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COUNTABLY S -CLOSED SPACES

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(Communicated by Július Korbaš)

ABSTRACT. In this paper we introduce the class of countably S -closed spaces which lies between the familiar classes of S -closed spaces and feebly compact spaces. We characterize countably S -closed spaces and study their basic properties. In addition, we investigate the relationship between countably S -closed spaces and feebly compact spaces. Several examples illustrate our results.

1. Introduction and preliminaries

In 1976, Thompson [8] introduced the class of S -closed spaces. A space X is called S -closed if every semi-open cover has a finite subfamily the closures of whose members cover X , or equivalently, if every regular closed cover of X has a finite subcover. Herrmann [3] proved that a Hausdorff space is S -closed if and only if it is quasi- H -closed and extremally disconnected. Recall that a space X is said to be quasi- H -closed if every open cover of X has a finite subfamily the closures of whose members cover X . If we replace in the definition of quasi- H -closedness “every open cover” by “every countable open cover” we obtain the important class of feebly compact spaces (also known as lightly compact spaces).

In this paper, we introduce and study a new class of spaces, namely countably S -closed spaces, i.e. spaces in which every countable regular closed cover has a finite subcover. In Section 2, we provide several characterizations of countably S -closed spaces and investigate their basic properties. It is pointed out that this class of spaces lies strictly between the class of S -closed spaces and the class of feebly compact spaces. In Section 3, we further explore the relationship between countably S -closed spaces and feebly compact spaces. In particular, the concept of km -perfect spaces is introduced. Finally, in Section 4, we present several examples to illustrate the results obtained in Section 2 and Section 3.

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For a subset A of a topological space (X, τ) we denote the closure of A and the interior of A by $\text{cl } A$ and $\text{int } A$, respectively. The subspace topology on A is denoted by $\tau|_A$. A subset G of (X, τ) is called regular open if $G = \text{int}(\text{cl } G)$. $F \subseteq X$ is said to be regular closed if $X - F$ is regular open, or equivalently, if $F = \text{cl}(\text{int } F)$. The families of regular open subsets and regular closed subsets of (X, τ) are denoted by $RO(X, \tau)$ and $RC(X, \tau)$, respectively. $RO(X, \tau)$ is a base for a coarser topology τ_s on X , called the semi-regularization topology on X . (X, τ) is said to be extremally disconnected, abbreviated e.d., if every regular open set is closed, or equivalently, if $RO(X, \tau) = RC(X, \tau)$. It is known that every dense subspace of an e.d. space is e.d. In order to facilitate the reading of this paper, we now summarize some well known results.

LEMMA 1.1. *Let (X, τ) be a space. Then*

- i) $RC(X, \tau) = RC(X, \tau_s)$;
- ii) (X, τ) is e.d. if and only if (X, τ_s) is e.d.;
- iii) if $A \subseteq X$ is locally dense, i.e. if $A \subseteq \text{int}(\text{cl } A)$, then

$$RC(A, \tau|_A) = \{F \cap A : F \in RC(X, \tau)\}.$$

A subset S of (X, τ) is called semi-open [5] (regular semi-open [1], respectively) if there is an open set U (a regular open set U , respectively) such that $U \subseteq S \subseteq \text{cl } U$. A space (X, τ) is called quasi- H -closed (feebly compact, S -closed [8], respectively) if every open cover (every countable open cover, every semi-open cover, respectively) of (X, τ) has a finite subfamily the closures of whose members cover X . Following H o d e l [4], a cellular family in a space (X, τ) is a collection of nonempty, pairwise disjoint open sets. We will denote the set of natural numbers by ω , and $\beta\omega$ is the Stone-Ćech compactification of ω . Finally, a sequence $\{A_n : n \in \omega\}$ of subsets of a set X is called decreasing (increasing, respectively) if $A_{n+1} \subseteq A_n$ ($A_n \subseteq A_{n+1}$, respectively) for each $n \in \omega$. Strictly decreasing sequences and strictly increasing sequences of subsets are defined in the obvious ways.

