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ON THE SEQUENTIAL ORDER

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ABSTRACT. We show that the sequential order in some distinguished convergence groups and convergence rings is ω_1.

A topological space is said to be *sequential* if each of its sequentially closed subsets is closed. In fact, every sequential space is the topological modification of a sequential convergence space, i.e., of a set X equipped with a convergence of sequences and the associated sequential closure operator cl assigning each subset A of X the set $\operatorname{cl} A$ of all limits of all convergent sequences ranging in A. In order to guarantee the usual properties of a closure it suffices to assume that each constant sequence $\langle x \rangle$ converges to x and if a sequence converges to some point, then each of its subsequences converges to this point as well. For every ordinal number α and for every subset A define $\alpha - \operatorname{cl} A$ as follows:

$$\begin{aligned}
&0 - \operatorname{cl} A = A, \\
&\alpha - \operatorname{cl} A = \operatorname{cl}(\beta - \operatorname{cl} A) \text{ if } \alpha = \beta + 1, \\
&\alpha - \operatorname{cl} A = \bigcup_{\beta < \alpha} \beta - \operatorname{cl} A \text{ if } \alpha \text{ is a limit ordinal.}
\end{aligned}$$

Then each $\alpha - \operatorname{cl}$ is a closure operator for X and since for each subset A of X we have $\operatorname{cl}(\omega_1 - \operatorname{cl} A) = \omega_1 - \operatorname{cl} A$, $\omega_1 - \operatorname{cl}$ is idempotent and hence topological. The least ordinal number α ($\alpha \leqq \omega_1$) such that $\alpha - \operatorname{cl} A$ is sequentially closed for all subsets A of X is said to be the *sequential order* of X.

The sequential order in the realm of sequential convergence spaces has been investigated by J. Novák in [18], where an example of a countable space (with unique limits) having the sequential order ω_1 can be found (cf. Example L_8). The literature on the sequential order in the realm of topological spaces is much more extensive and the sequential order within the theory of ordinal invariants has been thoroughly investigated in [13]. An often cited example by A. V. Arhangel'skiĭ and S. P. Franklin of a countable zero-dimensional Hausdorff homogeneous space, the sequential order of which is ω_1, has been

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given in [1]. G. H. Greco in [12] proved that the space known as the rational plane $\mathbb{Q}^2$ with the cross topology has (besides other nice properties) the sequential order ω_1, too.

The aim of the present paper is to prove that each of the sets $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, and $\mathbb{R}$ can be equipped with a distinguished sequential convergence with unique limits so that it is compatible with the group (ring) structure and the sequential order of the resulting space is ω_1. In particular, the rational torus $\mathbb{Q}/\mathbb{Z}$ becomes a countable sequentially compact convergence group with unique limits and the sequential order ω_1.

The proof by G. H. Greco and our proof are via transfinite induction and, as we point out, they follow the same scheme based on the fact that for a certain class of diagonal sequences either all subsequences converge to the same point or no subsequence converges at all. This in fact provides a control over the closure of countable unions of certain inductively defined sets.

1. By a *convergence group* we understand a group equipped with a *FLUSH-convergence* of sequences (besides the axioms on constant sequences and subsequences we assume unique limits, the Urysohn axiom and the sequential continuity of the algebraic operations). We say that a convergence group is *coarse* if the underlying group admits no strictly coarser (i.e. larger) FLUSH convergence. *Convergence rings* and *coarse convergence rings* are defined analogously. Using the Zorn-Kuratowski lemma, it can be shown that each group (ring) convergence can be enlarged to a coarse one. For further information on coarse groups and rings the reader is referred to [10], [5] and [6].

Denote by $\mathbb{Q}_c$ the group of rational numbers equipped with a coarse group convergence coarser than the usual metric one. It is known (cf. [5]) that $\mathbb{Q}_c$ is complete (every Cauchy sequence converges) and, even though the sequential limits are unique, no two points of $\mathbb{Q}_c$ can be separated by disjoint neighbourhoods (neither by $(\omega_1-\mathrm{cl})$-neighbourhoods nor by cl-neighbourhoods). In fact, for each $q \in \mathbb{Q}$ and for each interval $I = \{x \in \mathbb{Q};\ a \leqq x \leqq b\}$, $a, b \in \mathbb{Q}$, $a < b$, there is a one-to-one sequence $\langle x_n \rangle$ in I converging in Q_c to q. From the latter property it follows that the sequential order of $\mathbb{Q}_c$ is greater than one ([5]). We shall prove that the sequential order of $\mathbb{Q}_c$ is actually ω_1 (mentioned in [5] as a remark added in the proof).

