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WEAK ISOMETRIES AND DIRECT DECOMPOSITIONS OF DUALLY RESIDUATED LATTICE ORDERED SEMIGROUPS

MILAN JASEM

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ABSTRACT. In the paper the relations between weak isometries fixing zero in a dually residuated lattice ordered semigroup G and direct decompositions of G are established.

Introduction

Swamy [14], [15], [16] introduced and studied dually residuated lattice ordered semigroups (notation DRl-semigroups) as a common abstraction of Boolean rings and lattice ordered groups. It was a solution of Birkhoff's problem No. 115 [1]. In [13] Swamy introduced the notions of an autometrized algebra and an intrinsic metric. Isometries in lattice ordered groups, i.e. surjections preserving the intrinsic metric, were studied by Swamy [17], [18] and Powell [11] for the abelian case and by Jakubík [3], [4], [5] Holland [2] for the general case. Isometries in Riesz spaces and some types of partially ordered groups were dealt with by Trias [20], Jakubík and Kolibiar [6], Rachůnek [12] and the author [7], [8], [9], [10]. Results of [2], [3], [6], [7], [9], [10], [11] show that in lattice ordered groups, distributive multilattice groups and Riesz groups there exists a relation between isometries and direct decompositions. In [19] Swamy and Subba Rao investigated isometries in DRl-semigroups and proved that any isometry in a representable DRl-semigroup fixing zero is an involutory semigroup automorphism.

In this paper weak isometries (unlike isometries, surjectivity is not assumed here) in DRl-semigroups are studied. It is shown that the relation between isometries and direct decompositions exists in DRl-semigroups as well. Namely, the following results are established:

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(1) If f is a weak isometry in a DRl-semigroup G and $f(0) = 0$, then G is the direct product of a DRl-semigroup A and an l-group B and $f(x) = x_A + (0 - x_B)$ for each $x \in G$. (For the denotation, see below.)

(2) If a DRl-semigroup G is the direct product of a DRl-semigroup P and an l-group Q and we put $g(x) = x_P + (0 - x_Q)$ for each $x \in G$, then g is a weak isometry and $g(0) = 0$.

This generalizes some results of P o w e l l [11] concerning isometries and direct decompositions of abelian lattice ordered groups. Further, it is shown that any weak isometry in a DRl-semigroup fixing zero is an involutory semigroup automorphism. Thus the above mentioned result of S w a m y and S u b b a R a o is generalized as well.

Preliminaries

We review some notions and notations used in the paper.

A system $G = (G, +, \leq, -)$ is called a *dually residuated lattice ordered semigroup* if and only if

1. $(G, +, \leq)$ is a commutative lattice ordered semigroup with zero element 0, i.e. $(G, +)$ is a commutative semigroup with zero 0 and $(G, \leq)$ is a lattice with lattice operations $\wedge$ and $\vee$ such that $a + (b \vee c) = (a + b) \vee (a + c)$ and $a + (b \wedge c) = (a + b) \wedge (a + c)$ for each $a, b, c \in G$,
2. given a, b in G there exists a least x in G such that $b + x \geq a$, and this x is denoted by $a - b$,
3. $(a - b) \vee 0 + b \leq a \vee b$ for all $a, b \in G$,
4. $(a - a) \geq 0$ for each a $\in G$.

For any a and b in a DRl-semigroup G, we shall write $d(a, b) = (a - b) \vee (b - a)$ ($d(a, b)$ is called the *symmetric difference of a and b*). The symmetric difference satisfies the following conditions:

(i) $d(a, b) \geq 0$, $d(a, b) = 0$ if and only if $a = b$,
(ii) $d(a, b) = d(b, a)$,
(iii) $d(a, b) = d(a \vee b, a \wedge b)$.

Any DRl-semigroup is an autometrized algebra with the symmetric difference.

Let G be a DRl-semigroup. A mapping $f\colon G \to G$ is called a *weak isometry in* G if $d(x, y) = d\big(f(x), f(y)\big)$ for each $x, y \in G$. If a weak isometry f in G is a surjection, then f is called an *isometry*. A weak isometry (isometry) f in G is called a *weak 0-isometry* (*0-isometry*) if $f(0) = 0$.

From (i) it follows that any weak isometry in a DRl-semigroup is an injection.

The following assertions hold for any elements x, y, z of a DRl-semigroup G (see [14]):

(a) $x - x = 0$, $x - 0 = x$,
(b) $x \le y$ implies $x - z \le y - z$ and $z - y \le z - x$,
(c) $x \ge y$ implies $x - y \ge 0$,
(d) $x \le y$ if and only if $x - y \le 0$.

We shall often need these assertions and we shall apply them without special references.

Further, we shall often refer to the following assertions on DRl-semigroups from [14] (a, b, c stand for elements of a DRl-semigroup):

THEOREM 1. *Any DRl-semigroup can be equationally defined as an algebra with the binary operations* $+$, $\vee$, $\wedge$, $-$, *by replacing* (2) *by the equations:* $x + (y - x) \ge y$, $x - y \le (x \vee z) - y$ *and* $(x + y) - y \le x$.

LEMMA 2. $(a - b) \vee 0 + b = a \vee b$.

LEMMA 5. $a - (b \wedge c) = (a - b) \vee (a - c)$.

LEMMA 6. $a - (b + c) = (a - b) - c = (a - c) - b$.

LEMMA 8. $a \ge b$ *implies* $(a - b) + b = a$.

LEMMA 9. $a \vee b + a \wedge b = a + b$.

LEMMA 13. $a - (b - c) \le (a - b) + c$ *and* $(a + b) - c \le (a - c) + b$.

LEMMA 17. *For any positive integer* n, $na = 0$ *implies* $a = 0$.

Partially ordered semigroup H with a zero element is said to be the *direct product of its partially ordered subsemigroups* P *and* Q (notation $H = P \times Q$) if the following conditions are fulfilled:

(1) If $a \in P$ and $b \in Q$, then $a + b = b + a$;
(2) each element $c \in H$ can be uniquely represented in the form $c = c_1 + c_2$, where $c_1 \in P$, $c_2 \in Q$;
(3) if $g, h \in H$, $g = g_1 + g_2$, $h = h_1 + h_2$, where $g_1, h_1 \in P$, $g_2, h_2 \in Q$, then $g \le h$ if and only if $g_1 \le h_1$ and $g_2 \le h_2$.

If $H = P \times Q$, then for $x \in H$ we denote by x_P and x_Q the components of x in the direct factors P and Q, respectively.

Throughout this paper G will denote a DRl-semigroup. For $a, b \in G$, $U(a, b) = \{x \in G;\ x \ge a \text{ and } x \ge b\}$. If S is a subset of G, then we denote $S^+ = \{x \in S;\ x \ge 0\}$, $S^- = \{x \in S;\ x \le 0\}$.

1. Invertible elements

Let G_I be the set of all invertible elements of G. Swamy proved that G_I is an l-group ([15, Th. 1.1]). Now we establish some useful properties of invertible elements.

1.1. LEMMA. *Let* $y \in G_I$. *Then:*

(i) *the inverse of* y *is the element* $0 - y$ *and* $0 - (0 - y) = y$,

(ii) *for each* $x \in G$, $(x - y) + y = x$, $(x + y) - y = x$, $0 - (x + y) = (0-x)+(0-y)$, $0-(x-y) = y-x = y+(0-x)$, $x-y = x+(0-y)$, $x - (0 - y) = x + y$.

Proof.

(i) This follows from [15, Th. 1.1].

(ii) Let $y \in G_I$, $x \in G$. By [14, Th. 1], $(x - y) + y \geq x$. According to (i), [14, Lemmas 6 and 13] we obtain $[(x - y) + y] - x \leq [(x - y) - x] + y = [(x - x) - y] + y = (0 - y) + y = 0$. Then we have $(x - y) + y \leq x$. Therefore $(x - y) + y = x$.

From [14, Th. 1] it follows that $(x + y) - y \leq x$. According to (i), [14, Lemmas 6 and 13] we get $x - [(x+y) - y] \leq [x - (x+y)] + y = (0 - y) + y = 0$. Thus we have $x \leq (x + y) - y$. Therefore $x = (x + y) - y$.

In view of [14, Lemma 6] we have $[0-(x+y)] + y = [(0-x)-y] + y = 0-x$. Then $(0 - x) + (0 - y) = [0 - (x + y)] + y + (0 - y) = 0 - (x + y)$.