No separation axioms are assumed unless explicitly stated.

2. Characterizations and basic properties

We begin by defining the class of spaces we will study in this paper.

DEFINITION. A topological space (X, τ) is *countably S -closed* if every countable cover of regular closed sets has a finite subcover.

The following fundamental observation is easily verified.

PROPOSITION 2.1. *Every S -closed space is countably S -closed, and every countably S -closed space is feebly compact.*

Note that the converses of these implications are false, however. Example 4.1. provides a space which is countably S -closed but not S -closed, and in Example 4.3, we present several feebly compact spaces which are not countably S -closed.

In our next result we present a huge variety of characterizations of countably S -closed spaces.

THEOREM 2.2. *For a space (X, τ) the following are equivalent:*

- 1) (X, τ) is countably S -closed.
- 2) Every countable cover by semi-open sets has a finite subfamily the closures of whose members cover X .
- 3) Every countable cover by regular semi-open sets has a finite subfamily the closures of whose members cover X .
- 4) There is no strictly increasing sequence of regular closed sets whose union is X .
- 5) If $\{F_n : n \in \omega\}$ is a decreasing sequence of nonempty regular closed sets, then

$$\bigcap \{\text{int } F_n : n \in \omega\} \neq \emptyset.$$

- 6) If $\{G_n : n \in \omega\}$ is a decreasing sequence of nonempty regular open sets, then

$$\bigcap \{G_n : n \in \omega\} \neq \emptyset.$$

- 7) If $\{G_n : n \in \omega\}$ is a sequence of regular open sets satisfying the finite intersection property, then

$$\bigcap \{G_n : n \in \omega\} \neq \emptyset.$$

- 8) If $\{G_n : n \in \omega\}$ is a filter basis consisting of regular open sets, then

$$\bigcap \{G_n : n \in \omega\} \neq \emptyset.$$

Proof.

1) \iff 2) \iff 3): This is obvious since the closure of every semi-open set is regular closed. Furthermore, every regular closed set is regular semi-open and thus semi-open.

1) \implies 4): This is trivial.

4) \implies 1): Suppose that (X, τ) is not countably S -closed. Then there exists a countable regular closed cover $\{F_n : n \in \omega\}$ of X such that for a $k \in \omega$, $\bigcup\{F_n : 1 \leq n \leq k\} \neq X$. By induction we can construct a family $\{A_n : n \in \omega\}$ as follows: for $n = 1$ set $A_1 = F_1$. For $n \geq 2$ there must be a least $m \in \omega$ such that A_{n-1} is strictly contained in $F_1 \cup \dots \cup F_m \neq X$. Define A_n by $A_n = F_1 \cup \dots \cup F_m$. Since $\{A_n : n \in \omega\}$ is a strictly increasing sequence of regular closed sets whose union is X , we have a contradiction to 4).

1) \implies 5): Let $\{F_n : n \in \omega\}$ be a decreasing sequence of nonempty regular closed sets. Suppose that $\bigcap\{\text{int } F_n : n \in \omega\} = \emptyset$. Then $\{\text{cl}(X - F_n) : n \in \omega\}$ is a regular closed cover of X . By assumption, there exists $m \in \omega$ such that $X = \bigcup\{\text{cl}(X - F_i) : i = 1, \dots, m\} = \text{cl}(X - F_m)$. Hence $\text{int } F_m = \emptyset$, which gives a contradiction.

5) \implies 6): Set $F_n = \text{cl } G_n$ for all $n \in \omega$ and apply 5).

6) \implies 7): Let $\{G_n : n \in \omega\}$ be a sequence of regular open sets satisfying the finite intersection property. Set $U_n = G_1 \cap \dots \cap G_n$ for all $n \in \omega$. Then $\{U_n : n \in \omega\}$ is a decreasing sequence of nonempty regular open sets and $\bigcap\{G_n : n \in \omega\} = \bigcap\{U_n : n \in \omega\} \neq \emptyset$.

