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LINEAR FINITELY SEPARATED OBJECTS OF SUBCATEGORIES OF DOMAINS

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ABSTRACT. We enrich the category $\mathcal{F}rm$ of frames and frame homomorphisms with the important concept of approximation by specifying a full subcategory of *linear FS-frames*. We show that this subcategory is equivalent with the subcategory of linear FS-domains via the standard Stone duality. The Stone duality for sober spaces tells us that a distributive continuous lattice, i.e., a continuous frame can be viewed as the lattice of open sets of a locally compact space. The Stone duals of linear FS-domains considered as topological spaces, LFS-frames, may be replaced by their suitably taken distributive sublattices with an additional relation of approximation, thus discarding with infinitary operations. This is intended as a step towards the development of a *domain theory in logical form beyond the standard algebraic world.* Moreover, since any linear FS-frame is stably continuous and supercontinuous, we can characterize the full category of stably continuous supercontinuous frames to be equivalent to the subcategory of stable prelocales (distributive lattices with an approximation relation and stable approximable relations between them). Similarly, we may introduce the notion of a linear FS-preframe in the setting of the category of preframes and preframe homomorphisms. We can show that the category of linear FS-preframes is self-dual, and that any linear FS-preframe is a stably continuous complete lattice. A characterization of algebraic LFS-frames is given. The Stone duality in terms of abstract $\wedge$-semilattice bases is established.

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1. Introduction

In recent years, a new class of objects of Domain Theory – now commonly referred to as "FS-domains" – was extensively studied ([13], [1] and [2]) together with its counterparts in the categories of $\bigvee$-semilattices ([8], [9]) and Scott domains ([10]).

The present paper is an investigation on this subject in the category of frames and in the category of preframes. A good source of information about frames and domains are the classic text by Johnstone [12] and the book by Abramsky and Jung [2], where the interested reader can find unexplained terms and notation concerning the subject. Our terminology and notation agree with the book of Johnstone [12].

In Sections 2 and 5, we summarize some well-known results for supercontinuous frames and preframes. Section 3 is devoted to the study of the Stone duality for LFS-domains. We enrich the category $\mathcal{F}rm$ of frames and frame homomorphisms with the important concept of approximation by specifying a full subcategory of *linear FS-frames*. We show that this subcategory is equivalent with the subcategory of linear FS-domains via the standard Stone duality.

The Stone duals of linear FS-domains considered as topological spaces, LFS-frames, may be replaced by their suitably taken distributive sublattices with an additional relation of approximation, thus discarding with infinitary operations. This approach is studied in Section 4. It is intended as a step towards the development of a *domain theory in logical form beyond the standard algebraic world*. Moreover, since any linear FS-frame is stably continuous and supercontinuous, we can characterize the full subcategory of stably continuous supercontinuous frames to be equivalent to the subcategory of stable prelocales (distributive lattices with an approximation relation and stable approximable relations between them).

In Section 6, we investigate the category of LFS-preframes. We show that it is selfdual and give some sufficient characteristics of the LFS-property for preframes. The Stone duality in terms of abstract $\wedge$-semilattice bases is established.

Now, we begin by stating the definitions and basic properties of them, most of them well-known, which will be needed in the remainder of the article.

A poset D in which every directed subset has a supremum we call a *directed-complete partial order*, or *dcpo* for short. We write directed suprema as $\bigsqcup^{\uparrow} x_i$. Let D and E be dcpo's. A function $f\colon D \to E$ is *(Scott-) continuous* if it is monotone, and if for each directed subset A of D we have $f(\bigsqcup^{\uparrow} A) = \bigsqcup^{\uparrow} f(A)$.

Let x and y be elements of a dcpo D. We say that x *approximates* y if for all directed subsets A of D, $y \leq \bigsqcup^{\uparrow} A$ implies $x \leq a$ for some $a \in A$. This concept arose in the theory of continuous lattices ([6]), but it is also present in many arguments from topology and analysis, though not always fully explicit. We say that x is *compact* if it approximates itself.

We introduce the following notation for $x, y \in D$ and $A \subseteq D$:

$$
\begin{aligned}
x \ll y &\iff x \text{ approximates } y\,, \\
\mathop{\downarrow\!\!\!\downarrow} x &= \{y \in D \mid\ y \ll x\}\,, \\
\mathop{\uparrow\!\!\!\uparrow} x &= \{y \in D \mid\ x \ll y\}\,, \\
\mathop{\uparrow\!\!\!\uparrow} A &= \bigcup_{a \in A} \mathop{\uparrow\!\!\!\uparrow} a\,, \\
\mathsf{K}(D) &= \{x \in D \mid\ x \text{ compact}\}\,.
\end{aligned}
$$

The relation $\ll$ is traditionally called "way-below relation".

Now, we observe the following basic properties of approximation. Let D be a dcpo. Then the following is true for all $x, x', y, y' \in D$:

1. $x \ll y \implies x \leq y$;
2. $x' \leq x \ll y \leq y' \implies x' \ll y'$.

We say that a subset B of a dcpo D is a *basis* for D if for every element x of D the set $B_x = \mathop{\downarrow\!\!\!\downarrow} x \cap B$ contains a directed subset with supremum x. We call elements of B_x *approximants to x relative to B*.

A dcpo is called *continuous* or a *continuous domain* if it has a basis. It is called *algebraic* or an *algebraic domain* if it has a basis of compact elements.

Let D be a continuous domain, and let $M \subseteq D$ be a finite set each of whose elements approximates y. Then there exists $y' \in D$ such that $M \ll y' \ll y$ holds. If B is a basis for D, then y' may be chosen from B. We say, y' *interpolates* between M and y.

Let D be a dcpo. A subset A is called (*Scott-*) *closed* if it is a lower set and is closed under suprema of directed subsets. Complements of closed sets are called (*Scott-*) *open*; they are the elements of σ_D, the *Scott-topology* on D.

A Scott-open set O is necessarily an upper set such that every directed set whose supremum lies in O has a non-empty intersection with O.

We have for a dcpo D:

1. For elements $x, y \in D$ the following are equivalent:
   (a) $x \leq y$.
   (b) Every Scott-open set which contains x also contains y.
   (c) $x \in \mathrm{Cl}(\{y\})$.
2. The Scott-topology satisfies the $\mathbf{T}_0$ separation axiom.
3. $\langle D, \sigma_D \rangle$ is a Hausdorff ($= \mathbf{T}_2$) topological space if and only if the order on D is trivial.

So we can reconstruct the order between elements of a dcpo from the Scott-topology. The same is true for limits of directed sets.

For dcpo's D and E, a function f from D to E is Scott-continuous if and only if it is topologically continuous with respect to the Scott-topologies on D and E.

Let D be a dcpo and $f\colon D \to D$ be a Scott-continuous function. We say that f is *finitely separated* from the identity on D if there exists a finite set M such that for any $x \in D$ there is $m \in M$ with $f(x) \leq m \leq x$. We speak of *strong separation* if for each x there are elements $m, m' \in M$ with $f(x) \leq m \ll m' \leq x$.

A pointed dcpo D (a dcpo with a bottom element) is called an *FS-domain* if there is a directed collection $(f_i)_{i\in I}$ of continuous functions on D, each finitely separated from id_D, with the identity map as their supremum.

It is relatively easy to see that FS-domains are indeed continuous. Thus it makes sense to speak of **FS** as the full subcategory of **CONT**, the category of continuous domains, where the objects are the FS-domains. The category **FS** is closed under the formation of products, retracts and function spaces. It is cartesian closed.

Let us turn to the category of frames. A *frame* (or locale) is a complete lattice L in which $x \wedge \bigvee x_i = \bigvee x \wedge x_i$ for binary meet $\wedge$, arbitrary join $\bigvee$, and any $x, x_i \in L$, and a *frame homomorphism* $h\colon L \to M$ is a map between frames preserving finite meets, including the top element 1 and arbitrary joins, including the bottom element 0. The resulting category is denoted $\mathcal{F}rm$. A frame L is called *spatial* if for every $a, b \in L$, $a \not\leq b$ implies there is a frame morphism (point) $p\colon L \to \mathbf{2}$ such that $p(a) = 1$ and $p(b) = 0$; here $\mathbf{2}$ is the two-element Boolean algebra. Recall that the set $\mathrm{pt}\, L$ of all points can be equipped with the topology such that open sets are of the form $\{p \in \mathrm{pt}\, L : p(a) = 1$ for some $a \in L\}$. Spaces isomorphic to a space of this form are called *sober*, i.e., a sober space is defined as one which can be recovered from its lattice of opens in Stone duality. Moreover, continuous domains equipped with the Scott-topology are sober spaces. Now, let p be a point of L, $g\colon K \to L$ a frame homomorphism. Then $\mathrm{pt}\, g(p) = p \circ g$ is a point of K and $\mathrm{pt}\, g$ is a continuous map from $\mathrm{pt}\, L$ to $\mathrm{pt}\, K$. Note that, for each topological space X, we have a frame $\Omega(X)$ of all open subsets of X and, for each continuous map $f\colon X \to Y$, we have a frame homomorphism $\Omega f = f^{-1}\colon \Omega Y \to \Omega X$. Then the functors Ω and pt restrict to a dual equivalence between **CONT** and the category of completely distributive lattices.

Let L be a frame, $K \subseteq L$. We shall say that K is a *subframe* of L if K is closed wrt. arbitrary suprema and finite infima. Especially, $0, 1 \in K$. A frame L is said to be *compact* if 1 is a compact element of L.

2. Supercontinuous frames

DEFINITION 2.1. For a complete lattice L define a relation $\lll$ *totally bellow* on L by

$$x \lll y \quad \text{if} \quad \forall A{\subseteq}L \ \left(y \leq \bigvee A \implies \exists a{\in}A \ x \leq a\right).$$

Call L *supercontinuous* if for every $x \in L$, $x = \bigvee\{y \mid y \lll x\}$ holds.

We shall call an element $a \in L$ *supercompact* if $a \lll a$. The set of all supercompact elements of L will be denoted by $\mathrm{SK}(L)$. Call L *superalgebraic* if for every $x \in L$, $x = \bigvee\{y \leq x \mid y \lll y\}$ holds.

We say that a subset B of a complete lattice D is a *basis* for D if for every element x of D the set $B_x = \{y \in D : y \lll x\} \cap B$ contains a subset with supremum x. We call elements of B_x *total approximants to* x *relative to* B.

Recall that any superalgebraic lattice is supercontinuous.

THEOREM 2.2. *Let L be a complete lattice. Then the following conditions are equivalent:*

1. *L is $\bigvee\bigwedge$-embeddable into a powerset Boolean algebra B.*
2. *L is a superalgebraic lattice.*
3. *L^{op} is a superalgebraic lattice.*

Proof.