In the proof of the next theorem we shall use the following convention: by an interval we mean a set of the form $\{x \in \mathbb{Q};\ a \leqq x \leqq b\}$, $a, b \in \mathbb{Q}$, $a < b$, and if $q \in \mathbb{Q}$ and $\langle I_n \rangle$ is a sequence of intervals, then by $\langle I_n \rangle \to q$ we mean that for each $a \in \mathbb{Q}$, $a > 0$, we have $I_n \subset \{x \in \mathbb{Q};\ q - a \leqq x \leqq q + a\}$ for all but finitely many n, $n = 1, 2 \ldots$ (observe that if $q_n \in I_n$, then $\langle q_n \rangle$ converges in the usual metric to q).

THEOREM 1.1. *The sequential order of* $\mathbb{Q}_c$ *is* ω_1.

P r o o f . Clearly, it suffices to prove that for each ordinal number α, $\alpha < \omega_1$, the following proposition $\mathrm{P}(\alpha)$ holds true:

$\mathrm{P}(\alpha)$ For each $p \in \mathbb{Q}$ and for each interval I there is a set $A \subset I$ such that $\alpha - \mathrm{cl}\, A \subset I \setminus \{p\}$ and $(\alpha+1) - \mathrm{cl}\, A = \{p\} \cup \alpha - \mathrm{cl}\, A$.

Let $p \in \mathbb{Q}$ and let I be an interval. We proceed by transfinite induction.

1. Let $\alpha = 0$. As mentioned before (cf. condition (c) on page 476 in [5]), in I there is a one-to-one sequence $\langle x_n \rangle$ converging in $\mathbb{Q}_c$ to p. It suffices to put $A = \{x_n ;\ n = 1,2,\dots\} \setminus \{p\}$. Now, let $\alpha > 0$ and assume that $\mathrm{P}(\beta)$ holds for all ordinal numbers β, $\beta < \alpha$.

2. Let $\alpha = \beta + 1$. Choose $q \in I$, $q \neq p$, disjoint intervals $I_n \subset I$, $p \notin I_n$, $n = 0,1,\dots$, such that $\langle I_n \rangle \to q$ and a one-to-one sequence $\langle p_n \rangle$ in I_0 converging in $\mathbb{Q}_c$ to p. By the inductive assumption, for each n, $n = 1,2,\dots$, choose a set $A_n \subset I_n$ such that $\beta - \mathrm{cl}\, A_n \subset I_n \setminus \{p_n\}$ and $(\beta+1) - \mathrm{cl}\, A_n = \{p_n\} \cup \beta - \mathrm{cl}\, A_n$. It suffices to put $A = \{q\} \cup \bigcup_{n=1}^{\infty} A_n$.

3. Let α be a limit ordinal number. Let $\langle \alpha_n \rangle$ be an increasing sequence of ordinal numbers converging to α. Choose q, $\langle I_n \rangle$ and $\langle p_n \rangle$ as in the case of $\alpha = \beta + 1$. By the inductive assumption, for each n, $n = 1,2,\dots$, choose a set $A_n \subset I_n$ such that $\alpha_n - \mathrm{cl}\, A_n \subset I_n \setminus \{p_n\}$ and $(\alpha_n+1) - \mathrm{cl}\, A_n = \{p_n\} \cup \alpha_n - \mathrm{cl}\, A_n$. It suffices to put $A = \{q\} \cup \bigcup_{n=1}^{\infty} A_n$. This completes the proof.

Let $\mathbb{Q}/\mathbb{Z}$ be the rational torus and let d be the usual metric for $\mathbb{Q}/\mathbb{Z}$. Denote by $(\mathbb{Q}/\mathbb{Z})_c$ the rational torus equipped with a coarse group convergence coarser than the usual metric one.

THEOREM 1.2.