7) \implies 8): This is trivial since every filter base satisfies the finite intersection property.

8) \implies 1): Suppose that (X, τ) is not countably S -closed. Then there is a countable regular closed cover $\{F_n : n \in \omega\}$ of X without a finite subcover. For $n \in \omega$ define G_n by $G_n = X - (F_1 \cup \dots \cup F_n)$. Then G_n is nonempty and regular open for all $n \in \omega$. Furthermore, it is easily proved that $\{G_n : n \in \omega\}$ is a filter base with empty intersection, a contradiction to 8). \square

As an immediate consequence of Lemma 1.1 we note the following result.

LEMMA 2.3. *Let (X, τ) be a space and suppose that $X = A_1 \cup \dots \cup A_n \cup E$, where each A_i is a locally dense, countably S -closed subspace and $E \subseteq X$ is finite. Then (X, τ) is countably S -closed.*

LEMMA 2.4. *Let (X, τ) be a space. Suppose there exists $x_0 \in X$ having an open neighbourhood base $\{U_n : n \in \omega\}$ with the following properties:*

- i) each $\text{cl } U_{n+1}$ is strictly contained in U_n ,
- ii) $U_1 = X$,
- iii) $\{x_0\} = \bigcap\{U_n : n \in \omega\} = \bigcap\{\text{cl } U_n : n \in \omega\}$.

Then (X, τ) is not countably S -closed.

Proof. Let $\{\omega_k : k \in \omega\}$ be a partition of ω , where each ω_k is infinite. For every $k \in \omega$ let $G_k = \bigcup\{U_n - \text{cl } U_{n+1} : n \in \omega_k\}$. Then $\{G_k : k \in \omega\}$ is a cellular family. One checks easily that $x_0 \in \text{cl } G_k$ and $\bigcup\{\text{cl } U_n - \text{cl } U_{n+1} : n \in \omega_k\} \subseteq \text{cl } G_k$ for each $k \in \omega$. We now show that $\{\text{cl } G_k : k \in \omega\}$ covers

X . If $x \neq x_0$, then there exists $m \in \omega$ such that $x \in \text{cl}U_m - \text{cl}U_{m+1}$. There is some $k \in \omega$ such that $m \in \omega_k$ and so $x \in \text{cl}G_k$. Since $\{G_k : k \in \omega\}$ is a cellular family, $\{\text{cl}G_k : k \in \omega\}$ is a countable regular closed cover of X without a finite subcover. Thus (X, τ) is not countably S -closed. \square

COROLLARY 2.5.

1) *An infinite regular space which is first countable at some non-isolated point is not countably S -closed.*

2) *Suppose that (X, τ) is an infinite, regular feebly compact space and there exists $x_0 \in X$ such that $\{x_0\}$ is a G_δ -set but not open. Then (X, τ) is not countably S -closed.*

P r o o f. 1) is an immediate consequence of Lemma 2.4. To prove 2) observe that by [6; Proposition 2.2], (X, τ) is first countable at x_0 . Now apply 1). \square

We now focus on the fundamental properties of countably S -closed spaces. To begin with, recall that a topological property R is said to be semi-regular provided that a space (X, τ) has property R if and only if (X, τ_s) has property R . The property R is called contagious if a space (X, τ) has property R whenever a dense subspace of (X, τ) has property R . Our first result is an immediate consequence of Lemma 1.1.

PROPOSITION 2.6. *Let R be the property “countably S -closed”. Then R is both semi-regular and contagious.*

Recall that a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called irresolute if $f^{-1}(S)$ is semi-open in (X, τ) whenever S is semi-open in (Y, σ) . It is known that a function which is continuous, open and onto, is irresolute. Thompson [9] has shown that if (X, τ) is S -closed and $f : (X, \tau) \rightarrow (Y, \sigma)$ is irresolute and onto, then (Y, σ) is S -closed. The same idea works to prove our next result.