By [14, Lemma 6], $(y-x)-y = 0-x$. Then $y+(0-x) = [(y-x)-y] + y = y - x$. Also, by [14, Lemma 6], $[0 - (x - y)] - y = 0 - [(x - y) + y] = 0 - x$. From this we get $(0 - x) + y = \{[0 - (x - y)] - y\} + y = 0 - (x - y)$.

Because of $(x-y)+y = x$, we have $x+(0-y) = (x-y)+y+(0-y) = x-y$.

By [14, Lemma 6], $[x - (0 - y)] - y = x - [(0 - y) + y] = x$. Then $x + y = x - (0 - y)$.

1.2. LEMMA. *For each* $x \in G^-$ *the element* $0 - x$ *is the inverse of* x.

Proof. It is a direct consequence of [14, Lemma 8].

1.3. THEOREM. *Let* $G_n = \{x + (0 - y);\ x, y \in G^-\}$. *Then* $G_n = G_I$.

Proof. From 1.2 it follows that $G_n \subseteq G_I$. So it remains only to show that $G_I \subseteq G_n$.

Let $z \in G_I$, $v = (0-z)\vee 0$, $u = (0-z)\wedge 0$. By [14, Lemma 9], $u+v = 0-z$. In view of 1.1 and 1.2 from this we get $z = 0-(0-z) = 0-(u+v) = (0-u)+(0-v)$. Since $0 - v \leq 0$, we conclude $z \in G_n$. This ends the proof.

1.4. LEMMA. *Let* $x \in G$, $z = 0 \wedge x$, $y = (0 - x) \vee 0$. *Then* $z = 0 - y$, $y = 0 - z$, $0 - (0 - y) = y$.

Proof. By [14, Lemma 5], $0 - z = 0 \vee (0 - x) = y$. In view of 1.2 we get $y + z = 0$. Further, from [14, Lemma 6] it follows that $z = z - (z + y) = 0 - y$. Thus $y = 0 - (0 - y)$.

1.5. LEMMA. *Let* $x \in G$, $0 - (0 - x) = x$. *Then* $0 - x$ *is the inverse of* x.

Proof. Let $x \in G$, $0 - (0 - x) = x$, $x_1 = 0 \vee x$, $x_2 = x \wedge 0$. By [14, Lemma 9], $x_1 + x_2 = x$. Then 1.1 and 1.2 yield $x = 0 - [0 - (x_1 + x_2)] = 0 - [(0 - x_1) + (0 - x_2)] = [0 - (0 - x_1)] + [0 - (0 - x_2)] = [0 - (0 - x_1)] + x_2$. In view of 1.1 and 1.2 we have $x + (0 - x) = [0 - (0 - x_1)] + x_2 + (0 - x_1) + (0 - x_2) = 0$. This ends the proof.

1.6. COROLLARY. $G_I = \{x \in G;\ x = 0 - (0 - x)\}$.

Proof. This is a consequence of 1.1 and 1.5.

1.7. THEOREM. *Let* f *be a weak isometry in* G, $a \in G_I$. *Then the mapping* g *defined by* $g(x) = f(x) - a$ *for each* $x \in G$ *is a weak isometry as well. If* f *is an isometry, then* g *is also an isometry.*

Proof. Let $y, z \in G$. By 1.1 and [14, Lemma 6], $d(g(y), g(z)) = \{f(y) - [a + (f(z) - a)]\} \vee \{f(z) - [a + (f(y) - a)]\} = [f(y) - f(z)] \vee [f(z) - f(y)] = d(f(y), f(z)) = d(y, z)$. Thus g is a weak isometry. The rest follows by 1.1.

From 1.1 and 1.7 we obtain immediately:

1.8. COROLLARY. *If* f *is a weak isometry in* G *and* $f(0) \in G_I$, *then the mapping defined by* $g(x) = f(x) - f(0)$ *for each* $x \in G$ *is a weak* 0*-isometry and* $f(x) = g(x) + f(0)$ *for each* $x \in G$. *Moreover, if* f *is an isometry, then* g *is a* 0*-isometry.*

1.9. THEOREM. *Let* $G = P \times Q$ *be such direct product that* Q *is a group and* $a - b \in P$ *for each* $a, b \in P$. *Then:*

(i) $(x - y)_P = x_P - y_P$, $(x - y)_Q = x_Q - y_Q$ *and* $d(x, y) = d(x_P, y_P) + d(x_Q, y_Q)$ *for each* $x, y \in G$,

(ii) *the mapping* f *defined by* $f(z) = z_P + (0 - z_Q)$ *for each* $z \in G$ *is a* 0*-isometry in* G.

Proof.

(i) Let $x, y \in G$. By 1.1 and [14, Lemma 6], $(x - y) - x_Q = (x - x_Q) - (y_P + y_Q) = (x_P - y_P) - y_Q$. In view of 1.1 from this we get $x - y =$

$[(x-y)-x_Q]+x_Q=(x_P-y_P)+(0-y_Q)+x_Q=(x_P-y_P)+(x_Q-y_Q)$. Also, by 1.1, $x_Q-y_Q\in Q$. Thus $(x-y)_P=x_P-y_P$, $(x-y)_Q=x_Q-y_Q$. Then we have $d(x,y)_P\in U(x_P-y_P,\,y_P-x_P)$, $d(x,y)_Q\in U(x_Q-y_Q,\,y_Q-x_Q)$. Therefore $d(x,y)\geq [(x_P-y_P)+(y_Q-x_Q)]\vee[(y_P-x_P)+(x_Q-y_Q)]$. Then we have $d(x_P,y_P)+d(x_Q,y_Q)=[(x_P-y_P)+(x_Q-y_Q)]\vee[(x_P-y_P)+(y_Q-x_Q)]\vee[(y_P-x_P)+(x_Q-y_Q)]\vee[(y_P-x_P)+(y_Q-x_Q)]=d(x,y)\vee[(x_P-y_P)+(y_Q-x_Q)]\vee[(y_P-x_P)+(x_Q-y_Q)]=d(x,y)$.

(ii) Let $u,v\in G$. According to (i) and 1.1 we obtain $d(f(u),f(v))=d(u_P+(0-u_Q),\,v_P+(0-v_Q))=d(u_P,v_P)+d(0-u_Q,\,0-v_Q)=d(u_P,v_P)+[(0-u_Q)+v_Q]\vee[(0-v_Q)+u_Q]=d(u_P,v_P)+d(u_Q,v_Q)=d(u,v)$. Let $t\in G$. By 1.1, $f[t_P+(0-t_Q)]=t_P+[0+(0-t_Q)]=t_P+t_Q=t$. Since $f(0)=0$, we conclude that f is a 0-isometry.

2. The sets A and B

Throughout this section we suppose that f is a weak 0-isometry in G. We denote $A_1=\{x\in G^+;\ f(x)=x\}$, $B_1=\{x\in G^+;\ f(x)=0-x\}$, $A=\{x+(0-y);\ x,y\in A_1\}$, $B=\{x+(0-y);\ x,y\in B_1\}$.

2.1. Lemma. *Let $x\in G$.*

(i) *If $x\geq 0$, $f(x)\geq 0$, then $f(x)=x$;*
(ii) *if $x\geq 0$, $f(x)\leq 0$, then $f(x)=0-x$;*
(iii) *if $x\leq 0$, $f(x)\geq 0$, then $f(x)=0-x$;*
(iv) *if $x\leq 0$, $f(x)\leq 0$, then $f(x)=x$.*

Proof.

(i) If $x\geq 0$ and $f(x)\geq 0$, then $0-x\leq 0$, $0-f(x)\leq 0$. Thus $d(x,0)=d(f(x),f(0))$ yields $f(x)=x$.

Analogously we can verify (iii).

(ii) If $x\geq 0$, $f(x)\leq 0$, then from $d(x,0)=d(f(x),f(0))$ we obtain $x=0-f(x)$. Hence $0-x=0-(0-f(x))$. By 1.1 and 1.2, $f(x)=0-x$.

Analogously we can prove (iv).

2.2. Lemma. *Let $x\in G$.*

(i) *If $x\geq 0$, $f(x)\geq 0$, then $f(0-x)=0-x$;*
(ii) *if $x\geq 0$, $f(x)\leq 0$, then $f(0-x)=x$;*
(iii) *if $x\leq 0$, $f(x)\geq 0$, then $f(0-x)=x$;*
(iv) *if $x\leq 0$, $f(x)\leq 0$, then $f(0-x)=0-x$.*

Proof.