(1) $\implies$ (2): We assume that there is an embedding $e : L \hookrightarrow 2^M = B$ that preserves arbitrary joins and meets. So we have a left adjoint $c \colon B \to L$ which preserves arbitrary joins. For each atom $\{m\}$ of B, $m \in M$, we show that the element $c_m = c(\{m\})$ is supercompact in L. Notice that $c_m = \bigwedge\{x \in L : \{m\} \subseteq e(x)\}$. Assume $\emptyset \neq S \subseteq L$, $c_m \leq \bigvee S$. Then $\{m\} \leq e(c_m) \leq \bigvee e(S)$, i.e., there is $s \in S$ such that $\{m\} \leq e(c_m) \leq e(s)$. By an elementary computation, we can verify that the set $\{c_m\}_{m\in M}$ is $\bigvee$-dense in L.

(2) $\implies$ (1): Put $B = 2^S$, where S is the set of all nonzero supercompact elements of L, and define a map $e \colon L \to B$ by the prescription $e(a) = \{c \in S : c \leq a\}$. Obviously, e is injective, preserves arbitrary meets and the bottom element. We show that it preserves arbitrary nonempty joins. Namely, for $c \in S$ we have $c \in e\left(\bigvee_{i\in I} a_i\right) \iff c \leq a_{i_0}$ for some $i_0 \in I \iff c \in \bigcup_{i\in I} e(a_i)$.

(1) $\iff$ (3): By the duality argument. □

COROLLARY 2.3. *Let L be a complete lattice. L is superalgebraic if and only if it is isomorphic to the lattice $\mathcal{D}(P)$ of all down-sets of some poset P.*

Proof.

$\implies$: We put P to be the poset of all nonzero supercompact elements.

$\impliedby$: Clearly, the supercompact elements are exactly the principal ideals. The rest is evident. □

COROLLARY 2.4. *Every superalgebraic lattice is a spatial frame.*

THEOREM 2.5. *Let L be a complete lattice. The following conditions are equivalent:*

1. *L is a $\bigvee\bigwedge$-image of a superalgebraic lattice.*
2. *L is supercontinuous.*
3. *L^{op} is supercontinuous.*

Proof.

(1) $\Longrightarrow$ (2): Let $f\colon S \to L$ be a $\bigvee\bigwedge$-homomorphism of a superalgebraic lattice S onto L. Then f has a right adjoint u and a left adjoint l, i.e., for each $x \in L$ we have $u(x) = \bigvee\{s \in S : f(s) = x\}$, $l(x) = \bigwedge\{s \in S : f(s) = x\}$, $f \circ u(x) = x$ and $f \circ l(x) = x$. Denote $C_x := \{c \in S : c \leq l(x) \text{ and } c \lll c\}$. We show that $f(c) \lll x$ holds for each $c \in C_x$. Suppose $x \leq \bigvee\limits_{i \in I} x_i$. Since $f\bigl(l(x) \wedge \bigvee\limits_{i\in I} u(x_i)\bigr) = f\bigl(l(x)\bigr) \wedge \bigvee\limits_{i\in I} f\bigl(u(x_i)\bigr) = x \wedge \bigvee\limits_{i\in I} x_i = x$, we have $c \leq l(x) \leq \bigvee\limits_{i\in I} u(x_i)$, and, by the supercompactness of c, we obtain $c \leq u(x_j)$ for an element $j \in I$. But then $f(c) \leq f\bigl(u(x_j)\bigr) = x_j$. Consequently, $f(c) \lll x$. Now $x = f\bigl(l(x)\bigr) = f\bigl(\bigvee C_x\bigr) = \bigvee\{f(c) : c \in C_x\}$.

(2) $\Longrightarrow$ (1): Take $\tilde{\bigvee}\colon \mathcal{D}(L) \to L$ assigning to each down-set its join. $\mathcal{D}(L)$ is superalgebraic, and $\tilde{\bigvee}$ is a $\bigvee\bigwedge$-homomorphism. Indeed, $\tilde{\bigvee}\Bigl(\bigcup\limits_{i\in I} X_i\Bigr) = \bigvee\limits_{i\in I} \tilde{\bigvee}(X_i)$, and, for all $j \in I$, $\Bigl\{c : c \lll \bigwedge\limits_{i\in I} \tilde{\bigvee}(X_i)\Bigr\} \subseteq X_j$, which immediately yields $\tilde{\bigvee}\Bigl(\bigcap\limits_{i\in I} X_i\Bigr) = \bigwedge\limits_{i\in I} \tilde{\bigvee}(X_i)$.

(1) $\Longleftrightarrow$ (3): By the duality argument. □

We recall the following lemma.

LEMMA 2.6. *Let L be a supercontinuous lattice, $e\colon L \to L$ an idempotent $\bigvee$-preserving mapping. Then $e(L)$ is a supercontinuous lattice.*

Proof. It follows from [12; Chapter VII, Lemma 2.3]. □

THEOREM 2.7. *A complete lattice L is supercontinuous if and only if it is a retract of a superalgebraic lattice by $\bigvee$-preserving maps.*

Proof.

$\Longrightarrow$: Let L be a supercontinuous lattice, $\mathcal{D}(L)$ the lattice of its down-sets. Then the mappings $\tilde{\bigvee}\colon \mathcal{D}(L) \to L$ and $t\colon L \to \mathcal{D}(L)$ defined by the prescription $t(a) = \{x : x \lll a\}$ for all $a \in L$ are $\bigvee$-preserving, $\mathrm{id}_L = \tilde{\bigvee} \circ t$.

$\Longleftarrow$: The converse direction follows from the lemma and the fact that any superalgebraic lattice is supercontinuous. □

COROLLARY 2.8. *A complete lattice L is supercontinuous if and only if it is a retract of a powerset Boolean algebra by $\bigvee$-preserving maps.*

Recall that, in the presence of the Axiom of Choice, supercontinuous lattices are exactly completely distributive lattices.

THEOREM 2.9. *A complete lattice L is supercontinuous if and only if for any diagram*

$$\begin{tikzcd} & L \arrow[d, "f"] \\ A \arrow[r, "q"] & C \end{tikzcd}$$

of $\bigvee$-preserving mappings between complete lattices such that there is an order-preserving map $s\colon C \to A$, $\mathrm{id}_C = q \circ s$, we have a $\bigvee$-preserving morphisms $g\colon L \to A$ such that $q \circ g = f$, i.e., the following diagram commutes.

$$\begin{tikzcd} & L \arrow[dl, "g"'] \arrow[d, "f"] \\ A \arrow[r, "q"] & C \\ & C \arrow[ul, "s"] \arrow[u, "\mathrm{id}_C"'] \end{tikzcd}$$

Proof.

$\Longrightarrow$: Let L be a supercontinuous lattice. We put $g(b) = \bigvee\{s \circ f(x) : x \in t(b)\}$. We show that g is $\bigvee$-preserving. Let $X \subseteq L$. We put $Y = \bigcup\{t(x) : x \in X\}$. Then $g(\bigvee X) = g(\bigvee Y) = \bigvee\{s \circ f(y) :\ y \in Y\} = \bigvee_{x \in X} \bigvee\{s \circ f(y) : y \in t(x)\} = \bigvee g(X)$. The rest is evident.

$\Longleftarrow$: We have a $\bigvee$-preserving map $\tilde{\bigvee}\colon \mathcal{D}(L) \to L$, and since the map $d\colon L \to \mathcal{D}(L)$, $d(a) = \{x \in L :\ x \le a\}$ for all $a \in L$, is an order preserving map with $\mathrm{id}_L = \tilde{\bigvee} \circ d$, there is a $\bigvee$-preserving map $g\colon L \to \mathcal{D}(L)$ such that $\mathrm{id}_L = \tilde{\bigvee} \circ g$. The rest follows from Theorem 2.7. □

3. Linear FS-frames

DEFINITION 3.1. Let $\mathcal{C}$ be a subcategory of the category of dcpos and Scott-continuous mappings between them. An object A of $\mathcal{C}$ is said to be an *LFS-object* if $\mathrm{id}_A = \bigsqcup^{\uparrow} \mathcal{D}$ for some directed set $\mathcal{D}$ of $\mathcal{C}$-morphisms from A to A such that every $d \in \mathcal{D}$ is finitely separated from id_A, i.e., there is a finite subset $M_d \subseteq\subseteq A$ such that for all $a \in A$ there is $m_a \in M_d$ with $d(a) \le m_a \le a$. We shall denote, for all objects A, B from $\mathcal{C}$, $\mathcal{C}(A, B)$ the poset of all $\mathcal{C}$-morphisms from A to B.

Recall that FS-domains are LFS-objects in the category of dcpos and Scott-continuous mappings, FS-lattices are LFS-objects in the category of complete lattices and suprema-preserving mappings, and FS-frames are LFS-objects in the category $\mathcal{F}rm$ of frames. Evidently, any LFS-object in $\mathcal{C}$ is an FS-domain. We can then use all known corresponding results on FS-domains. Notice that, for all dcpos A, B, $[A \to B]$ is the dcpo of all Scott-continuous mappings from A to B, for all complete lattices A, B, $A \multimap B$ is the complete lattice of all suprema-preserving mappings from A to B, and, for all frames A, B, $\mathcal{F}rm(A, B)$ is the dcpo of all frame morphisms from A to B.

If moreover any object A of $\mathcal{C}$ is a distributive lattice, we may assume that the corresponding finite subset $M_d \subseteq\subseteq A$ is a finite $0-1$ sublattice of A, and the element m_a is the least one such that $d(a) \le m_a \le a$.