(i) *$(\mathbb{Q}/\mathbb{Z})_c$ is sequentially compact.*

(ii) *For each p, $q \in \mathbb{Q}/\mathbb{Z}$ and for each positive real number ε there is in $\mathbb{Q}/\mathbb{Z}$ a sequence $\langle q_n \rangle$ such that $\langle q_n \rangle$ converges in $(\mathbb{Q}/\mathbb{Z})_c$ to p and $d(q, q_n) < \varepsilon$, $n = 1,2,\dots$.*

(iii) *No two points of $(\mathbb{Q}/\mathbb{Z})_c$ can be separated by disjoint neighbourhoods.*

(iv) *The sequential order of $(\mathbb{Q}/\mathbb{Z})_c$ is ω_1.*

P r o o f .

(i) Let $\langle p_n \rangle$ be a sequence of points of $\mathbb{Q}/\mathbb{Z}$. Then there is a subsequence $\langle q_n \rangle$ of $\langle p_n \rangle$ which is Cauchy in the metric d. Since the convergence in $(\mathbb{Q}/\mathbb{Z})_c$ is coarser than that induced by d, $\langle q_n \rangle$ is also a Cauchy sequence in $(\mathbb{Q}/\mathbb{Z})_c$. By Proposition 3.1 in [5], $(\mathbb{Q}/\mathbb{Z})_c$ is complete. Thus $\langle q_n \rangle$ converges in $(\mathbb{Q}/\mathbb{Z})_c$.

Proposition (ii) can be proved virtually in the same way as condition (c) on page 476 in [5].

Proposition (iii) follows immediately from (ii).

Using (ii), proposition (iv) can be proved exactly in the same way as Theorem 1.1. This completes the proof of the theorem.

Denote by $\mathbb{Q}_r$ the ring of rational numbers equipped with a coarse ring convergence, coarser than the usual metric one. Basic properties of $\mathbb{Q}_r$ have been described in [6]. E.g., no Cauchy sequence of rational numbers converging in the real line to an algebraic number does converge in $\mathbb{Q}_r$ and hence the convergence in $\mathbb{Q}_r$ fails to be a coarse group convergence for $\mathbb{Q}$. Further, $\mathbb{Q}_r$ does not have a ring completion. Even though the convergence in $\mathbb{Q}_r$ is finer than that in $\mathbb{Q}_c$, it follows from the proof of Proposition 6 in [6] that for each $p, q \in \mathbb{Q}$ and for each positive real number ε there is a sequence $\langle q_n \rangle$ in $\mathbb{Q}$ such that $\langle q_n \rangle$ converges in $\mathbb{Q}_r$ to p and $|q - q_n| < \varepsilon$, $n = 1, 2, \dots$. Consequently, the proof of Theorem 1.1 yields the following result:

COROLLARY 1.3. *The sequential order of $\mathbb{Q}_r$ is ω_1.*

Remark 1.4. The nontrivial part ($\alpha > 0$) of the proof of Theorem 1.1 can be summarized as follows.

Let X be a set equipped with a sequential convergence. Assume that for each point $p \in X$, each "admissible" set $I \subset X$ and each nondecreasing sequence $\langle \alpha_n \rangle$ of isolated ordinal numbers $\alpha_n < \omega_1$ there are:

(a_1) A one-to-one sequence $\langle p_n \rangle$ of points of X, $p_n \neq p$, $n = 1, 2, \dots$;
(a_2) A sequence $\langle I_n \rangle$ of disjoint "admissible" sets $I_n \subset I$, $p \notin I_n$, $n = 1, 2, \dots$;
(a_3) A sequence $\langle A_n \rangle$ of sets $A_n \subset I_n$, $n = 1, 2, \dots$;
(a_4) A point $q \in X$, $q \neq p$;

such that

(b_1) The sequence $\langle p_n \rangle$ converges to p;
(b_2) $\alpha_n - \operatorname{cl} A_n \subset I_n \setminus \{p_n\}$, $(\alpha_n + 1) - \operatorname{cl} A_n = \{p_n\} \cup \alpha_n - \operatorname{cl} A_n$, $n = 1, 2, \dots$;
(b_3) Each sequence $\langle q_n \rangle$ of points $q_n \in I_n$, $n = 1, 2, \dots$, converges in X to q.

Then the sequential order of X is ω_1. Indeed, the set $A = \{q\} \cup \bigcup_{n=1}^{\infty} A_n$ satisfies condition P(α) both for isolated α (with $\alpha_n + 1 = \alpha$, $n = 1, 2, \dots$) and limit α (with $\alpha = \lim \alpha_n$).