(i) Let $x \in G^+$, $f(x) \geq 0$. Then $(0-x)-0 \leq 0$, $0-(0-x) \geq 0$. Thus from $d(0-x,0) = d(f(0-x), f(0))$ we get $0-(0-x) \geq f(0-x)$. By [14, Lemma 13], $0-(0-x) \leq x$. Thus $x \geq f(0-x)$. From this we obtain $x - f(0-x) \geq 0$, $f(0-x)-x \leq 0$. Since $0-x \leq x$, we have $x-(0-x) \geq 0$, $(0-x)-x \leq 0$. In view of 2.1 from $d(x,0-x) = d(f(x), f(0-x))$ we obtain $x-(0-x) = x - f(0-x)$. Thus $[x-(0-x)]-x = [x-f(0-x)]-x$. Then according to [14, Lemma 6] from this we obtain $0-(0-x) = 0-f(0-x)$. Since $0-(0-x) \in U(0, f(0-x))$, in view of [14, Lemma 2] we have $0-(0-x) \geq 0 \vee f(0-x) = [0-f(0-x)] \vee 0 + f(0-x)$. Hence $0-(0-x) \geq [0-(0-x)] + f(0-x)$. Then $[0-(0-x)] + (0-x) \geq [0-(0-x)] + (0-x) + f(0-x)$. Because of $0-x \leq 0$, in view of 1.2 we have $f(0-x) \leq 0$. According to 2.1 (iv) we obtain $f(0-x) = 0-x$.

(ii) Let $x \in G^+$, $f(x) \leq 0$. From $d(0,0-x) = d(f(0), f(0-x))$ we get $0-(0-x) \geq 0-f(0-x)$. By [14, Lemma 13], $x \geq 0-f(0-x)$. From this according to [14, Lemma 6] we obtain $0 \geq [0-f(0-x)]-x = (0-x)-f(0-x)$. Thus $f(0-x) \geq 0-x$, $f(0-x)-(0-x) \geq 0$. In view of 2.1 (ii) we have $x-(0-x) = d(x,0-x) = d(f(x), f(0-x)) = d(0-x, f(0-x)) = f(0-x)-(0-x)$. Then $[x-(0-x)] + (0-x) = [f(0-x)-(0-x)] + (0-x)$. By [14, Lemma 8], $f(0-x) = x$.

(iii) Let $x \in G^-$, $f(x) \geq 0$. From $d(0-x,0) = d(f(0-x), f(0))$ we get $0-x \geq f(0-x)$. Then according to 2.1 (iii) we get $(0-x)-x = d(x,0-x) = d(f(x), f(0-x)) = (0-x)-f(0-x)$. Then $[(0-x)-x] + 2x + f(0-x) = [(0-x)-f(0-x)] + f(0-x) + 2x$. By [14, Lemma 8], $f(0-x) = x$.

(iv) Let $x \in G^-$, $f(x) \leq 0$. Then from $d(0-x,0) = d(f(0-x), f(0))$ we get $0-x \geq 0-f(0-x)$. Then $0-[0-f(0-x)] \geq 0-(0-x)$. From this according to 1.1, 1.2 and [14, Lemma 13] we obtain $f(0-x) \geq x$. In view of 2.2 (iv) we have $(0-x)-x = d(x.0-x) = d(f(x), f(0-x)) = f(0-x)-x$. Then $[(0-x)-x]+x = [f(0-x)-x]+x$. By [14, Lemma 8], $0-x = f(0-x)$.

2.3. LEMMA. *Let $x,y \in A_1$. Then $f(x+y) = x+y$, $f(x-y) = x-y$, $f((0-x)+y) = (0-x)+y$, $f(0-(x+y)) = 0-(x+y)$.*

Proof. Let $x,y \in A_1$. Since $y \leq x+y$, in view of [14, Th. 1] from $d(x+y,y) = d(f(x+y), f(y))$ we obtain $x \geq y - f(x+y)$. By [14, Lemma 6], $0 \geq [y-f(x+y)] - x = (y-x) - f(x+y)$. Thus $f(x+y) \geq y-x \geq 0-x$. According to 2.2 (i), $(x+y)-(0-x) = d(x+y,\ 0-x) = d(f(x+y), f(0-x)) = f(x+y)-(0-x)$. Then $[(x+y)-(0-x)]+(0-x) = [f(x+y)-(0-x)]+(0-x)$. In view of [14, Lemma 8], $x+y = f(x+y)$.

In view of 2.2 (i) from $d(x-y,\, 0-y) = d\big(f(x-y), f(0-y)\big)$ we get $(x-y)-(0-y) \geq f(x-y)-(0-y)$. Thus $\big[(x-y)-(0-y)\big]+(0-y) \geq \big[f(x-y)-(0-y)\big]+(0-y)$. By 1.1 and 1.2, $x-y \geq f(x-y)$. Then $x-f(x-y) \geq x-(x-y)$. Further, from $d(x-y,x) = d\big(f(x-y), f(x)\big)$ we obtain $x-(x-y) \geq x-f(x-y)$. Thus $x-(x-y) = x-f(x-y)$. Then $\big[x-(x-y)\big]-y = \big[x-f(x-y)\big]-y$. From this according to [14, Lemma 6] we have $0 = (x-y)-f(x-y)$. Hence $f(x-y) \geq x-y$. Therefore $f(x-y) = x-y$.

By 2.2 (i), from $d\big((0-x)+y, 0-x\big) = d\big(f\big((0-x)+y\big), f(0-x)\big)$ we get $\big[(0-x)+y\big]-(0-x) \geq f\big((0-x)+y\big)-(0-x)$. Then $\big\{\big[(0-x)+y\big]-(0-x)\big\}+(0-x) \geq \big\{f\big((0-x)+y\big)-(0-x)\big\}+(0-x)$. By 1.1 and 1.2, $(0-x)+y \geq f\big((0-x)+y\big)$. Further, from $d\big((0-x)+y, y\big) = d\big(f\big((0-x)+y\big), f(y)\big)$ it follows that $y-\big((0-x)+y\big) \geq y-f\big((0-x)+y\big)$. By [14, Lemma 6], $0-(0-x) \geq y-f\big((0-x)+y\big)$. Then $\big(0-(0-x)\big)+(0-x)+f\big((0-x)+y\big) \geq \big[y-f\big((0-x)+y\big)\big]+f\big((0-x)+y\big)+(0-x)$. From this, according to [14, Th. 1 and Lemma 8], we obtain $f\big((0-x)+y\big) \geq y+(0-x)$. Therefore $f\big((0-x)+y\big) = (0-x)+y$.

Since $f(x+y) = x+y$, from 2.2 (i) it follows that $f\big(0-(x+y)\big) = 0-(x+y)$.

2.4. Lemma. *A_1 is a convex subsemigroup of G, $A^+ = A_1$ and $f(x) = x$ for each $x \in A$.*

Proof. In view of 2.3 we need to prove only the convexity of A_1. Let $x \leq z \leq y$ for some $x, y \in A_1$, $z \in G$. Since $z \geq 0$, from $d(z,0) = d\big(f(z), f(0)\big)$ we get $f(y) = y \geq z \geq f(z)$. Then from $d(y,z) = d\big(f(y), f(z)\big)$ we get $y-z = y-f(z)$. Thus $(y-z)-y = \big[y-f(z)\big]-y$. According to [14, Lemma 6], we obtain $0 \geq 0-z = 0-f(z)$. Hence $f(z) \geq 0$. Then from 2.1 (i) it follows that $f(z) = z$. Therefore $z \in A_1$.