Let us recall the following proposition from [8]:

LEMMA 3.2. *Let A be a dcpo. Then:*

1. *If $f \in [A \to A]$ is finitely separated from $\mathrm{id}\, A$, then $x \le f(y)$ implies $x \ll y$.*
2. *If A is an FS-domain, then $x \ll y$ implies $x \le f(y)$ for some f in $[A \to A]$ finitely separated from $\mathrm{id}\, A$;*
3. *in particular, every FS-domain is continuous.*

We shall prove the following easy observation:

LEMMA 3.3. *Let A be an LFS-object. Then $\mathrm{id}_A = \bigsqcup^{\uparrow} \mathcal{D}$ for some directed set $\mathcal{D}$ of $\mathcal{C}$-morphisms from A to A such that for all $d \in \mathcal{D}$ there is a finite subset $M_d \subseteq\subseteq A$ such that for all $a \in A$ there is $m_a \in M_d$ with $d(a) \ll m_a \ll a$.*

Proof. Recall that $\mathrm{id}_A = \bigsqcup^{\uparrow} \mathcal{E}$ for some directed set $\mathcal{E}$ of $\mathcal{C}$-morphisms from A to A such that all $d \in \mathcal{E}$ are finitely separated from the identity. Then we have that for all $d \in \mathcal{E}$ there is a finite subset $N_d \subseteq\subseteq A$ such that for all $a \in A$ there is $n_a \in N_d$ with $d(a) \le n_a \le a$ and

$$\begin{aligned}\mathrm{id}_A &= \mathrm{id}_A \cdot \mathrm{id}_A = \bigsqcup\nolimits^{\uparrow} \mathcal{E} \cdot \bigsqcup\nolimits^{\uparrow} \mathcal{E} \\ &= \bigsqcup_{d,e \in \mathcal{E}}^{\uparrow} d \cdot e = \bigsqcup_{d \in \mathcal{E}}^{\uparrow} d \cdot d = \bigsqcup_{d \in \mathcal{E}}^{\uparrow} d^2 = \bigsqcup_{d \in \mathcal{E}}^{\uparrow} d^3 .\end{aligned}$$

So we have that $d^2(a) \le d(n_a) \ll n_a \le a$, i.e., $d^3(a) \le d^2(n_a) \ll d(n_a) \ll n_a \le a$. We put then $\mathcal{D} = \{d^3 :\ d \in \mathcal{E}\}$, $M_{d^3} = \{d(n) :\ n \in N_d\}$, $d \in \mathcal{E}$. $\square$

DEFINITION 3.4. An element p in a lattice A is called $\vee$-*prime* if for all $x, y \in A$ with $p \leq x \vee y$ we must have $p \leq x$ or $p \leq y$. We shall denote by $\mathrm{Sp}(A)$ the poset of $\vee$-primes of A. We shall say that a complete lattice has *enough* $\vee$-*primes* if any its element is a supremum of $\vee$-primes below it.

PROPOSITION 3.5. *Let A be a complete lattice. Then the following conditions are equivalent:*

1. *A is a distributive LFS-lattice.*
2. *A is supercontinuous.*

Proof.

(1) $\implies$ (2): Let $0 \neq a \in A$. We know that $d(a) \ll a$ for each $d \in \mathcal{D}$. Moreover, there is a finite sublattice $M_d \subseteq\subseteq A$ and an element m_a such that $d(a) \leq m_a \leq a$, a_m is the infimum of all such elements in M_d. Let $x \in \mathrm{Sp}(M_d)$, $0 \neq x \leq a_m$. Then $x \lll a$. Namely, let $X \subseteq A$, $a \leq \bigvee X$. We may assume that $X \leq \downarrow(a)$. Then $d(a) \leq d\left(\bigvee X\right) = \bigvee_{y \in X} d(y) \leq \bigvee_{y \in X_0} m_y \leq a$ for some finite subset $X_0 \subseteq\subseteq X$. Then $x \leq a_m \leq \bigvee_{y \in X_0} m_y$, i.e., there is $y \in X$ such that $x \leq m_y \leq y$.

We have that $a = \bigvee_{d \in \mathcal{D}} d(a) \leq \bigvee\left\{x \in \mathrm{Sp}(M_d) :\ d \in \mathcal{D},\ x \leq m_a\right\} \leq a$.

(2) $\implies$ (1): We put $\mathcal{E} = \{M \subseteq\subseteq A :\ M$ is a finite distributive sublattice of $A\}$ and for all $M \in \mathcal{E}$ we shall define a map $d_M : A \to A$ as follows

$$d_M(a) = \bigvee\{m \in M :\ m \lll a\}$$

for all $a \in A$. We put $\mathcal{D} = \{d_M :\ M \in \mathcal{E}\}$. Then evidently each d_M is finitely separated from the identity, it preserves arbitrary suprema, $\mathcal{D}$ is directed and $\bigsqcup^{\uparrow}_{d \in \mathcal{D}} d = \mathrm{id}_A$. □

PROPOSITION 3.6. *Let A be a distributive LFS-lattice, $a \in A$ compact, i.e., there is $d \in \mathcal{D}$ such that $d(a) = a$. Then there is a finite subset F_0 of supercompact elements of A such that $a = \bigvee F_0$.*

Proof. If $a = 0$, we put $F_0 = \emptyset$. Now, let $0 \neq a \in A$, $a = \bigvee\{x \in A :\ x \lll a\} = \bigvee t(a)$. By compactness, there is an element $d \in \mathcal{D}$ such that $d(a) = a$ and $a = \bigvee\left\{d(x) \in A :\ x \in t(a)\right\}$, i.e., $a = \bigvee\left\{m \in \mathrm{Sp}(M_d) :\ m \in t(a)\right\}$, and there is a minimal finite subset $F_0 \subseteq \left\{m \in \mathrm{Sp}(M_d) :\ m \in t(a)\right\}$ such that $a = \bigvee F_0 = \bigvee d(F_0)$. We have that, for each $x \in F_0$, there is $y(x) \in F_0$ such that $x \leq d\left(y(x)\right) \leq y(x)$, i.e., $x = d\left(y(x)\right) = y(x)$, i.e., x is compact and $x \in M_d$, $x = m_x$. Let $x \leq u \vee v$. Then $x \leq d(u) \vee d(v) \leq m_u \vee m_v$, i.e., $x \leq m_u \leq u$ or $x \leq m_v \leq v$, i.e., $x \in \mathrm{Sp}(A)$, i.e., x is supercompact. □

COROLLARY 3.7. *Let A be an algebraic supercontinuous lattice. Then A is superalgebraic.*

LEMMA 3.8. *Let A and B be LFS-frames. Then:*

(i) *the poset $\mathcal{F}rm(A,B)$ is an FS-domain,*
(ii) *$A \oplus B$ is a linear FS-frame,*
(iii) *A is bicontinuous,*
(iv) *assuming the Axiom of Choice, A has enough coprimes,*
(v) *A is stably continuous, i.e., A is compact, and $a \ll b$, $a \ll c$ implies $a \ll b \wedge c$,*
(vi) *assuming the Axiom of Choice, A is linked, i.e., the Lawson topologies on A and A^{op} coincide.*

Proof.

(i): The proof follows the proof of Lemma 5 in [8]. We have: let $\mathcal{D} \subseteq \mathcal{F}rm(A,A)$ and $\mathcal{E} \subseteq \mathcal{F}rm(B,B)$ be directed sets with $\sqcup\mathcal{D} = \mathrm{id}\,A$ and $\sqcup\mathcal{E} = \mathrm{id}\,B$ such that all $f \in \mathcal{D}$ and $g \in \mathcal{E}$ are finitely separated from the respective identities. If M_f, respectively M_g, is a finite set separating $f \in \mathcal{D}$ from $\mathrm{id}\,A$, respectively $g \in \mathcal{E}$ from $\mathrm{id}\,B$, we are done if $(f \multimap g)^2$, where $f \multimap g = \lambda h\,.\,g \circ h \circ f \colon \mathcal{F}rm(A,B) \to \mathcal{F}rm(A,B)$, is separated from $\mathrm{id}_{\mathcal{F}rm(A,B)}$ by some finite set: then $\mathrm{id}_{\mathcal{F}rm(A,B)} = \sqcup\{(f \multimap g)^2 \mid f \in \mathcal{D},\ g \in \mathcal{E}\}$ as composition of frame morphisms is again a morphism of frames. We define an equivalence relation $\sim$ on $\mathcal{F}rm(A,B)$ by

$$h_1 \sim h_2 \quad :\Longleftrightarrow \quad \forall m{\in}M_f \quad \uparrow g\big(h_1(m)\big) \cap M_g = \uparrow g\big(h_2(m)\big) \cap M_g\,. \tag{1}$$

As M_f and M_g are finite, there are only finitely many equivalence classes on $\mathcal{F}rm(A,B)$. Let M be a non-redundant and complete set of representatives of these classes. We claim that the finite set $f \multimap g(M)$ separates $(f \multimap g)^2$ from $\mathrm{id}_{A \multimap B}$. Given $h \in \mathcal{F}rm(A,B)$, let $\bar{h}$ be the corresponding representative in M. For $a \in A$, we compute

$$\begin{aligned}
h(a) &\geq h(m_f) && \text{for some } m_f \in M_f \text{ with } f(a) \leq m_f \leq a\,,\\
&\geq m_g && \text{for some } m_g \in M_g \text{ with } g\big(h(m_f)\big) \leq m_g \leq h(m_f)\,,\\
&\geq g\big(\bar{h}(m_f)\big) && \text{as } g\big(h(m_f)\big) \leq m_g \text{ and } h \sim \bar{h}\,,\\
&\geq g\big(\bar{h}(f(a))\big) && \text{as } f(a) \leq m_f\,.
\end{aligned}$$

By symmetry, we obtain $\bar{h} \geq (f \multimap g)(h)$, so $h \geq f \multimap g(\bar{h}) \geq (f \multimap g)^2(h)$.

(ii): Let $\mathcal{D} \subseteq \mathcal{F}rm(A,A)$ and $\mathcal{E} \subseteq \mathcal{F}rm(B,B)$ be directed sets with $\bigsqcup^{\uparrow} \mathcal{D} = \mathrm{id}_A$ and $\bigsqcup^{\uparrow} \mathcal{E} = \mathrm{id}_B$ such that all $f \in \mathcal{D}$ and $g \in \mathcal{E}$ are finitely separated from the respective identities. We put $\mathcal{G} = \{f \oplus g : f \in \mathcal{D},$

$g \in \mathcal{E}\} \subseteq \mathcal{F}rm(A \oplus B, A \oplus B)$. Then $\bigsqcup^{\uparrow} \mathcal{G} = \mathrm{id}_{A\oplus B}$. For all $f \in \mathcal{D}$ and $g \in \mathcal{E}$ we put $N_{f\oplus g} = \{m \oplus n : \ m \in M_f\,,\ n \in M_g\}$, and let $M_{f\oplus g}$ be the finite subframe generated by $N_{f\oplus g}$. It is easy to see that $M_{f\oplus g}$ separates $f \oplus g$ from $\mathrm{id}_{A\oplus B}$.

(iii): It follows from 3.5 and 2.5.

(iv): Assuming the Axiom of Choice the proposition follows from (iii).

(v): We have $1 = f(1) \ll 1$ for all $f \in \mathcal{D}$, i.e., A is compact. Let $a \ll b$ and $a \ll c$. Then there are $f, g \in \mathcal{D}$ such that $a \leq f(b)$, $a \leq g(c)$. Then there is $h \in \mathcal{D}$ such that $a \leq h(b)$, $a \leq h(c)$, i.e., $a \leq h(b) \wedge h(c) = h(b \wedge c) \ll b \wedge c$.