PROPOSITION 2.7.

i) *Let (X, τ) be countably S -closed and let $f : (X, \tau) \rightarrow (Y, \sigma)$ be irresolute and onto. Then (Y, σ) is countably S -closed.*

ii) *Let (X, τ) be countably S -closed and let $f : (X, \tau) \rightarrow (Y, \sigma)$ be continuous, open and onto. Then (Y, σ) is countably S -closed.*

iii) *If a product of topological spaces is countably S -closed, then each factor space is countably S -closed.*

R e m a r k 2.8. The converse of Proposition 2.7. iii) is false. $\beta\omega$ is S -closed hence countably S -closed, but $\beta\omega \times \beta\omega$ is not countably S -closed as shown in Example 4.4.

PROPOSITION 2.9. *Let (X, τ) be countably S -closed.*

- i) *If $G \in RO(X, \tau)$, then $(G, \tau|_G)$ is countably S -closed.*
- ii) *If $F \in RC(X, \tau)$, then $(F, \tau|_F)$ is countably S -closed.*
- iii) *If $(A, \tau|_A)$ is a countably S -closed subspace of (X, τ) (here (X, τ) need not be countably S -closed), and if $A \subseteq T \subseteq \text{cl } A$, then $(T, \tau|_T)$ is countably S -closed.*
- iv) *Let (X, τ) be regular. If $p \in X$ is a non-isolated point, then $X - \{p\}$ is a countably S -closed subspace.*

P r o o f.

i) Let $\{A_n : n \in \omega\} \subseteq RC(G, \tau|_G)$ be a cover of G . By Lemma 1.1, for each $n \in \omega$ $A_n = G \cap F_n$ for some $F_n \in RC(X, \tau)$. Since $\{F_n : n \in \omega\} \cup \{X - G\}$ is a regular closed cover of (X, τ) , there exists $m \in \omega$ such that $X = (X - G) \cup F_1 \cup \dots \cup F_m$. Consequently, $G = A_1 \cup \dots \cup A_m$ and thus $(G, \tau|_G)$ is countably S -closed.

ii) Let $F \in RC(X, \tau)$. Then $\text{int } F \in RO(X, \tau)$ and $\text{int } F$ is dense in $(F, \tau|_F)$. By i) and Proposition 2.6, $(F, \tau|_F)$ is countably S -closed.

iii) Since A is dense in $(T, \tau|_T)$, by Proposition 2.6, $(T, \tau|_T)$ is countably S -closed.

iv) Let $D = X - \{p\}$. If D is finite, clearly $(D, \tau|_D)$ is countably S -closed. Suppose that D is infinite. Let $\{A_n : n \in \omega\} \subseteq RC(D, \tau|_D)$ be a cover of D . By Lemma 1.1, for each $n \in \omega$ $A_n = D \cap F_n$ for some $F_n \in RC(X, \tau)$. If $X \neq \bigcup \{F_n : n \in \omega\}$, then $\{p\}$ is a G_δ -set in (X, τ) and by Corollary 2.5, (X, τ) and $(D, \tau|_D)$ are finite spaces, which is a contradiction. Thus $X = \bigcup \{F_n : n \in \omega\}$, and there exists $m \in \omega$ such that $X = F_1 \cup \dots \cup F_m$ and $D = A_1 \cup \dots \cup A_m$, i.e. $(D, \tau|_D)$ is countably S -closed. \square

R e m a r k 2.10. The property “countably S -closed” is in general not hereditary with respect to open, dense or closed subspaces. $\beta\omega$ is S -closed hence countably S -closed. $\omega \subseteq \beta\omega$ is open and dense in $\beta\omega$ but clearly not countably S -closed. Moreover, we show in Example 4.5, that $\beta\omega - \omega$ fails to be countably S -closed.