2.5. Lemma. *Let $x \in G$, $f(x) = x$. Let $x_1 = 0 \vee x$, $x_2 = 0 \wedge x$. Then $f(x_1) = x_1$, $f(x_2) = x_2$, $x_1, 0-x_2 \in A_1$, $x_2, x \in A$.*

Proof. First we show that $f(x_2) = x_2$. From $d(x_2,0) = d\big(f(x_2), f(0)\big)$ we get $0-x_2 \geq f(x_2)$, $0-x_2 \geq 0-f(x_2)$. Further, from $d(x,x_2) = d\big(f(x), f(x_2)\big)$ we obtain $x-x_2 \geq f(x_2)-x$, $x-x_2 \geq x-f(x_2)$. By [14, Lemma 9], $x_1+x_2 = x$. Then from 1.1 and 1.2 it follows that $x-x_2 = x_1$. Thus $x_1 \geq f(x_2)-x$. According to [14, Lemma 6] we obtain $x_1-f(x_2) \geq \big(f(x_2)-x\big)-f(x_2) = 0-x$. In view of 1.1 and 1.2 from $0-x_2 \geq f(x_2)$ we get $x_1-f(x_2) \geq x_1-(0-x_2) = x_1+x_2 = x$. Then by [14, Th. 1], $2\big[x_1-f(x_2)\big] \geq x+(0-x) \geq 0$. According to [14, Lemma 16], $x_1-f(x_2) \geq 0$. Hence $x_1-f(x_2) \geq 0 \vee x = x_1$. From this by [14, Lemma 6] we get $\big[x_1-(0-x_2)\big]-f(x_2) \geq x_1-(0-x_2)$. Thus $x-f(x_2) \geq x$.

Further, in view of [14, Lemma 5] we have $x_1 - f(x_2) \geq 0 \vee (0 - x) = 0 - x_2$. From this according to 1.1, 1.2 and [14, Lemma 6] we obtain $0 \leq [x_1 - f(x_2)] - (0 - x_2) = [x_1 - (0 - x_2)] - f(x_2) = x - f(x_2)$. Hence $x - f(x_2) \geq 0 \vee x = x_1$. From this and [14, Lemma 6] it follows that $0 - f(x_2) = [x - f(x_2)] - x \geq x_1 - x = 0 - x_2$. Thus we have $0 - f(x_2) = 0 - x_2$. Because of $0 - x_2 \geq f(x_2)$, from [14, Lemma 2] it follows that $0 - x_2 \geq 0 \vee f(x_2) = [0 - f(x_2)] \vee 0 + f(x_2)$. Then $0 - x_2 \geq (0 - x_2) + f(x_2)$. Hence $(0 - x_2) + x_2 \geq (0 - x_2) + x_2 + f(x_2)$. By 1.2, $0 \geq f(x_2)$. Then 1.1, 1.2 and 2.1 yields $f(x_2) = x_2 = 0 - (0 - x_2)$. By 2.2 (iv), $f(0 - x_2) = 0 - x_2$. Therefore $(0 - x_2) \in A_1$, $x_2 \in A$.

Now we prove that $f(x_1) = x_1$. Since $x_1 \geq x$, $d(x_1, x) = d(f(x_1), f(x))$ yields $0 - x_2 \geq x - f(x_1)$. In view of [14, Th. 1] we have $(0 - x_2) + f(x_1) \geq x$. Further, from $d(x_1, 0) = d(f(x_1), f(0))$ we get $0 - f(x_1) \leq x_1 = x - x_2$. Then $[0 - f(x_1)] - x \leq (x - x_2) - x$. By [14, Lemma 6], $(0 - x) - f(x_1) \leq 0 - x_2$. In view of [14, Th. 1] we have $(0 - x) \leq [(0 - x) - f(x_1)] + f(x_1) \leq (0 - x_2) + f(x_1)$. Then $2[(0 - x_2) + f(x_1)] \geq x + (0 - x) \geq 0$. From [14, Lemma 16] it follows that $(0 - x_2) + f(x_1) \geq 0$. Thus $(0 - x_2) + f(x_1) \geq 0 \vee (0 - x) = 0 - x_2$. Hence $(0 - x_2) + x_2 + f(x_1) \geq (0 - x_2) + x_2$. By 1.2, $f(x_1) \geq 0$. Then according to 2.1 (i), $f(x_1) = x_1$. Thus $x_1 \in A_1$. Because of $x = x_1 + [0 - (0 - x_2)]$, we have $x \in A$.

2.6. LEMMA. $A = \{x \in G;\ f(x) = x\}$, *$A$ is a convex subset of G.*

P r o o f. First statement is a consequence of 2.4 and 2.5. Let $x_1 + (0 - x_2) \leq z \leq y_1 + (0 - y_2)$ for some $x_1, x_2, y_1, y_2 \in A_1$, $z \in G$. Then $0 \leq z - [x_1 + (0 - x_2)] \leq [y_1 + (0 - y_2)] - [x_1 + (0 - x_2)]$. It is easy to verify that $[y_1 + (0 - y_2)] - [x_1 + (0 - x_2)] \leq y_1 - (0 - x_2)$. By [14, Lemma 13], $y_1 - (0 - x_2) \leq y_1 + x_2$. Thus from 2.4 we get $z - [x_1 + (0 - x_2)] \in A_1$. Then also $\{z - [x_1 + (0 - x_2)]\} + x_1$ belongs to A_1. In view of [14, Lemma 8] we have $z = \{z - [x_1 + (0 - x_2)]\} + x_1 + (0 - x_2) \in A$. Therefore A is a convex subset of G.

2.7. THEOREM. *A is a DRl-semigroup.*

P r o o f. Let $u = u_1 + (0 - u_2)$, $v = v_1 + (0 - v_2)$, where $u_1, v_1, u_2, v_2 \in A_1$. By [14, Lemmas 6 and 13], $(0 - u_2) + (0 - v_2) \geq 0 - (u_2 + v_2)$. Then $u_1 + v_1 \geq u + v \geq u_1 + v_1 + [0 - (u_2 + v_2)]$. In view of 2.4 and 2.6 we have $u + v \in A$. Hence A is a subsemigroup of G. Further, it is easy to see that $u_1 - (0 - v_2) \geq u - v \geq (0 - u_2) - v_1$. Thus, in view of [14, Lemmas 6 and 13] we have $u_1 + v_2 \geq u - v \geq 0 - (u_2 + v_1)$. By 2.4 and 2.6, $u - v \in A$. Since $(0 \vee u) + (0 \vee v) \geq u \vee v \geq u$, $(0 \wedge u) + (0 \wedge v) \leq u \wedge v \leq u$, from 2.5 and 2.6 we obtain that $u \vee v$, $u \wedge v \in A$. Therefore A is a DRl-semigroup.

2.8. LEMMA. *Let $x \in B_1$. Then $x = 0 - (0 - x)$.*

Proof. It follows from the relation $d(x,0) = d(f(x), f(0))$.

2.9. LEMMA. *Let $x \in B$. Then $x = 0 - (0 - x)$.*

Proof. Let $x = y + (0 - z)$, where $y, z \in B_1$. By 1.1, 1.5, 2.8 and [14, Lemma 6], $0 - (0 - x) = 0 - \{[0 - (0 - z)] - y\} = 0 - (z - y) = y + (0 - z) = x$.

2.10. LEMMA. *Let $y \in B$. Then $0 - y$ is the inverse of y and $(x - y) + y = x$, $(x + y) - y = x$, $0 - (x + y) = (0 - x) + (0 - y)$, $0 - (x - y) = y - x = y + (0 - x)$, $x - y = x + (0 - y)$, $x - (0 - y) = x + y$ for each $x \in G$.*

Proof. It follows from 1.1, 1.5 and 2.9.

2.11. LEMMA. *Let $x, y \in B_1$. Then $f(x + y) = 0 - (x + y)$, $f(x - y) = 0 - (x - y)$.*

Proof. According to [14, Th. 1], from $d(x + y, y) = d(f(x + y), f(y))$ we get $x \geq (x + y) - y \geq f(x + y) - (0 - y)$. Then $x - y \geq [f(x - y) - (0 - y)] - y$. By 2.10 and [14, Lemma 6], $x \geq x - y \geq f(x + y) - [(0 - y) + y] = f(x + y)$. In view of 2.2 (ii) from $d(x + y, 0 - x) = d(f(x + y), f(0 - x))$ we have $(x + y) - (0 - x) = x - f(x + y)$. Then $[(x + y) - (0 - x)] + 2(0 - x) + (0 - y) + f(x + y) = [x - f(x + y)] + f(x + y) + 2(0 - x) + (0 - y)$. By 2.10 and [14, Lemma 8], $f(x + y) = (0 - x) + (0 - y) = 0 - (x + y)$.