(vi): It follows from Lemma 5 in [8]. □

COROLLARY 3.9. *Let A be an algebraic LFS-frame. Then A is superalgebraic and coherent.*

So we may for any compact supercontinuous frame A define a set A_{scm} of maximal supercompact elements. Recall that A_{scm} is finite and $\mathrm{Sp}(A) = \mathrm{Sp}(A) \cap \downarrow A_{scm}$. Moreover, in any LFS-frame A, there is a finite sublattice of compact elements of A containing A_{scm} such that it is generated as a join semilattice by a finite subset of supercompact elements. Let us denote this finite subframe of A as $T(A)$.

PROPOSITION 3.10. *Let A be a stably continuous frame, B be a continuous frame, and the dual $B^{\perp}$ be a frame. Let $F \subseteq \mathcal{F}rm(A, B)$ be a filtered family of frame morphisms. Then the infimum of F in $\mathcal{F}rm(A, B)$ equals the infimum of F in $A \multimap B$.*

Proof. The description of infima in $[A \to B]$ is well known (see [13]). If $F \subseteq \mathcal{F}rm(A, B)$ is given, then $g := \lambda a\,.\ \bigwedge^{\downarrow}_{f\in F} f(a) \colon A \to B$ is monotone, so $g^{\mathrm{c}} := \lambda a\,.\ \bigsqcup^{\uparrow}_{b \ll a} g(b) \colon A \to B$ is the greatest Scott-continuous function on A below g; therefore, g^{c} is the infimum of F in $[A \to B]$. Moreover, we have that $g(0) = 0$, $g(1) = 1$, g preserves finite infima and finite suprema. Namely, let $a, b \in A$. Then

$$g(a) \wedge g(b) = \bigwedge^{\downarrow}_{f\in F} f(a) \wedge \bigwedge^{\downarrow}_{f\in F} f(b) = \bigwedge^{\downarrow}_{f\in F} f(a \wedge b) = g(a \wedge b)\,.$$

Similarly, using the fact that F is filtered and $B^{\perp}$ is a frame, we have

$$g(a) \vee g(b) = \bigwedge^{\downarrow}_{f\in F} f(a) \vee \bigwedge^{\downarrow}_{h\in F} h(b) = \bigwedge^{\downarrow}_{d\in F} d(a \vee b) = g(a \vee b)\,.$$

We obtain that $g^{\mathrm{c}}(0) = 0$, $g^{\mathrm{c}}(1) = 1$ $(1 \ll 1)$ and g^{c} is monotone.

Now, we show that $g^{\mathrm{c}} \colon A \to B$ is in $A \multimap B$ if the function $g \colon A \to B$ preserves finite suprema: Since g^{c} is the infimum of F in $[A \to B]$, and since $A \multimap B$ is a subset of $[A \to B]$, we are done if g^{c} preserves finite suprema.

Given $x \ll g^c(a \vee b) = \bigsqcup_{y \ll a \vee b} g(y)$, we have $x \leq g(y)$ for some $y \ll a \vee b$. The set $\{a' \vee b' \mid a' \ll a \text{ and } b' \ll b\}$ is directed in A with supremum $a \vee b$, so $y \leq a' \vee b'$ for some $a' \ll a$ and $b' \ll b$. Then, $x \leq g(y) \leq g(a' \vee b')$; the latter equals $g(a') \vee g(b')$ by assumption, and it is a lower bound of $g^c(a) \vee g^c(b)$. Hence, $g^c(a \vee b) = g^c(a) \vee g^c(b)$ as B is continuous.

Let us show that g^c preserves finite infima. We easily have $g^c(a \wedge b) \leq g^c(a) \wedge g^c(b)$. Let $u \ll g^c(a) \wedge g^c(b)$. Then there are $a' \ll a$ and $b' \ll b$ such that $u \leq g(a') \wedge g(b') = g(a' \wedge b')$ and $a' \wedge b' \ll a \wedge b$. Then $u \leq g(a' \wedge b') \leq g^c(a \wedge b)$, i.e., $g^c(a \wedge b) = g^c(a) \wedge g^c(b)$. □

COROLLARY 3.11. *Let A be a stably continuous frame, $P = \mathrm{pt}\, A$ be a dcpo of points of A. Then P has all filtered infima.*

Proof. Let $F \subseteq P$ be a filtered subset. We may assume that $F \subseteq \mathcal{F}rm(A, \mathbf{2})$, and $\mathbf{2}$ evidently satisfies assumptions from 3.10. Then there is an element $p \in \mathcal{F}rm(A, \mathbf{2})$ such that $p = \bigwedge^{\downarrow}_{g \in F} g$. □

PROPOSITION 3.12. *Let A be a linear FS-frame. Then the poset $\mathbf{Pt}(A)$ of all points of A (prime elements) is an FS-domain.*

Proof. It follows immediately from 3.8 (i) applied on $B = \mathbf{2}$. We shall show how the way from LFS-frames to LFS-domains may be realized directly. Let $\mathbf{Pt}(A)$ be the set of all prime elements of A equipped with the reverse ordering. Recall that, for any frame map $f \colon A \to A$, we shall define a Scott-continuous map $\mathbf{Pt}(f) \colon \mathbf{Pt}(A) \to \mathbf{Pt}(A)$ as follows:

$$\mathbf{Pt}(f)(p) = \bigvee \{y \in A : \ f(y) \leq p\} = f^*(p).$$

Evidently, $\mathbf{Pt}(f)(p)$ is a prime element of A. Recall that if f is finitely separated from $\mathrm{id}\, A$ by M in the category of $\bigvee$-lattices, then f^* is finitely separated from $\mathrm{id}_{A^\perp}$ by the set $f^*(M)$. By 3.3, we know that $f^*(p) \ll_{A^{op}} f^*(m) \ll_{A^{op}} p$. Let us take the set $\mathcal{D}$ as in 3.3. We have that $d^2(a) \leq d(n_a) \ll n_a \leq a$ for all $d \in \mathcal{D}$, $a \in A$, and for suitable elements $n_a \in N_d$. This implies $d(p) \leq n_p \leq d^*(n_p) \leq d^*(p)$. Applying d^* again we obtain $p \leq d^*(d(p)) \leq d^*(n_p) \leq d^*\big(d^*(n_p)\big) \leq d^*\big(d^*(p)\big)$. Observe that $d^*\big(d^*(n_p)\big) \ll_{op} d^*(n_p) = \bigwedge\{q \in \mathbf{Pt}(L) : \ q \geq d^*(n_p)\}$. Then there is a finite subset $M_{d,p}$ such that $M_{d,p} \subseteq\subseteq \{q \in \mathbf{Pt}(L) : \ q \geq d^*(n_p)\}$ and $d^*\big(d^*(n_p)\big) \leq_{op} \bigwedge M_{d,p}$. This implies $\bigwedge M_{d,p} \leq d^*\big(d^*(n_p)\big) \leq d^*\big(d^*(p)\big)$, i.e., $p \leq q_{d,p} \leq d^*\big(d^*(p)\big)$ for some $q_{d,p} \in M_{d,p}$. So we have, for all n_p, a prime element $q_{d,p}$, i.e., we may put $P_d = \{q_{d,p} : \ n_p \in N_d\}$. Evidently, this subset finitely separates $\mathbf{Pt}(d^2) = (d^2)^*$ from the identity in $\mathbf{Pt}(A)$. □

PROPOSITION 3.13. *Let A be an FS-domain. Then the lattice $\sigma(A)$ of all Scott-open subsets of A is a linear FS-frame.*

Proof. Let A be an FS-domain, and let $\mathcal{D} \subseteq [A \to A]$ be the directed system of $\bigsqcup^{\uparrow}$-preserving maps which are finitely separated from the identity id_A on A. Let $f \in \mathcal{D}$, and let M_f be the corresponding finite subset of A. We shall show that f^{-1} is finitely separated from the identity $\mathrm{id}_{\sigma(A)}$ on $\sigma(A)$.

Evidently, f^{-1} is a frame morphisms. For each $a \in A$ we put $m(a) = \bigcup\{\uparrow m :\ m \in M_f,\ \exists y\ \ a \ll f(y) \leq m\}$. We have $m(a) \subseteq \uparrow a$. Now, let $y \in f^{-1}(\uparrow a)$. Then there is $z \in A$ such that $z \ll y$ and $z \in f^{-1}(\uparrow a)$, i.e., $f(z) \in \uparrow a$. Then there is an element $m \in M_f$ such that $a \ll f(z) \leq m \leq z \ll y$, i.e., $y \in \uparrow m \subseteq m(a)$. We have, for all $a \in A$, that $f^{-1}(\uparrow a) \subseteq m(a) \subseteq \uparrow a$.

Now, let U be a Scott-open subset of A. We put $m(U) = \bigcup\{\uparrow m :\ m \in M_f,\ \exists a{\in}U\ \exists y\ \ a \ll f(y) \leq m\}$, i.e., $m(U) = \bigcup\{m(a) :\ a \in U\}$. We have $f^{-1}(U) \subseteq m(U) \subseteq U$, i.e., f^{-1} is finitely separated from the identity. Here we put $M_{f^{-1}} = \{m(U) :\ U \in \sigma(A)\}$.

Recall that $f \leq g$ if and only if $f^{-1} \subseteq g^{-1}$, i.e., the set $\mathcal{C} = \{f^{-1} :\ f \in \mathcal{D}\}$ is a directed subset of $\mathcal{F}rm(A, A)$. We have to prove that, for all $a \in A$, $\uparrow a = \bigsqcup^{\uparrow}_{f \in \mathcal{D}} f^{-1}(\uparrow a)$. Namely, let $y \in \uparrow a$, i.e., there is $f \in \mathcal{D}$ and $z \in A$ such that $a \ll z \leq f(y) \ll y$, i.e., $y \in f^{-1}(\uparrow a)$ for some $f \in \mathcal{D}$. We have that, for all Scott-open subsets U of A, $U = \bigsqcup^{\uparrow}_{f \in \mathcal{D}} f^{-1}(U)$, i.e., $\mathrm{id}_{\sigma(A)} = \bigsqcup^{\uparrow}_{f \in \mathcal{D}} f^{-1}$. □

THEOREM 3.14. (**STONE DUALITY FOR FS-DOMAINS**) *The functors Ω and* pt *restrict to a dual equivalence between the category of FS-domains and the category of LFS-frames.*

Proof. Apply 3.12, 3.13 and [2; Theorem 7.2.28]. □

PROPOSITION 3.15. (**STONE DUALITY FOR ALGEBRAIC FS-DOMAINS**) *The functors Ω and* pt *restrict to a dual equivalence between the category of algebraic FS-domains and the category of algebraic LFS-frames.*

Proof. Apply 3.14 and the fact that a domain is algebraic if and only if its lattice of Scott-open subsets is algebraic. □

So we have the following interesting fact.