It is easy to verify that if we leave out condition (a_4) and replace condition (b_3) by

(b_3') No diagon.' sequence $\langle q_n \rangle$, $q_n \in I_n$, $n = 1, 2, \dots$, has a convergent subsequence;

then the sequential order of X is ω_1, too. Now, the set $A = \bigcup_{n=1}^{\infty} A_n$ does the trick.

This modified scheme generalizes the proof of Theorem in [12], asserting that the sequential order of $\mathbb{Q}^2$ equipped with the cross topology is ω_1. The role of the "admissible" sets I_n is played there by the discs Ω_n "tangent at x_n $(= p_n)$ to the line through x $(= p)$ and points x_n". We shall use the latter scheme in the next section to prove that both the group categorical completion of $\mathbb{Q}$ and the ring categorical completion of $\mathbb{Q}$ have the sequential order ω_1.

2. There are striking differences between the completion theories of topological groups (rings) and convergence groups (rings). Every abelian convergence group has a completion and it can have several nonhomeomorphic ones ([19], [7]), whereas there are convergence groups having no two-sided completion ([9]). As shown in [14] (see also [15]), if a convergence group has a completion, then it has the categorical one, yielding the epireflection into the subcategory of complete groups. The categorical completion, also called the Novák completion, has been explicitly constructed for abelian sequential convergence groups in [19] (see also [8]). The necessary and sufficient condition for the Novák completion of a Fréchet abelian group to be Fréchet has been given in [16] and a nontrivial example of a Fréchet abelian group the Novák completion of which is Fréchet has been given in [4]. As shown in [3], a sequential convergence commutative ring need not have a completion, but in special cases, e.g. for $\mathbb{Q}$ equipped with the usual metric convergence, the categorical ring completion has been constructed in [17]. We shall prove that both the group categorical completion $\nu\mathbb{Q}$ of $\mathbb{Q}$ and the ring categorical completion $\varrho\mathbb{Q}$ of $\mathbb{Q}$ have the sequential order ω_1.

The Novák completion $\nu\mathbb{Q}$ of $\mathbb{Q}$ can be briefly described as follows (cf. [19]). The underlying group of $\nu\mathbb{Q}$ is the group $\mathbb{R}$ of real numbers and the convergence in $\nu\mathbb{Q}$ is the Urysohn modification of the convergence in which a sequence $\langle z_n \rangle$ converges to z if and only if there is a Cauchy sequence $\langle q_n \rangle$ of rational numbers such that $z_n = q_n - q + z$, where q is the real number to which $\langle q_n \rangle$ converges in the real line. Thus, the sequence $\langle \pi/n \rangle$ does not converge in $\nu\mathbb{Q}$ to 0 (since $\pi/n - \pi/m$ is not a rational number for $n \neq m$) and, in fact, the following proposition holds ([19]).

PROPOSITION 2.1. *Let $\langle q_n \rangle$ be a sequence of real numbers such that $z_n - z_m$ is not a rational number whenever $n \neq m$. Then no subsequence of $\langle z_n \rangle$ converges in $\nu\mathbb{Q}$.*

THEOREM 2.2. *The sequential order of $\nu\mathbb{Q}$ is ω_1.*

Proof. Let H be a set of irrational numbers such that $\{1\} \cup H$ is a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. Given an infinite countable set $B \subset H$ and $r \in \mathbb{R}$, define

$I(B, r)$ to be the set of all real numbers of the form $r + \sum_{i=1}^{k}(b_i - q_i)$, where k is a natural number ($k \neq 0$), $q_i \in \mathbb{Q}$, $b_i \in B$, $b_i \neq b_j$ for $i \neq j$, $i,j = 1,\dots,k$; each set $I(B, r)$ will be called admissible.

Clearly, it suffices to prove that for each ordinal number α, $\alpha < \omega_1$, the following proposition $\mathrm{R}(\alpha)$ holds true:

$\mathrm{R}(\alpha)$ For each $p \in \mathbb{R}$ and for each admissible set $I(B, p)$ there is a set $A \subset I(B, p)$ such that $\alpha - \mathrm{cl}\, A \subset I(B, p) \setminus \{p\}$ and $(\alpha + 1) - \mathrm{cl}\, A = \{p\} \cup \alpha - \mathrm{cl}\, A$.