By 2.2 (ii) from $d(x - y, 0 - y) = d(f(x - y), f(0 - y))$ we get $(x - y) - (0 - y) \geq y - f(x - y)$. Then $[(x - y) - (0 - y)] - x \geq [y - f(x - y)] - x$. In view of [14, Lemma 6] from this we obtain $0 \geq (y - x) - f(x - y)$. Thus $f(x - y) \geq y - x \geq 0 - x$. Then $d(x, x - y) = d(f(x), f(x - y))$ yields $x - (x - y) = f(x - y) - (0 - x)$. Hence $[x - (x - y)] - x = [f(x - y) - (0 - x)] - x$. Finally, according to 2.10 and [14, Lemma 6] from this we obtain $0 - (x - y) = f(x - y)$.

2.12. LEMMA. *B_1 is a convex subsemigroup of G, $B^+ = B_1$ and $f(x) = 0 - x$ for each $x \in B$.*

Proof. In view of 2.10 and 2.11 it remains to prove only the convexity of B_1. Let $x \leq z \leq y$ for some $x, y \in B_1$, $z \in G$. Then $d(z, 0) = d(f(z), f(0))$ yields $z \geq 0 - f(z)$. In view of [14, Lemma 13] from this we get $f(z) \geq 0 - (0 - f(z)) \geq 0 - z \geq 0 - y = f(y)$. Then from $d(y, z) = d(f(y), f(z))$ we obtain $y - z = f(z) - (0 - y)$. Thus $(y - z) - y = [f(z) - (0 - y)] - y$. According to 2.10 and [14, Lemma 6] we have $0 - z = f(z)$. Hence $z \in B_1$.

2.13. LEMMA. *Let* $x \in G$, $f(x) = 0 - x$. *Let* $x_1 = 0 \vee x$, $x_2 = 0 \wedge x$. *Then* $f(x_1) = 0 - x_1$, $f(x_2) = 0 - x_2$, $x_1, 0 - x_2 \in B_1$, $x_2, x \in B$.

Proof. By [14, Lemma 9], $x = x_1 + x_2$. In view of 1.2 from $x_1 = 0 \vee x$ we get $x_1 + (0 - x_2) = x_1 \vee (0 - x_2)$. From $x_2 = 0 \wedge x$ according to [14, Lemma 5] we obtain $0 - x_2 = 0 \vee (0 - x)$. Then we have $x_1 + (0 - x_2) \in U(x, 0 - x)$. Let $v \in G$, $v \in U(x, 0 - x)$. Then $2v \geq x + (0 - x) \geq 0$. By [14, Lemma 16], $v \geq 0$. Thus $v \geq 0 \vee x = x_1$, $v \geq 0 \vee (0 - x) = 0 - x_2$. Hence $v \geq (0 - x_2) \vee x_1 = x_1 + (0 - x_2)$. Therefore $x_1 + (0 - x_2) = x \vee (0 - x)$.

Since $0 - x_1 \leq 0$, $0 - x_1 \leq 0 - x$, from the relations $x_2 \leq 0$, $x_2 \leq x$ we obtain $(0 - x_1) + x_2 \leq x$, $(0 - x_1) + x_2 \leq 0 - x$. From this we derive $0 - [(0 - x_1) + x_2] \geq (0 - x) \vee (0 - (0 - x))$. Further, from $d(0, x) = d(f(0), f(x))$ we have $x \vee (0 - x) = (0 - x) \vee [0 - (0 - x)]$. Therefore $x_1 + (0 - x_2) \leq 0 - [(0 - x_1) + x_2]$. From this according to 1.1 and 1.2 we can easily get that $x_1 \leq 0 - (0 - x_1)$. But according to [14, Lemma 13], $0 - (0 - x_1) \leq x_1$. Thus $0 - (0 - x_1) = x_1$. By 1.5, $0 - x_1$ is the inverse of x_1. Since x_1 and x_2 are invertible, x is invertible as well. Thus by 1.1, $0 - (0 - x) = x$. Further, according to 1.1, 1.2 and [14, Lemma 6] from $x = x_1 + x_2$ we get $x_1 = x - x_2$, $x_1 - x = 0 - x_2$.

Now we prove that $f(x_2) = 0 - x_2$. From the relation $d(x_2, 0) = d(f(x_2), f(0))$ we get $0 - x_2 \geq f(x_2)$, $0 - x_2 \geq 0 - f(x_2)$. Then $(0 - x_2) + x_2 + f(x_2) \geq [0 - f(x_2)] + f(x_2) + x_2$. By 1.1, 1.2 and [14, Th. 1], $f(x_2) \geq x_2$. This implies $x_1 + f(x_2) \geq x$. Further, from $d(x, x_2) = d(f(x), f(x_2))$ we get $x_1 = x - x_2 \geq (0 - x) - f(x_2)$. From this according to [14, Th. 1] we obtain $x_1 + f(x_2) \geq 0 - x$. Therefore $x_1 + f(x_2) \geq x \vee (0 - x) = x_1 + (0 - x_2)$. Because of $0 - x_2 \geq f(x_2)$, we have $x_1 + (0 - x_2) \geq x_1 + f(x_2)$. Thus $x_1 + f(x_2) = x_1 + (0 - x_2)$. Then $f(x_2) = 0 - x_2$. By 2.2 (iii), $f(0 - x_2) = x_2 = 0 - (0 - x_2)$. Therefore $0 - x_2 \in B_1$, $x_2 \in B$.

Finally we show that $f(x_1) = 0 - x_1$. From $d(x_1, 0) = d(f(x_1), f(0))$ we get $x_1 \geq f(x_1)$, $x_1 \geq 0 - f(x_1)$. Thus $x \geq f(x_1) + x_2$. In view of [14, Lemma 6] we have $0 - x \leq (0 - x_2) - f(x_1)$. From $d(x_1, x) = d(f(x_1), f(x))$ we obtain $0 - x_2 \geq f(x_1) - (0 - x)$. By [14, Lemma 6], $(0 - x_2) - f(x_1) \geq 0 - (0 - x) = x$. Thus $(0 - x_2) - f(x_1) \geq x \vee (0 - x) = x_1 + (0 - x_2)$. Because of $x_1 \geq 0 - f(x_1)$, according to 1.1 and [14, Lemma 6] we have $x_1 + (0 - x_2) \geq [0 - f(x_1)] + (0 - x_2)$ $= [0 - f(x_1)] - x_2 = (0 - x_2) - f(x_1)$. Then $x_1 + (0 - x_2) = (0 - x_2) - f(x_1)$. From 1.1 and [14, Lemma 6] it follows that $x_1 = x_1 + (0 - x_2) + x_2 = \{[0 - f(x_1)] - x_2\} + x_2 = 0 - f(x_1)$. Since $x_1 \geq 0 \vee f(x_1)$, [14, Lemma 2] implies $x_1 \geq [0 - f(x_1)] \vee 0 + f(x_1)$. Thus $x_1 \geq x_1 + f(x_1)$. Hence $0 \geq f(x_1)$. By 2.1 (ii), $f(x_1) = 0 - x_1$. Therefore $x_1 \in B_1$. Since $x = x_1 + [0 - (0 - x_2)]$, we have $x \in B$.

2.14. LEMMA. *B is an l-group and a convex subset of G. Furthermore* $B = \{x \in G,\ f(x) = 0 - x\}$.

Proof. Let $x, y \in B$. Thus $x = x_1 + (0 - x_2)$, $y = y_1 + (0 - y_2)$ for some $x_1, x_2, y_1, y_2 \in B_1$. According to 2.10 and 2.12 we have $x + y = x_1 + y_1 + [0 - (x_2 + y_2)] \in B$, $0 - x = (0 - x_1) + [0 - (0 - x_2)] = x_2 + (0 - x_1) \in B$. By 2.10, $0 - x$ is the inverse of x. Hence B is a group. In view of 2.12 and 2.13 it is easy to see that B is an l-group.

Let $g \geq d \geq h$ for some $g, h \in B$, $d \in G$. Then $g + (0 - h) \geq d + (0 - h) \geq 0$. By 2.12, $d + (0 - h)$ belongs to B_1. Then $d = d + (0 - h) + h \in B$. Therefore B is a convex subset of G. The last proposition follows from 2.12 and 2.13.

2.15. LEMMA. *Let $x \in A_1$, $y \in B_1$. Then* $f(x + y) = x + (0 - y)$, $f(x - y) = x + y$, $f((0 - x) + (0 - y)) = (0 - x) + y$, $f((0 - x) - y) = (0 - x) + y$, $f((0 - x) + y) = (0 - x) + (0 - y)$.