PROPOSITION 3.16. *Let A be a FS-domain. Then A has all filtered infima.*

Proof. We put $L = \sigma(A)$. Apply 3.11 on L (by 3.8, we know that L is stably continuous). □

LEMMA 3.17. *Let A be a frame, $a \in A$. Then:*

(i) *if A is a continuous frame, then $\downarrow(a)$ is continuous,*
(ii) *if A is a continuous frame, then $\uparrow(a)$ is continuous,*
(iii) *if A is an algebraic frame, then $\downarrow(a)$ is algebraic,*
(iv) *if A is an algebraic frame, then $\uparrow(a)$ is algebraic,*
(v) *if A is a linear FS-frame and a is a compact element, then $\downarrow(a)$ is a linear FS-frame,*
(vi) *if A is a linear FS-frame, then $\uparrow(a)$ is a linear FS-frame.*

Proof.

(i–iv): It is trivial.

(v): Let a be a compact element, $\mathcal{D}$ be the corresponding directed set of frame morphisms which are finitely separated from the identity id_A on A and $\mathrm{id}_A = \bigsqcup^{\uparrow} \mathcal{D}$. Then there is, by compactness of a, an element $g \in \mathcal{D}$ with $a = g(a)$. We put $\mathcal{E} = \{f \in \mathcal{D} : f \geq g\}$. Evidently, $\mathrm{id}_A = \bigsqcup^{\uparrow} \mathcal{E}$. We define $\mathcal{H} = \{h \in \mathcal{F}rm(\downarrow(a), \downarrow(a)) : h = f|_{\downarrow(a)},\ f \in \mathcal{E}\}$. Evidently, $\mathcal{H}$ is a directed set of frame morphisms finitely separated from $\mathrm{id}_{\downarrow(a)}$. Namely, any $h \in \mathcal{H}$ is well-defined, preserves arbitrary suprema and finite infima, and $h(0) = 0$, $h(1_{\downarrow(a)}) = f(a) = a = 1_{\downarrow(a)}$ for any $f \in \mathcal{D}$. We put $M_h = \{m \in M_f : m \leq a\}$, and let $x \in \downarrow(a)$. Then there is $m \in M_f$ such that $h(x) = f(x) \leq m \leq x \leq a$, i.e., there is $m \in M_h$ such that $h(x) \leq m \leq x$.

It remains to show that $\mathrm{id}_{\downarrow(a)} = \bigsqcup^{\uparrow} \mathcal{H}$. We have, for all $x \in \downarrow(a)$, $x = \bigsqcup^{\uparrow}_{f\in\mathcal{E}} f(x) = \bigsqcup^{\uparrow}_{h\in\mathcal{H}} h(x)$, i.e., $\downarrow(a)$ is a linear FS-frame.

(vi): Let $a \in A$, $\mathcal{D}$ be the corresponding directed set of frame morphisms which are finitely separated from the identity id_A on A and $\mathrm{id}_A = \bigsqcup^{\uparrow} \mathcal{D}$. We define $\mathcal{H} = \{h \in \mathcal{F}rm(\uparrow(a), \uparrow(a)) : h = f|_{\uparrow(a)} \vee a,\ f \in \mathcal{D}\}$. Evidently, $\mathcal{H}$ is a directed set of frame morphisms finitely separated from $\mathrm{id}_{\uparrow(a)}$. Namely, any $h \in \mathcal{H}$ is well-defined, preserves arbitrary suprema and finite infima and $h(0_{\uparrow(a)}) = f(a) \vee a = a = 0_{\uparrow(a)}$, $h(1_{\downarrow(a)}) = f(1) \vee a = 1_{\downarrow(a)}$ for some $f \in \mathcal{E}$. We put $M_h = \{n \in \downarrow(a) : n = m \vee a,\ m \in M_f\}$, and let $x \in \uparrow(a)$. Then there is $m \in M_f$ such that $h(x) = f(x) \vee a \leq m \vee \leq x \vee a = x$, i.e., there is $n \in M_h$ such that $h(x) \leq n \leq x$.

We show that $\mathrm{id}_{\uparrow(a)} = \bigsqcup^{\uparrow} \mathcal{H}$. We have, for all $x \in \uparrow(a)$, $x = \bigsqcup^{\uparrow}_{f\in\mathcal{E}} f(x) = \bigsqcup^{\uparrow}_{f\in\mathcal{E}} f(x) \vee a = \bigsqcup^{\uparrow}_{h\in\mathcal{H}} h(x)$, i.e., $\uparrow(a)$ is a linear FS-frame. □

4. Abstract bases and the Stone duality

First, let us recall some definitions from [2].

DEFINITION 4.1. An (*abstract*) *basis* is given by a set B together with a transitive relation $\prec$ on B, such that

$$z \prec x \implies \exists y{\in}B \;\; z \prec y \prec x \tag{INT}$$

holds for all elements $x, z \in B$.

Examples of abstract bases are concrete bases of continuous domains, of course, where the relation $\prec$ is the restriction of the order of approximation. Axiom (INT) is satisfied because of the interpolation property of $\ll$. Similarly, any basis in the sense of Definition 2.1 equipped with the relation $\lll$ is an abstract base.

Other examples are partially ordered sets, where (INT) is satisfied because of reflexivity. We may identify posets as being exactly the bases of supercompact elements of superalgebraic frames.

Note that a subset $U \subseteq B$ is *lower Scot-closed* if

1. $x \in U$, $y \prec x$ implies $y \in U$,
2. $x \in U$ implies there is $y \in U$ such that $x \prec y$ ("roundness").

DEFINITION 4.2. For a basis $\langle B, \prec\rangle$ let $\mathcal{D}(B)$ be the set of all lower Scott-closed subsets ordered by inclusion. It is called the *lower completion* of B. Furthermore, let $i\colon B \to \mathcal{D}(B)$ denote the function which maps $x \in B$ to $\downarrow x = \{y \in B : \ y \prec x\}$. If we want to stress the relation with which B is equipped, then we write $\mathcal{D}(B, \prec)$ for the lower completion.

Recall that, in the following, if we have both $\prec$ and $\leq$ on a set B, we shall always understand by $\downarrow(x)$ the set $\{y \in B : \ y \prec x\}$.

PROPOSITION 4.3. *Let $\langle B, \prec\rangle$ be an abstract basis.*

1. *The lower completion of B is a complete lattice.*
2. *$A \lll A'$ holds in $\mathcal{D}(B)$ if and only if there are $x \prec y$ in B such that $A \subseteq i(x) \subseteq i(y) \subseteq A'$.*
3. *$A \ll A'$ holds in $\mathcal{D}(B)$ if and only if there are $x_j \prec y_j$, $j = 1, \dots, n$, in B such that $A \subseteq \bigcup_{j=1}^{n} i(x_j) \subseteq \bigcup_{j=1}^{n} i(y_j) \subseteq A'$.*
4. *$\mathcal{D}(B)$ is a supercontinuous frame and a basis of $\mathcal{D}(B)$ is given by $i(B)$.*
5. *If $\prec$ is reflexive, then $\mathcal{D}(B)$ is superalgebraic.*
6. *If $\langle B, \prec\rangle$ is a poset, then B, $\mathrm{SK}\big(\mathcal{D}(B)\big)$, and $i(B)$ are all isomorphic.*

P r o o f. (1) holds because, clearly, the union of lower Scott-closed sets is a lower Scott-closed set. Roundness implies that every $A \in \mathcal{D}(B)$ can be written as

$\bigcup_{x\in A} \downarrow x$. This proves (2), (3), and also (4). The fifth observation follows from the characterization of the order of approximation. The last part holds because of 2.3 and that there is only one basis of supercompact elements for a superalgebraic frame. □

Our 'completion' has a weak universal property:

PROPOSITION 4.4. *Let $\langle B, \prec\rangle$ be an abstract basis, and let D be a complete lattice. For every monotone function $f\colon B \to D$ there is a largest $\bigvee$-preserving morphism $\hat{f}\colon \mathcal{D}(B) \to D$ such that $\hat{f}\circ i$ is below f. It is given by $\hat{f}(A) = \bigvee f(A)$. If the relation $\prec$ is reflexive, then $\hat{f}\circ i$ equals f. Moreover, if D is a frame, and f such that the following holds:*

$$f(y) = f(z) \implies \exists x\prec y,\ x\prec z \quad f(x) = f(y) = f(z)\,. \tag{MP}$$

then there is a largest frame morphism $\hat{f}\colon \mathcal{D}(B) \to D$ such that $\hat{f}\circ i$ is below f given by the same formula as above.

Proof. Let us first check that $\hat{f}$ is a morphism of $\bigvee$-semilattices (frames). Let $(A_i)_{i\in I}$ be a collection of lower Scott-closed sets. We can calculate: $\hat{f}\Big(\bigvee_{i\in I} A_i\Big)$ $= \hat{f}\Big(\bigcup_{i\in I} A_i\Big) = \bigvee\Big\{f(x) \mid x \in \bigcup_{i\in I} A_i\Big\} = \bigvee_{i\in I}\bigvee\{f(x) \mid x\in A_i\} = \bigvee_{i\in I}\hat{f}(A_i)$.
Now, let $A, B \in \mathcal{D}(B)$. Evidently, $f(A_1\cap A_2) \subseteq f(A_1)\cap f(A_2)$. Assume $a\in f(A_1)\cap f(A_2)$. Then there are $y\in A_1$, $z\in A_2$ such that $f(y) = f(z) = a$, i.e., there is $x\in B$ such that $x\prec y$, $x\prec z$, $f(x) = f(y) = f(z)$, i.e., $x\in A_1\cap A_2$.

Since f is assumed to be monotone, $f(x)$ is an upper bound for $f(\downarrow x)$. This proves that $\hat{f}\circ i$ is below f. If, on the other hand, $g\colon \mathcal{D}(B) \to D$ is another $\bigvee$-semilattice (frame) morphism with this property, then we have $g(A) = g\Big(\bigcup_{x\in A}\downarrow x\Big) = \bigvee_{x\in A} g(\downarrow x) = \bigvee_{x\in A} g(i(x)) \le \bigvee_{x\in A} f(x) = \hat{f}(A)$.

