipartite graph $H \hookrightarrow S$. In fact, the embedding r can be obtained as the derived embedding of the embedding k of the n-fold K_2 into the sphere (see [3, p. 210]). Since k^* is an embedding of C_n into the sphere and r^* covers k^*, then r^* must be bipartite if n is even. Now define a voltage assignment mapping $\psi\colon D(K_{n,n,n}) \to \mathbb{Z}_{2n}$ as follows. Set $\psi(e) = 1$ if an arc e of $K_{n,n} \cong \langle A \cup B\rangle$ has the chosen orientation and set $\psi(e^{-1}) = -1$ for the arc e^{-1}. Let ϱ_u be the local rotation of arcs emanating from a vertex u of C determined by the embedding $j\colon K_{n,n,n} \hookrightarrow S$. Let $(e_0, e_1, \ldots, e_{2n-1})$ be one of the rotations ϱ_u, ϱ_u^{-1} which is consistent with the orientation on the boundary of a face of r containing the vertex u. Then put $\psi(e_i) = i$ and $\psi(e_i^{-1}) = -i$ for all $i = 0, \dots, 2n-1$.

THEOREM 2. *Let ψ be the voltage assignments on the graph $K_{n,n,n}$, $n \geq 2$ even, with values in the cyclic group $\mathbb{Z}_{2n}$ defined above. Then $G = K_{n,n,n} \times_\psi \mathbb{Z}_{2n}$ is a locally-C_{2n}-graph.*

P r o o f. Since the sum of assignments of arcs on each triangle-face of j is 0, then the derived embedding i is unbranched, and consequently, it must be a $2n$-valent triangulation. To complete the proof it is sufficient to show that each cycle $((u, x), (v, y)(w, z))$ of length 3 in G forms a face-boundary. By the definition of G we have that (uvw) is a cycle of length 3 in $K_{n,n,n}$ and $\psi(uv) + \psi(vw) + \psi(wu) = 0$. We may suppose that $u \in C$, $v \in A$, $w \in B$. By the definition of ψ we have $\psi(vw) = 1$ or $\psi(vw) = -1$, and consequently, $\psi(uv)$ and $\psi(uw)$ differ by 1. Then either $\varrho_u(v) = w$ or $\varrho_u(w) = v$. In both cases we see that (uvw) forms the boundary of a triangle face in j, hence $((u, x)(v, y)(w, z))$ forms a face boundary. The assertion follows from Theorem 1. □

Concluding remark. A triangulation T is called a *clean* triangulation if every cycle of length 3 in T forms a face-boundary. N. H a r t s f i e l d and G. R i n g e l [4] investigated the problem of determining of the minimum number $\mathcal{T}(S_p)$ of triangles of a clean triangulation of surface of genus p. They proved $\lim\limits_{p\to\infty} \dfrac{\mathcal{T}(S_p)}{p} = 4$. Let T_k be the triangulation obtained using our construction for $n = 2k$, denote by $\mathcal{T}(T_k)$ the number of triangles of T_k and by p_k the genus of the underlying surface. Then it is easy to compute $\mathcal{T}(T_k) = 32k^3$, $p_k = 8k^3 - 12k^2 + 1$, and hence, $\lim\limits_{k\to\infty} \dfrac{\mathcal{T}(T_k)}{p_k} = 4$. Thus the sequence $\{T_k\}$ is extremal in sense of R i n g e l and H a r t s f i e l d [4].

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