Proof. Let $x \in A_1$, $y \in B_1$. In view of [14, Th. 1] from $d(x + y, y) = d(f(x + y), f(y))$ we get $x \geq (x + y) - y \geq f(x + y) - (0 - y)$. Then $x - y \geq [f(x + y) - (0 - y)] - y$. By 2.10 and [14, Lemma 6], $x - y \geq f(x + y)$. From [14, Th. 1] and the relation $d(x + y, x) = d(f(x + y), f(x))$ we infer that $y \geq (x + y) - x \geq x - f(x + y)$. From this according to [14, Lemma 6] we obtain $0 \geq [x - f(x + y)] - y = (x - y) - f(x + y)$. Then $f(x + y) \geq x - y$. Therefore $f(x + y) = x - y = x + (0 - y)$.

According to [14, Lemma 13] from $d(x, x - y) = d(f(x), f(x - y))$ we obtain $y \geq x - f(x - y)$. By [14, Lemma 6], $0 \geq [x - f(x - y)] - y = (x - y) - f(x - y)$. Thus $f(x - y) \geq x - y$. From this and the relation $d(x - y, x + y) = d(f(x - y), f(x + y))$ we get $f(x - y) - (x - y) = (x + y) - (x - y)$. Then $[f(x - y) - (x - y)] + (x - y) = [(x + y) - (x - y)] + (x - y)$. By [14, Lemma 8], $f(x - y) = x + y$.

In view of 2.2 (i), 2.8 and [14, Lemma 6] from $d((0 - x), (0 - x) + (0 - y)) = d(f(0 - x), f((0 - x) + (0 - y)))$ we get $y = (0 - x) - [(0 - x) + (0 - y)] \geq f((0 - x) + (0 - y)) - (0 - x) \geq f((0 - x) + (0 - y))$. According to 2.2 (ii) and [14, Lemma 6] from $d((0 - x) + (0 - y),\ 0 - y) = d(f((0 - x) + (0 - y)), f(0 - y))$ we obtain $0 - (0 - x) = y - f((0 - x) + (0 - y))$. Then $[0 - (0 - x)] + (0 - x) + f((0 - x) + (0 - y)) = [y - f((0 - x) + (0 - y))] + f((0 - x) + (0 - y)) + (0 - x)$. By [14, Lemma 8], $f((0 - x) + (0 - y)) = (0 - x) + y$.

According to 2.2 (i) and [14, Lemma 13] from $d((0 - x) - y, 0 - x) = d(f((0 - x) - y), f(0 - x))$ we obtain $y \geq f((0 - x) - y) - (0 - x) \geq f((0 - x) - y)$. In view of 2.2 (ii) from $d((0 - x) - y,\ 0 - y) = d(f((0 - x) - y), f(0 - y))$ we get

$(0-y)-[(0-x)-y]=y-f((0-x)-y)$. Then $\{(0-y)-[(0-x)-y]\}+[(0-x)-y]+f((0-x)-y)+y=[y-f((0-x)-y)]+f((0-x)-y)+[(0-x)-y]+y$. By 2.10 and [14, Lemma 8], we have $f((0-x)-y)=(0-x)+y$.

In view of 2.2 (i), from $d((0-x)+y,\ 0-x)=d(f((0-x)+y),f(0-x))$ we obtain $[(0-x)+y]-(0-x)\geq f((0-x)+y)-(0-x)$. Then $\{[(0-x)+y]-(0-x)\}+(0-x)\geq[f((0-x)+y)-(0-x)]+(0-x)$. By 1.1 and 1.2, $(0-x)+y\geq f((0-x)+y)$. Then from $d((0-x)+y,(0-x)+(0-y))=d(f((0-x)+y),f((0-x)+(0-y)))$ we have $[(0-x)+y]-[(0-x)+(0-y)]=[(0-x)+y]-f((0-x)+y)$. Then $\{[(0-x)+y]-[(0-x)+(0-y)]\}+(0-x)+(0-y)+[0-(0-x)]+(0-y)+f[(0-x)+y]=\{[(0-x)+y]-f((0-x)+y)\}+f((0-x)+y)+(0-x)+(0-y)+[0-(0-x)]+(0-y)$. In view of 2.10 and [14, Lemma 8] from this we get $f((0-x)+y)=(0-x)+(0-y)$.

2.16. Lemma. *Let $x\in A$, $y\in B_1$. Then $f(x-y)=x+y$.*

Proof. Let $x=a_1+(0-a_2)$ for some $a_1,a_2\in A_1$ and let $y\in B_1$. Since $x-y\leq a_1-y$, according to 2.10, 2.15 and [14, Lemma 6] from $d(a_1-y,\ x-y)=d(f(a_1-y),f(x-y))$ we obtain $(a_1+y)-f(x-y)\leq(a_1-y)-(x-y)=a_1-x=0-(0-a_2)$. Thus $[(a_1+y)-f(x-y)]+(0-a_2)\leq[0-(0-a_2)]+(0-a_2)$. By 1.1, 1.2 and [14, Lemma 13], $[a_1+(0-a_2)+y]-f(x-y)\leq 0$. Then $f(x-y)\geq a_1+(0-a_2)+y\geq(0-a_2)+y$.

Since $x-y\geq(0-a_2)-y$, in view of 2.10, 2.15 and [14, Lemma 6] from $d(x-y,\ (0-a_2)-y)=d(f(x-y),f((0-a_2)-y))$ we get $f(x-y)-[(0-a_2)+y]=(x-y)-[(0-a_2)-y]=x-\{[(0-a_2)-y]+y\}=x-(0-a_2)$. Then $\{f(x-y)-[(0-a_2)+y]\}+(0-a_2)+y=[x-(0-a_2)]+(0-a_2)+y$. By [14, Lemma 8], $f(x-y)=x+y$.

2.17. Lemma. *Let $x\in A_1$, $y\in B$. Then $f((0-x)+y)=(0-x)+(0-y)$.*

Proof. Let $x\in A_1$ and $y=b_1+(0-b_2)$ for some $b_1,b_2\in B_1$. In view of 2.15 and [14, Lemma 13] from $d((0-x)+y,\ (0-x)+(0-b_2))=d(f((0-x)+y),f((0-x)+(0-b_2)))$ we get $b_1\geq[(0-x)+y]-[(0-x)+(0-b_2)]\geq[(0-x)+b_2]-f((0-x)+y)$. From this according to 2.10 and [14, Lemma 6] we obtain $0\geq\{[(0-x)+b_2]-f((0-x)+y)\}-b_1=\{[(0-x)+b_2]-b_1\}-f((0-x)+y)=[(0-x)+b_2+(0-b_1)]-f((0-x)+y)$. Hence $f((0-x)+y)\geq(0-x)+b_2+(0-b_1)\geq(0-x)+(0-b_1)$. Then from $d((0-x)+y,\ (0-x)+b_1)=d(f((0-x)+y),f((0-x)+b_1))$, 2.15 and [14, Lemma 6] it follows that $f((0-x)+y)-[(0-x)+(0-b_1)]=[(0-x)+b_1]-[(0-x)+b_1+(0-b_2)]=$

$0-(0-b_2)$. Thus $\{f((0-x)+y)-[(0-x)+(0-b_1)]\}+[(0-x)+(0-b_1)] = (0-x)+(0-b_1)+[0-(0-b_2)]$. By 2.10 and [14, Lemma 8], $f((0-x)+y) = (0-x)+(0-y)$.

2.18. THEOREM. *Let* $x \in A$, $y \in B$. *Then* $f(x+y) = x+(0-y)$.

Proof. Let $x = a_1+(0-a_2)$, $y = b_1+(0-b_2)$ for some $a_1, a_2 \in A_1$, $b_1, b_2 \in B_1$. In view of 2.10 and 2.16 from $d(x+y,\, x+(0-b_2)) = d(f(x+y),\, f(x+(0-b_2)))$ it follows that $(x+y)-[x+(0-b_2)] \geq (x+b_2)-f(x+y)$. By [14, Lemma 13], $b_1 \geq (x+b_2)-f(x+y)$. According to 2.10 and [14, Lemma 6] $0 \geq [(x+b_2)-f(x+y)]-b_1 = [(x+b_2)-b_1]-f(x+y) = \{x+[0-(0-b_2)]+(0-b_1)\}-f(x+y) = [x+(0-y)]-f(x+y)$. Therefore $f(x+y) \geq x+(0-y) \geq (0-a_2)+(0-y)$. Then from 2.17 and the relation $d(x+y,(0-a_2)+y) = d(f(x+y), f((0-a_2)+y))$ we obtain $(x+y)-[(0-a_2)+y] = f(x+y)-[(0-a_2)+(0-y)]$. In view of [14, Lemma 8] we have $\{(x+y)-[(0-a_2)+y]\}+(0-a_2)+(0-y) = f(x+y)$. Finally, according to 1.1, 1.2, 2.10 and [14, Lemma 6] we have $f(x+y) = \{[(x+y)-y]-(0-a_2)\}+(0-a_2)+(0-y) = x+(0-y)$.