Let $p \in \mathbb{R}$ and let $I(B, p)$ be an admissible set. We proceed by transfinite induction.

1. Let $\alpha = 0$. Choose $b \in B$ and let $\langle q_n \rangle$ be a one-to-one sequence of rational numbers converging in $\nu\mathbb{Q}$ to b. It suffices to put $A = \{p + (b - q_n);\ n = 1, 2, \dots\}$. Now, let $\alpha > 0$ and assume that $\mathrm{R}(\beta)$ holds for all ordinal numbers β, $\beta < \alpha$.

2. Let $\alpha > 0$. Assume $\alpha = \beta + 1$. There is a finite sub et $B_p \subset H$ such that p is a $\mathbb{Q}$-linear combination of elements of $\{1\} \cup B_p$. Choose $b \in B \setminus B_p$. Let $\langle q_n \rangle$ be a one-to-one sequence of rational numbers such that $|b - q_n| < 1/100n$, $n = 1, 2, \dots$. Put $p_n = p + 1/n + b - q_n$, $n = 1, 2, \dots$. Let $B \setminus (\{b\} \cup B_p)$ be the union of disjoint infinite sets B_n, $n = 1, 2, \dots$. Consider the admissible sets $I(B_n, p_n)$. Then $I(B_n, p_n) \cap I(B_m, p_m) = \emptyset$ whenever $n \neq m$, $I(B_n, p_n) \subset I(B, p) \setminus \{p\}$ and $p_n \notin I(B_n, p_n)$ for each n, $n = 1, 2, \dots$, and no diagonal sequence $\langle x_n \rangle$, $x_n \in I(B_n, p_n)$, $n = 1, 2, \dots$, has a convergent subsequence. By the inductive assumption, for each n, $n = 1, 2, \dots$, choose a set $A'_n \subset I(B_n, p_n)$ such that $\beta - \mathrm{cl}\, A'_n \subset I(B_n, p_n) \setminus \{p_n\} = I(B_n, p_n)$ and $(\beta + 1) - \mathrm{cl}\, A'_n = \{p_n\} \cup \beta - \mathrm{cl}\, A'_n$. It suffices to put $A_n = A'_n \cap (p + 1/n - 1/10n,\ p + 1/n + 1/10n)$ and $A = \bigcup_{n=1}^{\infty} A_n$ (cf. Remark 1.4).

For a limit ordinal number α the assertion $\mathrm{R}(\alpha)$ can be proved analogously. We omit the details. This completes the proof.

The categorical ring completion $\varrho\mathbb{Q}$ of $\mathbb{Q}$ equipped with the usual metric convergence has $\mathbb{R}$ as the underlying ring and the convergence in $\varrho\mathbb{Q}$ is the Urysohn modification of the convergence in which a sequence $\langle z_n \rangle$ converges to z if and only if there are a natural number k, real numbers a_i and Cauchy sequences $\langle q_{in} \rangle$ of rational numbers, $i = 1, \dots, k$, such that for each n, $n = 1, 2, \dots$, we have $z_n = \sum_{i=1}^{k} q_{in} a_i$ and $z = \sum_{i=1}^{k} q_i a_i$, where q_i is the limit of $\langle q_{in} \rangle$ in the real line. Observe that a convergent sequence $\langle z_n \rangle$ in $\varrho\mathbb{Q}$ is in some finite dimensional subspace of the vector space $\mathbb{R}$ over the scalar field $\mathbb{Q}$. Clearly, if a sequence converges to a point in $\nu\mathbb{Q}$, then it converges to the same point in $\varrho\mathbb{Q}$ as well.

THEOREM 2.3. *The sequential order of* $\mathbb{Q}$ *is* ω_1.

Proof. The proof is analogous to that of Theorem 2.2. Indeed, the sequence converges to a point in $\varrho\mathbb{Q}$ whenever it does in $\nu\mathbb{Q}$ and no subsequence of any diagonal sequence $\langle x_n\rangle$, $x_n \in I(B_n, p_n)$, $n = 1, 2, \dots$, does converge in $\varrho\mathbb{Q}$. Thus, for all ordinal numbers α, $\alpha < \omega_1$, the assertion $\mathrm{R}(\alpha)$ holds in $\varrho\mathbb{Q}$, too.