3. Countably S -closed spaces versus feebly compact spaces

In this section, we focus on the relationship between countably S -closed spaces and feebly compact spaces. We already pointed out in Proposition 2.1 that every countably S -closed space is feebly compact whereas the converse does not hold in general (see Example 4.3). Therefore it is quite natural to search for a condition (P) such that a space is countably S -closed if and only if it is feebly compact and satisfies (P). For the class of S -closed spaces there exists the

following interesting result [3]: A Hausdorff space is S -closed if and only if it is quasi- H -closed and e.d. Unfortunately, there is no analogous result for the class of countably S -closed spaces. It is obvious that every feebly compact e.d. space is countably S -closed but in Example 4.2, we show that there exist countably S -closed, compact Hausdorff spaces which are not e.d. However, we are able to characterize the class of spaces which are countably S -closed and e.d.

DEFINITION. A space (X, τ) is called *km-perfect* if for each $U \in RO(X, \tau)$ and each $x \notin U$ there is a sequence $\{G_n : n \in \omega\}$ of open sets such that $\bigcup\{G_n : n \in \omega\} \subseteq U \subseteq \bigcup\{\text{cl } G_n : n \in \omega\}$ and $x \notin \bigcup\{\text{cl } G_n : n \in \omega\}$.

Our next result shows that there is a variety of spaces which are *km-perfect*. Recall that a space (X, τ) is said to be perfect (*RC*-perfect [6], respectively) if every open set is a countable union of closed sets (regular closed sets, respectively).

THEOREM 3.1. *If a space (X, τ) is either*

- i) *e.d., or*
- ii) *hereditarily Lindelöf and Hausdorff, or*
- iii) *second countable and Hausdorff, or*
- iv) *RC-perfect, or*
- v) *regular and perfect,*

then it is km-perfect.

Proof. Let $U \in RO(X, \tau)$ and $x \notin U$.

i) Suppose that (X, τ) is e.d. Then U is closed and we are done by setting $G_n = U$ for each $n \in \omega$.

ii) If (X, τ) is hereditarily Lindelöf and Hausdorff, for each $y \in U$ there is an open set V_y such that $y \in V_y \subseteq U$ and $x \notin \text{cl } V_y$. Then $\{V_y : y \in U\}$ is an open cover of U which possesses a countable subcover $\{V_{y_n} : n \in \omega\}$. Then $U = \bigcup\{V_{y_n} : n \in \omega\}$ and $x \notin \text{cl } V_{y_n}$ for each $n \in \omega$, proving that (X, τ) is *km-perfect*.

iii) This follows from ii) since every second countable space is hereditarily Lindelöf.

iv) If (X, τ) is *RC*-perfect, then $U = \bigcup\{F_n \in RC(X, \tau) : n \in \omega\}$. Thus $\bigcup\{\text{int } F_n : n \in \omega\} \subseteq U = \bigcup\{F_n : n \in \omega\}$ and $x \notin F_n$ for each $n \in \omega$.

v) Suppose that (X, τ) is regular and perfect. Then $U = \bigcup\{A_n : n \in \omega\}$, where each A_n is closed. For each $n \in \omega$, $x \notin A_n$ and by regularity there exists an open set G_n with $A_n \subseteq G_n \subseteq U$ and $x \notin \text{cl } G_n$. Hence (X, τ) is *km-perfect*. □

The importance of the class of *km-perfect* spaces is illustrated by

THEOREM 3.2. *Let (X, τ) be countably S -closed and km -perfect. Then (X, τ) is e.d.*

Proof. Let $U \in RO(X, \tau)$ and $x \notin U$. Let $\{G_n : n \in \omega\}$ be a sequence of open sets with $\bigcup\{G_n : n \in \omega\} \subseteq U \subseteq \bigcup\{\text{cl}G_n : n \in \omega\}$ and $x \notin \bigcup\{\text{cl}G_n : n \in \omega\}$. By Lemma 1.1, $\{U \cap \text{cl}G_n : n \in \omega\} \subseteq RC(U, \tau|_U)$ is a cover of U . By Proposition 2.9, $(U, \tau|_U)$ is countably S -closed so there exists $m \in \omega$ such that $U \subseteq \text{cl}G_1 \cup \dots \cup \text{cl}G_m$. Since $x \in X - (\text{cl}G_1 \cup \dots \cup \text{cl}G_m)$, we have $x \notin \text{cl}U$. Thus U is closed, i.e. (X, τ) is e.d.