3. Direct decomposition corresponding to a weak 0-isometry

Let f, A_1, B_1, A, B be as in Section 2.

3.1. LEMMA. *Let* $x \in G^+$, $x_1 = 0 \vee f(x)$, $x_2 = 0 \vee (0-f(x))$. *Then* $x = x_1+x_2$, $f(x) = x_1-x_2 = x_1+(0-x_2)$, $f(x_1) = x_1$, $f(x_2) = 0-x_2$.

Proof. First we prove that $x = x_1+x_2$. From $d(x,0) = d(f(x), f(0))$ we get $x = f(x) \vee (0-f(x))$. Since $x_1+x_2 \in U(f(x), 0-f(x))$, we have $x_1+x_2 \geq x$.

Let $\bar{x}_2 = x-x_1$. Clearly $x \geq x_1$, $x \geq x_2$, $\bar{x}_2 \geq 0$. From [14, Lemma 8] it follows that $\bar{x}_2+x_1 = x$. Since $x \in U(0, 0-f(x))$, in view of [14, Th. 1] we have $x+f(x) \in U(f(x),0)$. Thus $x+f(x) \geq 0 \vee f(x) = x_1$. Then according to [14, Lemma 13], $0 \leq [x+f(x)]-x_1 \leq (x-x_1)+f(x)$. Again, by [14, Lemma 13], $0-f(x) \leq [(x-x_1)+f(x)]-f(x) \leq [f(x)-f(x)]+(x-x_1) = \bar{x}_2$. Thus $\bar{x}_2 \geq 0 \vee (0-f(x)) = x_2$. Then $x = x_1+\bar{x}_2 \geq x_1+x_2$. Therefore $x = x_1+x_2$.

Let $z = 0 \wedge f(x)$. Then according to 1.4 we have $z = 0-x_2$, $x_2 = 0-z$, $0-(0-x_2) = x_2$. Then from 1.5 it follows that $0-x_2$ is the inverse of x_2. By 1.1 and [14, Lemma 9], $f(x) = x_1+z = x_1+(0-x_2) = x_1-x_2$.

Now we verify that $f(x_2) = 0-x_2$. From 1.1, and $d(x,x_2) = d(f(x), f(x_2))$ we get $x_1 = x-x_2 \geq f(x_2)-f(x)$. Then $x_1-f(x_2) \geq [f(x_2)-f(x)]-f(x_2)$.

By [14, Lemma 6], $x_1 - f(x_2) \geq 0 - f(x)$. From $d(x_2,0) = d(f(x_2), f(0))$ we obtain $x_2 \geq f(x_2)$, $x_2 \geq 0 - f(x_2)$. Then $x_1 - f(x_2) \geq x_1 - x_2 = f(x)$. Thus $x_1 - f(x_2) \geq f(x) \vee [0 - f(x)] = x_1 + x_2$. According to [14, Lemma 13], from the relation $x_2 \geq 0 - f(x_2)$ we get $x_1 + x_2 \geq x_1 + (0 - f(x_2)) \geq x_1 - f(x_2)$. Therefore $x_1 - f(x_2) = x_1 + x_2$. In view of [14, Lemma 6] we have $0 - f(x_2) = [x_1 - f(x_2)] - x_1 = (x_1 + x_2) - x_1 = (x_1 + x_2) - [x_1 + x_2 + (0 - x_2)] = 0 - (0 - x_2) = x_2$. Since $x_2 \geq f(x_2)$, according to [14, Lemma 2] we obtain $x_2 \geq 0 \vee f(x_2) = (0 - f(x_2)) \vee 0 + f(x_2)$. Thus $x_2 \geq x_2 + f(x_2)$. Hence $0 \geq f(x_2)$. Then 2.1 (ii) yields $f(x_2) = 0 - x_2$. Further, in view of [14, Th. 1] we have $\bar{x}_2 = (x_1 + x_2) - x_1 \leq x_2$. Therefore $x_2 = \bar{x}_2 = x - x_1$.

Finally, we prove that $f(x_1) = x_1$. From $d(x, x_1) = d(f(x), f(x_1))$ we get $x_2 = x - x_1 \geq f(x) - f(x_1)$. In view of [14, Th. 1] we have $x_2 + f(x_1) \geq [f(x) - f(x_1)] + f(x_1) \geq f(x)$. Further, $d(x_1,0) = d(f(x_1), f(0))$ yields $x_1 \geq f(x_1)$, $x_1 \geq 0 - f(x_1)$. By [14, Lemma 13], $f(x_1) \geq 0 - [0 - f(x_1)] \geq 0 - x_1$. From this according to 1.1 and 1.5 we obtain $x_2 + f(x_1) \geq (0 - x_1) + x_2 = (0-x_1)+[0-(0-x_2)] = 0-f(x)$. Thus $x_2+f(x_1) \geq f(x) \vee [0-f(x)] = x_1+x_2$. From this we get $f(x_1) \geq x_1$. Therefore $f(x_1) = x_1$. This completes the proof.

3.2. Lemma. *Let $x \in G^+$ and let $x = g + h = x_1 + x_2$ for some $g, x_1 \in A_1$, $h, x_2 \in B_1$. Then $x_1 = g$, $x_2 = h$.*

Proof. By 2.18, $f(x) = g + (0 - h) = x_1 + (0 - x_2)$. Then from this, 2.10 and [14, Lemma 6] we get $(0 - x_2) = x_1 - (x_1 + x_2) = x_1 - (g + h) = (x_1 - g) - h$, $x_2 = 0 - (0 - x_2) = x_1 - [x_1 + (0 - x_2)] = (x_1 - g) - (0 - h)$. From this according to 2.10 and [14, Lemma 6] we obtain $x_2 - h = [(x_1 - g) - (0 - h)] - h = [(x_1 - g) - h] - (0 - h) = (0 - x_2) - (0 - h) = [0 - (0 - h)] - x_2 = h - x_2$. Since $x_2, h \in B$, in view of 2.10 we have $2(x_2 - h) = (x_2 - h) + h + (0 - x_2) = x_2 + (0 - x_2) = 0$. By [14, Lemma 17], $x_2 - h = 0$. From this according to 2.10 we obtain $x_2 = h$. Then $x_1 + x_2 = g + x_2$ yields $x_1 = g$.

From 3.1 and 3.2 we immediately obtain:

3.3. Lemma. *For each $x \in G^+$ there exist uniquely determined elements $x_1 \in A_1$, $x_2 \in B_1$ such that $x = x_1 + x_2$.*

3.4. Theorem. *Let $x \in A_1$, $y \in B_1$. Then $x = 0 \vee (x - y)$, $0 - y = 0 \wedge (x - y)$.*

Proof. Let $x \in A_1$, $y \in B_1$ and let $z = x + y$, $z_1 = 0 \vee f(z)$, $z_2 = 0 \vee (0 - f(z))$. Then the desired result follows from 1.4, 3.1 and 3.3.

3.5. LEMMA. *For each* $x \in G^-$ *there exist uniquely determined elements* $x_1, x_2 \in G^-$ *such that* $x = x_1 + x_2$, $f(x_1) = x_1$, $f(x_2) = 0 - x_2$. *Moreover,* $x_1 = 0 - \bar{x}_1$, $x_2 = 0 - \bar{x}_2$, *where* $\bar{x}_1 \in A_1$, $\bar{x}_2 \in B_1$.