If $\prec$ is a preorder, then we can show that $\hat{f}\circ i = f$: $\hat{f}\big(i(x)\big) = \hat{f}(\downarrow x) = \bigvee f(\downarrow x) = f(x)$. □

Assume that B and B' are two abstract bases, and $f\colon B\to B'$ is a monotone map.

By the extension of f to $\mathcal{D}(B)$, we mean the map $\widehat{i'\circ f}\colon \mathcal{D}(B) \to \mathcal{D}(B)$.

PROPOSITION 4.5. *Let D be a supercontinuous frame with basis B, i.e., for every element x of D the set $B_x = \{y \lll x : y\in D\}\cap B$ contains a subset with supremum x.*

Viewing $\langle B, \lll \rangle$ as an abstract basis, we have the following:

1. *$\mathcal{D}(B)$ is isomorphic to D. The isomorphism $\sigma \colon \mathcal{D}(B) \to D$ is the extension $\hat{e}$ of the embedding of B into D. Its inverse β maps elements $x \in D$ to B_x.*
2. *For every frame E and frame morphism $f \colon D \to E$ such that g satisfies (MP) we have $f = \hat{g} \circ \beta$, where g is the restriction of f to B.*

P r o o f . First, let us prove that $\langle B, \lll \rangle$ is an abstract basis. Evidently, the relation $\lll$ is transitive and we can always interpolate. Now, we have to check the isomorphism. In a supercontinuous frame we have $x = \bigvee B_x$ for all elements, so $\sigma \circ \beta = \mathrm{id}_D$. Composing the maps the other way round we need to see that every $c \in B$ which totally approximates $\bigvee A$, where A is an lower Scott-closed set in $\langle B, \lll \rangle$, actually belongs to A. We interpolate: $c \lll d \lll \bigvee A$, and using the defining property of the totally bellow property, we find $a \in A$ above d. Therefore c totally approximates a and belongs to A.

The calculation for (2) is straightforward:

$$f(x) = f\Bigl(\bigvee B_x\Bigr) = \bigvee f(B_x) = \hat{g}(B_x) = \hat{g}\bigl(\beta(x)\bigr) .$$

□

Let us introduce the morphisms between approximable bases.

DEFINITION 4.6. A relation R between abstract bases B and C is called *approximable relation* if the following conditions are satisfied:

1. $\forall x{\in}B \ \forall y, y'{\in}C \ \ (xRy \succ y' \implies xRy')$;
2. $\forall x{\in}B \ \forall y{\in}C \ \ \bigl(xRy \implies (\exists z{\in}C \ \ xRz \text{ and } z \succ y)\bigr)$;
3. $\forall x, x'{\in}B \ \forall y{\in}C \ \ (x' \succ xRy \implies x'Ry)$;
4. $\forall x{\in}B \ \forall y{\in}C \ \ \bigl(xRy \implies (\exists z{\in}B \ \ x \succ zRy)\bigr)$.

Recall that it is well known that abstract bases and approximable relations form a category with respect to composition of relations. We then have the following.

THEOREM 4.7. *The category of abstract bases and approximable relations is equivalent to* **SUPERCONT**, *the category of supercontinuous frames and $\bigvee$-preserving mappings.*

P r o o f . Following 4.3 and 4.5, we have established the equivalence on the corresponding objects. Now, let $\langle B_1, \prec_1 \rangle$, $\langle B_2, \prec_2 \rangle$ be abstract bases, $R \subseteq B_1 \times B_2$ an approximable relation. Then we shall define a frame morphism $f_R \colon \mathcal{D}(B_1) \to \mathcal{D}(B_2)$ as follows:

$$f_R(U) = \bigl\{ b \in B_2 : \ (\exists a{\in}U)(aRb) \bigr\} .$$

Evidently, f_R is correctly defined and $f_R(\emptyset) = \emptyset$. Let us prove that f_R preserves arbitrary nonempty suprema (unions). Evidently, f_R is monotone, i.e., $U \subseteq V$ implies $f_R(U) \subseteq f_R(V)$. Now, let $b \in f_R\Big(\bigcup\limits_{j\in J} U_j\Big)$. Then there is an element $a \in \bigcup\limits_{j\in J} U_j$, aRb. So we can choose an index $j_0 \in J$ such that $a \in U_{j_0}$. The rest is evident.

Now, let $g\colon L_1 \to L_2$ be any $\bigvee$-preserving function between supercontinuous frames. Let $\langle B_1, \prec_1\rangle$, $\langle B_2, \prec_2\rangle$ be the corresponding bases. We shall define $R_g \subseteq B_1 \times B_2$. We put $aR_g b$ if and only if there is an element $u \in L_1$ such that $u \lll a$ and $b \lll g(u)$. Let us show that R_g is an approximable relation. It is enough to check the property 2, the rest is easy. Let $x \in B_1$, $y \in B_2$ such that for $xR_g y$, i.e., there is $u \in L_1$ such that $u \lll x$ and $y \lll g(u)$. Then evidently, since B_1 is a base of L_1, there is an element $w \in B_1$ such that $u \lll w \lll x$ and $y \lll g(w)$, i.e., by (INT), there is an element $z \in B_2$ such that $y \lll z \lll g(w)$. □

In particular:

Theorem 4.8. *The category of preorders and approximable relations is equivalent to* **SUPERALG**, *the category of superalgebraic frames and* $\bigvee$*-preserving mappings.*

Stone duality for stable prelocales.

First, we shall need some new notions.

Definition 4.9. A *stable prelocale* $(B, \prec, \vee, \wedge, 0, 1)$ is given by a set B together with a transitive relation $\prec$ on B and lattice operations $\vee$ and $\wedge$ on B, least element 0 and largest element 1 with respect to the lattice ordering, such that $\prec$ is an abstract base on B, and the following conditions are satisfied:

(LAT1) $M \prec x \implies \bigvee M \prec x$,
(LAT2) $y \prec N \implies y \prec \bigwedge N$,
(LAT3) $x \prec \bigvee M \implies$ there is $M' \subseteq\subseteq B$ such that for all $n \in M'$ there is $m \in M$ satisfying $n \prec m$, $x \prec \bigvee M'$.
(LAT4) $\leq \circ \prec \circ \leq \subseteq \prec \subseteq \leq$ and $\downarrow\colon B \to 2^B$ is an injective mapping.
(LAT5) $(B, \vee, \wedge, 0, 1)$ is a distributive lattice.
(LAT6) $x \prec y \implies$ there is $M \subseteq\subseteq \mathrm{Sp}(B)$ such that $x \prec \bigvee M \prec y$.

holds for all elements x, y and finite subsets M, N of B.

DEFINITION 4.10. A relation R between stable prelocales B and C is called *stable approximable relation* if the following conditions are satisfied:

1. $\forall x{\in}B\ \forall y,y'{\in}C\ \ (xRy \succ y' \implies xRy')$;
2. $\forall x{\in}B\ \forall M{\subseteq\subseteq}C\ \ \big(\forall y{\in}M\ \ xRy \implies$ there is an element $z \in C$ such that $xRz \succ \bigvee M\big)$;
3. $\forall x,x'{\in}B\ \forall y{\in}C\ \ (x' \succ xRy \implies x'Ry)$;
4. $\forall y{\in}C\ \forall N{\subseteq\subseteq}B\ \ \big(\forall x{\in}N\ \ xRy \implies$ there is an element $w \in B$ such that $\bigwedge N \succ wRy\big)$;
5. $\forall x{\in}B\ \forall y{\in}C\ \ \big(xRy \implies$ there are subsets $M = \{m_i : i \in I\} \subseteq\subseteq \mathrm{Sp}(B)$, $N = \{n_i : i \in I\} \subseteq\subseteq \mathrm{Sp}(C)$, m_iRn_i for all $i \in I$ and $x \succ \bigvee M\ R \bigvee N \succ y\big)$.

Evidently, any directed union of stable approximable relations is a stable approximable relation. Similarly as in [1] for domain prelocales we shall show that stable prelocales and stable approximable relations form a category.

PROPOSITION 4.11. *Stable approximable relations are closed under composition.*

Proof. Let $R\colon A \to B$, $S\colon B \to C$ are stable approximable relations. We shall show that $R \circ S$ is again a stable approximable relation. Evidently, the properties 1 and 3 are satisfied. Let us prove the condition 2. Let $x \in A$, $M \subseteq\subseteq C$, $xR \circ SM$. Then there is a subset $N \subseteq\subseteq B$, $N = \{n_y : y \in M\}$, xRn_ySy for all $y \in M$. We can find an element $z \in B$ such that $xRz \succ \bigvee N$, i.e., zSy for all $y \in M$. Again, we can find an element $w \in C$ such that $zRw \succ \bigvee M$ and $xRzSw$. The condition 4 may be proved dually. Let us prove the condition 5. Assume $xRzSy$. Then there are sets $M_1 = \{m_i^1 : i \in I_1\} \subseteq\subseteq \mathrm{Sp}(A)$, $N_1 = \{n_i^1 : i \in I_1\} \subseteq\subseteq \mathrm{Sp}(B)$, $m_i^1Rn_i^1$ for all $i \in I_1$ and $x \succ \bigvee M_1\ R \bigvee N_1 \succ z$ and $M_2 = \{m_j^2 : j \in I_2\} \subseteq\subseteq \mathrm{Sp}(B)$, $N_2 = \{n_i^2 : j \in I_2\} \subseteq\subseteq \mathrm{Sp}(C)$, $m_j^2Sn_j^2$ for all $j \in I_2$ and $z \succ \bigvee M_2\ S \bigvee N_2 \succ y$.

Then for each $j \in I_2$ there is an index $i_j \in I_1$ such that $m_{i_j}^1Rm_j^2Sn_j^2$, i.e., $m_{i_j}^1R \circ Sn_j^2$. Evidently, then $x \succ \bigvee M_1'\ R \circ S\ \bigvee N_2 \succ y$, here $M_1' = \{m_{i_j}^1 : j \in I_2\}$. □

Identities with respect to this composition are given by

$$a \ \mathrm{id}_A\ b \iff a \succ b\,.$$

So we may define a category $\mathcal{StPre}$ of stable prelocales and stable approximable relations.

Now, let us describe the passage from a stable prelocale to a stably continuous supercontinuous frame.