COROLLARY 3.3.

i) *A km -perfect space is countably S -closed if and only if it is feebly compact and e.d.*

ii) *A countably S -closed space is e.d. if and only if it is km -perfect.*

In order to characterize countably S -closed spaces in terms of feebly compact spaces satisfying an additional condition, we need

LEMMA 3.4. *For a space (X, τ) the following are equivalent:*

- 1) *(X, τ) is feebly compact.*
- 2) *Every locally finite cellular family is finite.*
- 3) *If $\{U_n : n \in \omega\}$ is a decreasing sequence of nonempty open sets (regular open sets, respectively), then $\bigcap\{\text{cl}U_n : n \in \omega\} \neq \emptyset$.*
- 4) *If $\{F_n : n \in \omega\}$ is a decreasing sequence of nonempty regular closed sets, then $\bigcap\{F_n : n \in \omega\} \neq \emptyset$.*

Proof.

1) \iff 2) \iff 3) can be found in [7; p. 50], and 3) \iff 4) is obvious.

Using Theorem 2.2 and Lemma 3.4, the next result is immediate.

THEOREM 3.5. *A space (X, τ) is countably S -closed if and only if it is feebly compact, and whenever $\{F_n : n \in \omega\}$ is a decreasing sequence of nonempty regular closed sets with nonempty intersection, then $\bigcap\{\text{int}F_n : n \in \omega\} \neq \emptyset$.*

THEOREM 3.6. *For a space (X, τ) the following are equivalent:*

- 1) *(X, τ) is countably S -closed.*
- 2) *Every cellular family $\{U_\lambda : \lambda \in \Lambda\}$ satisfying $\text{cl}\left(\bigcup\{U_\lambda : \lambda \in \Lambda\}\right) = \bigcup\{\text{cl}U_\lambda : \lambda \in \Lambda\}$ is finite.*

Proof.

1) \implies 2): Let $\{G_\lambda : \lambda \in \Lambda\}$ be a cellular family with $\text{cl}\left(\bigcup\{G_\lambda : \lambda \in \Lambda\}\right) = \bigcup\{\text{cl}G_\lambda : \lambda \in \Lambda\}$. Suppose that Λ is infinite. Pick a countably infinite subset $\Lambda_1 \subseteq \Lambda$ and let $\Lambda_2 = \Lambda - \Lambda_1$. Set $U_\lambda = G_\lambda$ for each $\lambda \in \Lambda_1$, $U^* = \bigcup\{G_\lambda : \lambda \in \Lambda_2\}$ and $V = \text{int}\left(\text{cl}\left(\bigcup\{G_\lambda : \lambda \in \Lambda\}\right)\right)$. Because of $\text{cl}\left(\bigcup\{G_\lambda : \lambda \in \Lambda\}\right) = \bigcup\{\text{cl}G_\lambda : \lambda \in \Lambda\}$ we have $V \subseteq \bigcup\{\text{cl}U_\lambda : \lambda \in \Lambda_1\} \cup \text{cl}U^*$. Since V is a regular open subset, it follows from Proposition 2.9. that $(V, \tau|_V)$ is countably S -closed. By Lemma 1.1, $\{\text{cl}U_\lambda \cap V : \lambda \in \Lambda_1\} \cup \{\text{cl}U^* \cap V\} \subseteq RC(V, \tau|_V)$, hence there is a finite subset $\{\lambda_1, \dots, \lambda_m\} \subseteq \Lambda_1$ such that $V \subseteq \text{cl}U_{\lambda_1} \cup \dots \cup \text{cl}U_{\lambda_m} \cup \text{cl}U^*$. If $\lambda \in \Lambda_1 - \{\lambda_1, \dots, \lambda_m\}$, then $G_\lambda \subseteq V$ and $G_\lambda \cap (U_{\lambda_1} \cup \dots \cup U_{\lambda_m} \cup U^*)$ is empty, thus G_λ is empty, which is a contradiction. Hence Λ has to be finite.