Remark 2.4. It is known that if X is a group (ring, vector space, etc.) equipped with a compatible sequential convergence, then the first countable filter modification functor γ yields a filter convergence compatible with the algebraic operations of X (cf. [2]). The resulting filter convergence space has the same closure operator and hence all propositions concerning the sequential order proved in the present paper remain valid when applying the functor γ. For interesting applications of γ see [2], [9], [6], [11], [17].

REFERENCES

[1] ARHANGEL'SKIĬ, A. V.—FRANKLIN, S. P.: *New ordinal invariants for topological spaces*, Michigan Math. J. **15** (1968), 313–320.

[2] BEATTIE, R.—BUTZMANN, H.-P.: *Sequentially determined convergence spaces*, Czechoslovak Math. J. **37** (1987), 231–247.

[3] CONTESSA, M.—ZANOLIN, F.: *Example of a commutative convergence ring which has no completion*, Boll. Un. Mat. Ital. A (5) **18** (1981), 467–472.

[4] DIKRANJAN, D.: *Non-completeness measure of convergence Abelian groups.* In: General Topology and its Relations to Modern Analysis and Algebra VI. (Proc. Sixth. Prague Topological Sympos., 1986), Heldermann Verlag, Berlin, 1988, pp. 125–134.

[5] DIKRANJAN, D.—FRIČ, R.—ZANOLIN, F.: *On convergence groups with dense coarse subgroups*, Czechoslovak Math. J. **37** (1987), 471–479.

[6] FRIČ, R.: *Rational with exotic convergences*, Math. Slovaca **39** (1989), 141–147.

[7] FRIČ, R.—KENT, D. C.: *Completion of pseudo-topological groups*, Math. Nachr. **99** (1980), 99–103.

[8] FRIČ, R.—KOUTNÍK, V: *Completions of convergence groups.* In: General Topology and its Relations to Modern Analysis and Algebra VI. (Proc. Sixth. Prague Topological Sympos., 1986), Heldermann Verlag, Berlin, 1988, pp. 187–201.

[9] FRIČ, R.—ZANOLIN, F.: *A convergence group having no completion.* In: Convergence Structures and Applications, II. (Proc. Schwerin Conference) Abh. Akad. Wiss. DDR, Abt. Math.-Naturwiss.-Technik 2 1984, Akademie-Verlag, Berlin, 1984, pp. 47–48.

[10] FRIČ, R.—ZANOLIN, F.: *Coarse convergence groups.* In: Convergence Structures 1884. (Proc. Conf. on Convergence, Bechyně 1984), Akademie-Verlag, Berlin, 1985, pp. 107–114.

[11] FRIČ, R.—ZANOLIN, F.: *Coarse sequential convergence in groups, etc.*, Czechoslovak Math. J. **40** (1990), 459–467.

[12] GRECO, G. H.: *The sequential defect of the cross topology is* ω_1, Topology Appl. **19** (1985), 91–94.

[13] KANNAN, V.: *Ordinal invariants in topology*, Mem. Amer. Math. Soc. **32** no. 245 (1981).

[14] KNEIS, G.: *Eine allgemeine Theorie der Vervollständigung und Čech-Stone-Kompaktifizierung. (Dr. Sc. Nat. Dissertation)*, Ernst-Moritz-Arndt-Universität Greifswald, Greifswald, 1983.

[15] KNEIS, G.: *Completion of sequential convergence groups.* In: Proc. Conf. Topology and Measure IV, Greifswald 1984, Part 1. Wiss. Beitr. der Ernst-Moritz-Arnd Universität Greifswald, pp. 125–132.

[16] KOUTNÍK, V.: *Completeness of sequential convergence groups*, Studia Math. **77** (1984), 454–464.

[17] KOUTNÍK, V,—NOVÁK, J.: *Completion of a class of convergence rings.* (To appear).

[18] NOVÁK, J.: *On convergence spaces and their sequential envelopes*, Czechoslovak Math. J. **15** (1965), 74–100.

[19] NOVÁK, J.: *On completions of convergence commutative groups.* In: General Topology and its Relations to Modern Analysis and Algebra III. (Proc. Third Prague Topological Sympos., 1971), Academia, Praha, 1972, pp. 335–340.

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