PROPOSITION 4.12. *Let B be a stable prelocale. Then, if we put $\mathcal{I}d(B)$ to be the set of all lower Scott-closed subsets closed under finite suprema — the Scott-closed ideals, $\mathcal{I}d(B)$ is a stably continuous supercontinuous frame.*

Proof. Evidently, any intersection of Scott-closed ideals is a Scott-closed ideal, i.e., $\mathcal{I}d(B)$ is a complete lattice. So we may define a map $\chi : \mathcal{D}(B) \to \mathcal{D}(B)$ such that $\chi(U) = \{x \in B : x \le \bigvee M, M \subseteq\subseteq U\}$. Evidently, χ is an idempotent, extensive and order preserving map. We shall prove that $\chi(U) \cap \chi(V) = \chi(U \cap V)$. Let $x \in \chi(U) \cap \chi(V)$. Then there are finite sets M, N, $M \subseteq\subseteq U$, $N \subseteq\subseteq V$, such that $x \le \bigvee M$, $x \le \bigvee N$. Since B is distributive, we have that $M \wedge N \subseteq U \cap V$ and $x \le \bigvee M \wedge N$. Then χ is a nucleus on $\mathcal{D}(B)$, and its image is exactly the frame of Scott-closed ideals.

We shall prove that $\mathcal{I}d(B)$ is a stably continuous supercontinuous frame. Recall that by (LAT4) and roundness, any Scott-closed ideal I satisfies $x \in I$, $y \le x$ implies $y \in I$, and by (LAT3) and (LAT6), any Scott-closed ideal I is a supremum of all principal ideals $\downarrow a$, $a \in I \cap \mathrm{Sp}(B)$. Let $a \in I \cap \mathrm{Sp}(B)$. We shall prove that $\downarrow a \lll_{\mathcal{I}d(B)} I$.

First, note that $\bigvee\limits_{\alpha \in \Lambda} I_\alpha = \{x \le y_1 \vee \dots y_n : y_1 \in I_{\alpha_1}, \dots, y_n \in I_{\alpha_n}, n \in \mathbb{N}\}$, i.e., $\bigvee\limits_{\alpha \in \Lambda} I_\alpha = \bigcup\{\downarrow(y_1 \vee \dots y_n) : y_1 \in I_{\alpha_1}, \dots, y_n \in I_{\alpha_n}, n \in \mathbb{N}\}$ by the property (LAT1), (LAT2), (LAT4) and the definition of a Scott-closed ideal. Then we have that $a \le y_1 \vee \dots y_n$, i.e., $a \le y_j$ for some j, i.e., $a \in I_{\alpha_j}$, i.e., $\downarrow a \subseteq I_{\alpha_j}$. So we have proved that $\mathcal{I}d(B)$ is supercontinuous. Let us prove the stable continuity. Evidently, $B = \downarrow 1$ is a compact element of $\mathcal{I}d(B)$. Recall that $I \ll_{\mathcal{I}d(B)} J$ if and only if $I \subseteq \downarrow a \subseteq J$ for some element $a \in J$. Now, let $I_i \subseteq \downarrow_i a \subseteq J_i$, $i = 1, 2$. Then $I_1 \cap I_2 \subseteq \downarrow a_1 \cap \downarrow a_2 = \downarrow(a_1 \wedge a_2) \subseteq J_1 \cap J_2$ by (LAT2). □

PROPOSITION 4.13. *Let L be a stable continuous supercontinuous frame. Then, if we put $\mathcal{B}(L) = B$ to be the distributive sublattice of L generated by* $\mathrm{Sp}(L)$, $\mathcal{B}(L) = (B, \ll)$ *is a stable prelocale.*

Proof. We know that any stably continuous supercontinuous frame is totally bellow generated by its $\vee$-prime elements. We shall define on B a relation $\prec_B$ such that

$$x \prec_B y \iff x \ll y .$$

Recall that $\prec_B$ is transitive and satisfies the interpolation property (both is evident).

Now, we shall prove that B is a stable prelocale. (LAT1) is satisfied by the fact that the relation $\ll$ preserves finite suprema, and (LAT2) holds trivially by stable continuity. (LAT3) follows from the fact that the $\vee$-prime elements of L form a basis for L. The fourth clause follows from the fact that $\le \circ \ll \circ \le$

$\subseteq\ll\subseteq\leq$. (LAT5) and (LAT6) follow from the definition of B and the fact that the $\vee$-prime elements form a basis of L. □

THEOREM 4.14. *The category of stable prelocales and stable approximable relations is equivalent to* **STSUPERCONT**, *the category of stably continuous supercontinuous frames and frame morphisms.*

Proof. Let us show that our construction from 4.12 and 4.13 is functorial.

Now, let $g\colon L_1 \to L_2$ be any frame morphism between stably continuous supercontinuous frames. Let $\langle \mathcal{B}(L_1), \prec_1\rangle$, $\langle \mathcal{B}(L_2), \prec_2\rangle$ be the corresponding bases, we put $B_1 = \mathcal{B}(L_1)$ and $B_2 = \mathcal{B}(L_2)$. We shall define $\mathcal{B}(g) = R_g \subseteq B_1 \times B_2$. We put aR_gb if and only if there is an element $u \in L_1$ such that $u \ll a$ and $b \ll g(u)$.

Let us show that R_g is a stable approximable relation.

1. We have $u \ll a$, and $b' \ll b \ll g(u)$ implies $u \ll a$ and $b' \ll g(u)$, i.e., aR_gb'.

2. If M is empty, we have always that $0 \ll g(0)$, $0 \ll x$, i.e., for all $xR_g0 \gg \bigvee M$ for all $x \in B_1$. Let us assume that M is nonempty, i.e., let $x \in B_1$ such that for all $y \in M$ we have xR_gy, i.e., there are $u_y \in L_1$ such that $u_y \ll x$ and $y \ll g(u_y)$. Then evidently, by (INT) and (LAT1), there is an element $w \in B_1$ such that $u_y \ll w \ll x$ and $y \ll g(w)$, i.e., again by (INT) there is an element $z \in B_2$ such that $y \ll z \ll g(w)$ for all $y \in M$, i.e., $\bigvee M \ll z \ll g(w)$, xR_gz.

3. We have $u \ll x \ll x'$, $y \ll g(u)$ implies $u \ll x'$, $y \ll g(u)$, i.e. $x'R_gy$.

4. If N is empty, we have always that $y \ll g(1) = 1$ for every $y \in B_2$, i.e., for all $1R_gy$ for all $y \in B_2$. Let us assume that N is nonempty, i.e., let $y \in B_2$ such that for all $x \in N$ we have xR_gy, i.e., there are $u_x \in L_1$ such that $u_x \ll x$ and $y \ll g(u_x)$. Then evidently, by stable continuity of L_1 and L_2, we have that $u = \bigwedge\{u_x :\ x \in N\} \ll \bigwedge N$ and $y \ll g(u)$, i.e., by (INT) and (LAT2) there is an element $w \in B_1$ such that $u \ll w \ll \bigwedge N$, wR_gy.

5. Let $u \ll x$, $y \ll g(u)$. Then there is a set $N \subseteq\subseteq \mathrm{Sp}(L_2)$ such that $y \ll \bigvee N \ll g(u)$. Then there is, for all $n \in N$, an element $z_n \in \mathrm{Sp}(L_1)$ such that $n \ll g(z_n)$ and $z_n \ll u \ll x$, i.e., there is an element $w_n \in \mathrm{Sp}(L_1)$ such that $z_n \ll w_n \ll x$, i.e., w_nR_gn and $x \gg \bigvee w_nR_g\bigvee n \gg y$.

Now, let $\langle B_1, \prec_1\rangle$, $\langle B_2, \prec_2\rangle$ be stable prelocales, $R \subseteq B_1 \times B_2$, a stable approximable relation. Then we shall define a frame morphism $f_R\colon \mathcal{I}d(B_1) \to \mathcal{I}d(B_2)$ as follows:

$$f_R(U) = \chi\bigl(\{b \in B_2 :\ (\exists a \in U \cap \mathrm{Sp}(B_1))(aRb)\}\bigr).$$

Evidently, $f_R(\{0\}) = \{0\}$ and $f_R(B_1) = B_2$ by the condition 5. Let us prove that f_R preserves arbitrary nonempty suprema. Evidently, f_R is monotone, i.e., $U \subseteq V$ implies $f_R(U) \subseteq f_R(V)$. Now, let $b \in f_R\Bigl(\bigvee\limits_{j\in J} U_j\Bigr)$. Then $b \leq \bigvee\limits_{k=1}^{m} b_k$

such that for each b_k there is an element $a_k \in \mathrm{Sp}(B_1) \cap \bigvee_{j\in J} U_j$, $a_k R b_k$. Then by $\vee$-primeness of a_k, there is an element $u_k \in U_{j_k}$ such that $a_k \prec u_k$, i.e., $b_k \in f_R(U_{j_k})$. This gives us that $b \leq \bigvee b_k \in \bigvee_{j\in J} f_R(U_j)$.

We have to prove that $f_R(U\cap V) = f_R(U)\cap f_R(V)$. Now, let $y\in f_R(U)\cap f_R(V)$. Then $b \leq \bigvee_{k=1}^{m} b_k^1$ such that for each b_k^1 there is an element $a_k^1 \in \mathrm{Sp}(B_1) \cap U$, $a_k^1 R b_k^1$, and $b \leq \bigvee_{l=1}^{n} b_l^2$ such that for each b_l^2 there is an element $a_l^2 \in \mathrm{Sp}(B_1)\cap V$, $a_l^2 R b_l^2$. By the property 4, we have that $a_k^1 \wedge a_l^2 R b_k^1 \wedge b_l^2$ for all k, l. Applying the property 5, we obtain finite subsets $M_{kl} = \{m_i : i \in I_{kl}\} \subseteq\subseteq \mathrm{Sp}(B_1)$, $N_{kl} = \{n_i : i \in I_{kl}\} \subseteq\subseteq \mathrm{Sp}(B_2)$, $m_i R n_i$ for all $i \in I_{kl}$, and $a_k^1 \wedge a_l^2 \succ \bigvee M_{kl}\ R\ \bigvee N_{kl} \succ b_k^1 \wedge b_l^2$. Then evidently, $M_{kl} \subseteq\subseteq U \cap V \cap \mathrm{Sp}(B_1)$, i.e., $N_{kl} \subseteq\subseteq f_R(U \cap V)$, i.e., $b \leq \bigvee_{k,l} b_k^1 \wedge b_l^2 \prec \bigvee_{k,l} N_{k,l}$.

It is easy to see that $\mathcal{I}d(\mathcal{B}(L)) \equiv L$. □

COROLLARY 4.15. *The category of reflexive stable prelocales and stable approximable relations is equivalent to* **STSUPERALG**, *the category of coherent superalgebraic frames and frame morphisms.*

P r o o f . Evidently, for a reflexive stable prelocale B, $\mathcal{I}d(B)$ is a coherent superalgebraic frame ($I \in \mathrm{SK}(\mathcal{I}d(B))$ if and only if $I = \downarrow a$ for some $a \in \mathrm{Sp}(B)$). The other direction is evident. □

DEFINITION. A stable approximable relation $R \subseteq \succ_B$ on a stable prelocale B is said to be *finitely separated from the identity* if there is a finite subset $M \subseteq\subseteq B$ such that aRb implies that we can find an element $m \in M$ such that $a \succ m \succ b$.