Proof. Since $0 - x \geq 0$, for the elements $\bar{x}_1 = 0 \vee f(0-x)$ and $\bar{x}_2 = 0 \vee (0 - f(0-x))$ from 3.1 we obtain $(0-x) = \bar{x}_1 + \bar{x}_2$, $\bar{x}_1 \in A_1$, $\bar{x}_2 \in B_1$. According to 1.1, 1.2 and 2.10 we get $x = 0 - (0-x) = 0 - (\bar{x}_1 + \bar{x}_2) = (0 - \bar{x}_1) + (0 - \bar{x}_2)$. Let $x_1 = 0 - \bar{x}_1$, $x_2 = 0 - \bar{x}_2$. Thus $x_1 \leq 0$, $x_2 \leq 0$. By 2.2 and 2.8, $f(x_1) = x_1$, $f(x_2) = \bar{x}_2 = 0 - (0 - \bar{x}_2) = 0 - x_2$.

Let $x = y_1 + y_2$, $f(y_1) = y_1$, $f(y_2) = 0 - y_2$ for some $y_1, y_2 \in G^-$. By 1.1 and 1.2, $0 - x = (0 - y_1) + (0 - y_2)$. According to 1.1, 1.2 and 2.2 we have $f(0 - y_1) = 0 - y_1$, $f(0 - y_2) = y_2 = 0 - (0 - y_2)$. Since $0 - y_1$, $0 - y_2 \in G^+$, from 3.3 it follows that $0 - y_1 = \bar{x}_1$, $0 - y_2 = \bar{x}_2$. In view of 1.1 and 1.2 we have $y_1 = 0 - (0 - y_1) = x_1$. Similarly $y_2 = x_2$.

3.6. LEMMA. *Let* $z \in G$. *Then there exist* $z_1 \in A$, $z_2 \in B$ *such that* $z = z_1 + z_2$.

Proof. Let $z \in G$, $x = z \vee 0$, $y = z \wedge 0$. According to 3.1 and 3.5 there exist x_1, $y_1 \in A_1$, $x_2, y_2 \in B_1$ such that $x = x_1 + x_2$, $y = (0 - y_1) + (0 - y_2)$. By [14, Lemma 9], $z = x + y$. If we put $z_1 = x_1 + (0 - y_1)$, $z_2 = x_2 + (0 - y_2)$, then z_1 and z_2 have the desired properties.

3.7. LEMMA. *Let* $x \in G$, $x = g + h = x_1 + x_2$, *where* $g, x_1 \in A$, $h, x_2 \in B$. *Then* $x_1 = g$, $x_2 = h$.

The proof of this lemma follows on the same lines employed in the proof of Lemma 3.2.

3.8. LEMMA. *For each* $x \in G$ *there exist uniquely determined elements* $x_1 \in A$, $x_2 \in B$ *such that* $x = x_1 + x_2$.

Proof. This is a consequence of 3.6 and 3.7.

3.9. LEMMA. *Let* $x, y \in G$ *and let* $x = x_1 + x_2$, $y = y_1 + y_2$, $x + y = (x+y)_1 + (x+y)_2$, $x - y = (x-y)_1 + (x-y)_2$, *where* $x_1, y_1, (x+y)_1, (x-y)_1 \in A$, $x_2, y_2, (x+y)_2, (x-y)_2 \in B$. *Then* $(x+y)_1 = x_1 + y_1$, $(x+y)_2 = x_2 + y_2$, $(x-y)_1 = x_1 - y_1$, $(x-y)_2 = x_2 - y_2$.

Proof. According to 2.7, 2.14 and 3.8 from the relation $(x+y)_1 + (x+y)_2 = x + y = x_1 + y_1 + x_2 + y_2$ we obtain $(x+y)_1 = x_1 + y_1$, $(x+y)_2 = x_2 + y_2$. In view of 2.10 and [14, Lemma 6] we have $(x - y) - x_2 = [(x_1 + x_2) - x_2] - (y_1 + y_2) = (x_1 - y_1) - y_2 = (x_1 - y_1) + (0 - y_2)$. By 2.10 we have $(x - y) = [(x-y) - x_2] + x_2 = (x_1 - y_1) + (x_2 - y_2)$. According to 2.7, 2.10 and 2.14, $(x_1 - y_1) \in A$, $(x_2 - y_2) \in B$. Then from 3.8 it follows that $(x - y)_1 = x_1 - y_1$, $(x - y)_2 = x_2 - y_2$.

3.10. LEMMA. *Let $x, y \in G$, $x = x_1 + x_2$, $y = y_1 + y_2$, where $x_1, y_1 \in A$, $x_2, y_2 \in B$. Then $x \le y$ if and only if $x_1 \le y_1$ and $x_2 \le y_2$.*

Proof. The "if" part is obvious, so we prove "only if" part. Since $x - y \le 0$, according to 3.5 we obtain $x - y = (x - y)_1 + (x - y)_2$, where $(x - y)_1 \in A$, $(x - y)_2 \in B$, $(x - y)_1 \le 0$, $(x - y)_2 \le 0$. In view of 3.9 we have $(x - y)_1 = x_1 - y_1 \le 0$, $(x - y)_2 = x_2 - y_2 \le 0$. Thus $x_1 \le y_1$, $x_2 \le y_2$.

3.11. THEOREM. *G is the direct product of the DRl-semigroup A and the l-group B and $f(x) = x_A + (0 - x_B)$ for each $x \in G$.*

Proof. This follows from 2.7, 2.14, 2.18, 3.8 and 3.10.

3.12. THEOREM. *Any weak 0-isometry in G is an involutory semigroup automorphism.*

Proof. The assertion is a consequence of 2.10 and 3.11.

REFERENCES

[1] BIRKHOFF, G.: *Lattice Theory. Amer. Math. Soc. Colloq. Publ. 25.* (3rd ed.), Amer. Math. Soc., Providence, RI, 1967.

[2] HOLLAND, CH.: *Intrinsic metrics for lattice ordered groups*, Algebra Universalis **19** (1984), 142–150.

[3] JAKUBÍK, J.: *Isometries of lattice ordered groups*, Czechoslovak Math. J. **30** (1980), 142–152.

[4] JAKUBÍK, J.: *On isometries of non-abelian lattice ordered groups*, Math. Slovaca **31** (1981), 171–175.

[5] JAKUBÍK, J.: *Weak isometries of lattice ordered groups*, Math. Slovaca **38** (1988), 133–138.

[6] JAKUBÍK, J.—KOLIBIAR, M.: *Isometries of multilattice groups*, Czechoslovak Math. J. **33** (1983), 602–612.

[7] JASEM, M.: *Isometries in Riesz groups*, Czechoslovak Math. J. **36** (1986), 35–43.

[8] JASEM, M.: *On weak isometries in multilattice groups*, Math. Slovaca **40** (1990), 337–340.

[9] JASEM, M.: *Weak isometries and isometries in partially ordered groups.* (Submitted).

[10] JASEM, M.: *Isometries in non-abelian multilattice groups*, Czechoslovak Math. J., (To appear).

[11] POWELL, W. B.: *On isometries in abelian lattice ordered groups*, J. Indian Math. Soc. **46** (1982), 189–194.

[12] RACHŮNEK, J.: *Isometries in ordered groups*, Czechoslovak Math. J. **34** (1984), 334–341.

[13] SWAMY, K. L. N.: *A general theory of autometrized algebras*, Math. Ann. **157** (1964), 65–74.

[14] SWAMY, K. L. N.: *Dually residuated lattice ordered semigroups*, Math. Ann. **159** (1965), 105–114.

[15] SWAMY, K. L. N.: *Dually residuated lattice ordered semigroups. II*, Math. Ann. **160** (1965), 64–71.

[16] SWAMY, K. L. N.: *Dually residuated lattice ordered semigroups. III*, Math. Ann. **167** (1966), 71–74.

[17] SWAMY, K. L. N.: *Isometries in autometrized lattice ordered groups*, Algebra Universalis **8** (1978), 59–64.

[18] SWAMY, K. L. N.: *Isometries in autometrized lattice ordered groups. II*, Math. Seminar Notes Kobe Univ. **5** (1977), 211–214.

[19] SWAMY, K. L. N.—SUBBA RAO, B. V.: *Isometries in dually residuated lattice ordered semigroups*, Math. Seminar Notes Kobe Univ. **8** (1980), 369–379.

[20] TRIAS, J.: *Lattice isometries in Riesz spaces*, Preprint, Univ. Politecnica Barcelona. (1981).

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Department of Mathematics
Faculty of Chemical Technology
Slovak Technical University
Radlinského 9
812 37 Bratislava
Slovakia