2) \implies 1): If (X, τ) is not countably S -closed, then by 4), there is a strictly increasing sequence $\{F_n : n \in \omega\}$ of regular closed sets whose union is X . Define $U_1 = \text{int}F_1$ and $U_n = \text{int}F_n - F_{n-1}$ for each $n \geq 2$. It is easily checked that $\{U_n : n \in \omega\}$ is an infinite cellular family satisfying $\text{cl}\left(\bigcup\{U_n : n \in \omega\}\right) = \bigcup\{\text{cl}U_n : n \in \omega\}$, which gives a contradiction. Hence (X, τ) is countably S -closed. \square

Note that condition 2) of the above theorem is a generalization of the condition 2) of Lemma 3.4 because every locally finite family is closure-preserving. Thus it might be interesting to know whether the condition “Every closure-preserving cellular family is finite” defines a new class of spaces between the countably S -closed spaces and the feebly compact spaces.

Recall that a space (X, τ) is called a P -space if every G_δ -set is open. Note that if X is an uncountable set endowed with the co-countable topology τ , then (X, τ) is a countably S -closed P -space. There is an interesting characterization of countably S -closed P -spaces which seems to be worth mentioning.

PROPOSITION 3.7. *A P -space (X, τ) is countably S -closed if and only if every dense subspace is feebly compact.*

Proof. Let (X, τ) be countably S -closed and let $D \subseteq X$ be dense. If $\{U_n : n \in \omega\} \subseteq \tau$ is a cover of D , then $\bigcup\{\text{cl}U_n : n \in \omega\}$ is closed and thus equal to X . Hence there exists $m \in \omega$ such that $X = \text{cl}U_1 \cup \dots \cup \text{cl}U_m$. Consequently, $(D, \tau|_D)$ is feebly compact. To prove the converse, let $\{F_n : n \in \omega\}$ be a regular closed cover of (X, τ) . Then $\bigcup\{\text{int}F_n : n \in \omega\}$ is dense and, by assumption, feebly compact. This clearly implies that (X, τ) is covered by finitely many F_n , i.e. (X, τ) is countably S -closed. \square

Remark 3.8. Closing this section, we briefly discuss countably S -closed spaces in relationship to first countability and second countability. It is well-known that every first countable, e.d. Hausdorff space is discrete (see [11; p. 301]), and thus every first countable, e.d., countably S -closed Hausdorff space has to be finite. Moreover, since every second countable Hausdorff space is km -perfect, it follows by Theorem 3.2 and the preceding observation that every second countable, countably S -closed Hausdorff space is finite. This result is false, however, in the absence of Hausdorffness since the space obtained by taking the cofinite topology on a countably infinite set is obviously non-Hausdorff, second countable and countably S -closed.

4. Examples

Example 4.1. Let $X = \beta\omega - \{p\}$, where $p \in \beta\omega - \omega$. It is well known (see e.g. [11; p. 301]) that X is countably compact, and hence feebly compact, but not compact. Since X is e.d., X is countably S -closed. However, X fails to be S -closed since a regular S -closed space is compact.

Example 4.2. Let (Y, σ) be a space such that $Y - \{p\}$ is a countably S -closed subspace for some non-isolated point $p \in Y$. Let Y_1 and Y_2 denote two disjoint copies of $Y - \{p\}$. For any subset $A \subseteq Y$ we will denote the corresponding subsets of Y_1 and Y_2 by A_1 and A_2 , respectively. Now let $X = Y_1 \cup Y_2 \cup \{p\}$. We define a topology τ on X in the following way. For any $x \in X$, if $x \in Y_1$ ($x \in Y_2$, respectively), then the basic open neighbourhoods of x in (X, τ) are of the form V_1 (V_2 , respectively), where V is an open subset of $Y - \{p\}$. For every open neighbourhood W of p in (Y, σ) , a basic open neighbourhood of p in (X, τ) is $\{p\} \cup (W - \{p\})_1 \cup (W - \{p\})_2$. It is easy to see that both Y_1 and Y_2 are regular open subsets of (X, τ) and homeomorphic to $Y - \{p\}$. By Lemma 2.3, (X, τ) is countably S -closed but not e.d. since neither Y_1 nor Y_2 are closed in (X, τ) .

In particular, if (Y, σ) is $\beta\omega$, then the resulting space is a compact, countably S -closed Hausdorff space which is not S -closed since it fails to be e.d.

Example 4.3. Here we present some familiar spaces which are feebly compact but not countably S -closed.

i) Isbell's space Ψ ([2; p. 79]) is a locally compact, feebly compact, perfect Hausdorff space hence also completely regular. It is also first countable and thus cannot be countably S -closed by Corollary 2.5.

ii) ω_1 , the space of all countable ordinals with the order topology is regular, first countable and countably compact, thus feebly compact. By Corollary 2.5, ω_1 is not countably S -closed.

iii) Let D be an infinite set with the discrete topology. Let (X, τ) denote the one-point-compactification of D , where $X = D \cup \{a\}$ and $a \notin D$ is the only non-isolated point of (X, τ) . Then (X, τ) is a compact Hausdorff space, hence feebly compact. Let $\{D_n : n \in \omega\}$ be a partition of D , where each D_n is infinite. For each $n \in \omega$, if $F_n = D_n \cup \{a\}$, then $F_n \in RC(X, \tau)$. Clearly, $\{F_n : n \in \omega\}$ is a regular closed cover of (X, τ) without a finite subcover. Thus (X, τ) is not countably S -closed.

Example 4.4. $\beta\omega \times \beta\omega$ is not countably S -closed. Consider $W = \{(n, n) : n \in \omega\} \subseteq \beta\omega \times \beta\omega$. It is known that W is a regular open subset of $\beta\omega \times \beta\omega$. By Proposition 2.9. i), $\beta\omega \times \beta\omega$ cannot be countably S -closed since W is also an infinite discrete subspace of $\beta\omega \times \beta\omega$.

Example 4.5. $\omega^* = \beta\omega - \omega$ is not countably S -closed.

Let $f : \omega \rightarrow [0, 1]$ be a function which maps ω onto the rationals of the unit interval $[0, 1]$. If $\beta f : \beta\omega \rightarrow [0, 1]$ denotes the Stone-extension of f , then βf is continuous and onto. Let $g : \omega^* \rightarrow [0, 1]$ denote the restriction of βf to ω^* , i.e. $g = \beta f|_{\omega^*}$. Note that for each irrational number $t \in [0, 1]$ we have $g^{-1}(\{t\}) \neq \emptyset$. By Corollary 2.5, $[0, 1]$ is not countably S -closed so there exists a regular closed cover $\{A_n : n \in \omega\}$ of $[0, 1]$ without a finite subcover. Clearly, each A_n is a zero-set in $[0, 1]$, and so each $g^{-1}(A_n)$ is a nonempty zero-set in ω^* . By [10; p. 78], $\{g^{-1}(A_n) : n \in \omega\}$ is a countable regular closed cover of ω^* . Let $m \in \omega$. Then there is an irrational number $t \in [0, 1] - (A_1 \cup \dots \cup A_m)$. Since $g^{-1}(\{t\}) \neq \emptyset$, we have $\omega^* \neq g^{-1}(A_1) \cup \dots \cup g^{-1}(A_m)$. This proves that ω^* is not countably S -closed.

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