A stable prelocale B is said to be an *LFS-prelocale* if $\succ_B$ is a directed union of stable approximable relations finitely separated from the identity.

THEOREM 4.17. *The category of LFS-prelocales and stable approximable relations is equivalent to the category of LFS-frames and frame morphisms.*

P r o o f . It follows immediately from the definitions and 4.14. □

COROLLARY 4.18. *The category of reflexive LFS-prelocales and stable approximable relations is equivalent to the category of algebraic LFS-frames and frame morphisms.*

5. The category of preframes

Recall that a *preframe* (see [3], [11]) is a partially ordered set A in which all finitary infima and all directed suprema exist, and for any $x \in A$ and directed subset $D \subseteq A$

$$x \wedge \bigsqcup^{\uparrow} D = \bigsqcup^{\uparrow}\{x \wedge t : \ t \in D\}.$$

Note that a preframe need not have the smallest element although it has a largest one, the infimum of the empty set. A *preframe* morphism is a map between preframes preserving all finitary infima and all directed suprema. The resulting category will be called $\mathcal{P}re\mathcal{F}rm$. Evidently, $\mathcal{P}re\mathcal{F}rm$ is then a subcategory of the category of dcpos and Scott-continuous mappings and $\mathcal{F}rm$ is a subcategory of $\mathcal{P}re\mathcal{F}rm$. An LFS-object in $\mathcal{P}re\mathcal{F}rm$ is called an *LFS-preframe*. Evidently, any LFS-preframe is an FS-domain.

For $A, B \in \mathcal{P}re\mathcal{F}rm$, let $A \multimap B$ be the poset of all preframe maps $f\colon A \to B$, ordered pointwise. Define $\mathbf{1} := \{1\}$ and $\perp := \{0 < 1\}$ ($\perp = \mathbf{2}$).

Lemma 5.1. *The category $\mathcal{P}re\mathcal{F}rm$ is closed under $\mathbf{1}$, $\perp$ and $\multimap$.*

Proof. Let A, B be objects in $\mathcal{P}re\mathcal{F}rm$. We know that $A \multimap B \subseteq [A \to B]$, and the supremum s in $[A \to B]$ of a directed subset D of $A \multimap B$ exists and is the pointwise one. We have to show that s preserves finite infima. Evidently, $s(1) = 1$. Now, let $x, y \in A$. Then

$$s(x) \wedge s(y) = \bigsqcup^{\uparrow}_{d\in D} d(x) \wedge \bigsqcup^{\uparrow}_{e\in D} e(y) = \bigsqcup^{\uparrow}_{d\in D} d(x \wedge y) = s(x \wedge y)$$

by the preframe distributive law and the directness of D. The function $\lambda x\,.\,1_B\colon A \to B$ is the top of $A \multimap B$. Now, let $f, g \in A \multimap B$. Then evidently $f \wedge g$ preserves finite infima, and we have

$$\begin{aligned}(f \wedge g)(\bigsqcup^{\uparrow} S) &= f(\bigsqcup^{\uparrow} S) \wedge g(\bigsqcup^{\uparrow} S) = \bigsqcup^{\uparrow} f(S) \wedge \bigsqcup^{\uparrow} g(S)\\ &= \bigsqcup^{\uparrow}_{s\in S} \bigsqcup^{\uparrow}_{t\in S} f(s) \wedge g(t) = \bigsqcup^{\uparrow}(f \wedge g)(S),\end{aligned}$$

i.e., $f \wedge g \in A \multimap B$. Finally, $\mathbf{1}$ and $\perp$ are preframes. □

The following propositions are well known (see [3], [11]).

Lemma 5.2. *The category $\mathcal{P}re\mathcal{F}rm$ has arbitrary products and coproducts.*

Proof. Evidently, a cartesian product of a system of preframes is a preframe as well. Let $(A_i)_{i\in I}$ be any family of preframes, $A \subseteq \prod_{i\in I} A_i$ a subset consisting of all those $a = (a_i)_{i\in I}$ whose support $\operatorname{spt}(a) = \{i \in I : \ a_i < 1_i\}$, 1_i the top element of A_i, is finite. A is closed under finite infima and directed suprema in $\prod_{i\in I} A_i$ and hence a subpreframe of the latter. We have preframe maps $k_i\colon A_i \to A$ defined by

$$k_i(x)_j = \begin{cases} x & \text{if } j = i\,, \\ 1_j & \text{otherwise}\,, \end{cases}$$

and it is easy to see that they are the coproduct injections. □

COROLLARY 5.3. *Finite products and finite coproducts coincide in $\mathcal{PreFrm}$.*

LEMMA 5.4.

1. *The forgetful functor from $\mathcal{PreFrm}$ to $\mathcal{Set}$ has a left adjoint. Moreover, the monadic length of the adjunction is 2.*
2. *The free preframe over a meet-semilattice S is the ideal completion $\mathrm{Idl}(S)$.*
3. *The free frame over a meet-semilattice S is the set $\mathcal{D}(S)$ of lower closed sets of S.*
4. *The free frame over a preframe A is the set $\Upsilon(A)$ of Scott-closed subsets of A.*

P r o o f. See [3] and [11]. □

PROPOSITION 5.5. *Let S be a meet-semilattice, and let R be a set (coverage) each of whose elements has the form (X,a) where $X = (x_i)_{i\in P}$ is a monotone net in S, and a is an upper bound in S for $\{x_i : i \in P\}$. Then the preframe presentation*

$$\mathcal{PreFrm}\big\langle S(\textit{qua meet-semilattice}) \mid \textstyle\bigsqcup^{\uparrow} X = a \ \ \big((X,a)\in R\big)\big\rangle$$

exists.

P r o o f. See [11]. □

So we have the following

THEOREM 5.6.

1. *$\mathcal{PreFrm}$ has equalizers and coequalizers.*
2. *$\mathcal{PreFrm}$ has arbitrary limits and colimits.*

The category SET of sets and set-theoretic functions is cartesian closed, and the functions

$$\begin{aligned} &\mathit{curry}\colon C^{A\times B} \to (C^B)^A\,, && \mathit{curry} := \lambda f\,.\,\lambda a\,.\,\lambda b\,.\,f\langle a,b\rangle \\ &\mathit{uncurry}\colon (C^B)^A \to C^{A\times B}\,, && \mathit{uncurry} := \lambda g\,.\,\lambda\langle a,b\rangle\,.\,g(a)(b) \end{aligned} \tag{2}$$

are mutually inverse bijections. This provides us with the concept of a *bimorphism* if we characterize the set $\mathit{uncurry}\big(A \multimap (B \multimap C)\big)$ in $C^{A\times B}$.

DEFINITION 5.7. For objects A, B and C in $\mathcal{P}re\mathcal{F}rm$, a set-theoretic function f of type $f\colon A \times B \to C$ is a *bimorphism* if and only if

$$\begin{aligned} &\forall a{\in}A: \quad \lambda b\,.\, f\langle a,b\rangle\colon B \to C \text{ is a preframe morphism,}\\ &\forall b{\in}B: \quad \lambda a\,.\, f\langle a,b\rangle\colon A \to C \text{ is a preframe morphism.}\end{aligned}$$

We denote by $\mathrm{Bil}(A{\times}B,C)$ the poset of all bilinear functions $f\colon A \times B \to C$ in the *pointwise* order.

LEMMA 5.8. *For objects A, B and C in $\mathcal{P}re\mathcal{F}rm$, $\mathrm{Bil}(A{\times}B,C)$ is indeed an object in $\mathcal{P}re\mathcal{F}rm$.*

Proof. Let $f,g \in \mathrm{Bil}(A{\times}B,C)$. Let $a \in A$, $b,c \in B$, $S \subseteq B$, S directed. Then

$$\begin{aligned}(f \wedge g)\big(a,{\textstyle\bigsqcup^{\uparrow}} S\big) &= f\big(a,{\textstyle\bigsqcup^{\uparrow}} S\big) \wedge g\big(a,{\textstyle\bigsqcup^{\uparrow}} S\big) = \bigsqcup_{s\in S}^{\uparrow} f(a,s) \wedge \bigsqcup_{t\in S}^{\uparrow} g(a,t)\\ &= \bigsqcup_{s\in S}^{\uparrow} (f \wedge g)(a,s)\,,\end{aligned}$$

$$\begin{aligned}\big(f(a,b) \wedge g(a,b)\big) \wedge \big(f(a,c) \wedge g(a,c)\big) &= \big(f(a,b\wedge c) \wedge g(a,b\wedge c)\big)\\ &= (f \wedge g)(a,b\wedge c)\,.\end{aligned}$$

The rest is evident.

Similarly, let $D \subseteq \mathrm{Bil}(A{\times}B,C)$, D directed. Let $a \in A$, $b,c \in B$, $S \subseteq B$, S directed. Then

$$\begin{aligned}\big({\textstyle\bigsqcup^{\uparrow}} D\big)\big(a,{\textstyle\bigsqcup^{\uparrow}} S\big) &= \bigsqcup_{d\in D}^{\uparrow} d\big(a,{\textstyle\bigsqcup^{\uparrow}} S\big) = \bigsqcup_{d\in D}^{\uparrow}\bigsqcup_{s\in S}^{\uparrow} d(a,s)\\ &= \bigsqcup_{s\in S}^{\uparrow} \big({\textstyle\bigsqcup^{\uparrow}} D\big)(a,s)\,,\\ \big({\textstyle\bigsqcup^{\uparrow}} D\big)(a,b) \wedge \big({\textstyle\bigsqcup^{\uparrow}} D\big)(a,c) &= \bigsqcup_{d\in D}^{\uparrow} d(a,b) \wedge \bigsqcup_{t\in D}^{\uparrow} t(a,c) = \bigsqcup_{d\in D}^{\uparrow}\bigsqcup_{t\in D}^{\uparrow} d(a,b) \wedge t(a,c)\\ &= \bigsqcup_{d\in D}^{\uparrow} d(a,b\wedge c)\,.\end{aligned}$$

□

Recall that a bilinear map $f\colon A \times B \to C$ need not be a morphism in $\mathcal{P}re\mathcal{F}rm$, nor is a map $g \in A \times B \multimap C$ bilinear in general. If we restrict the maps *curry* and *uncurry* to $\mathrm{Bil}(A{\times}B,C)$ and $A \twoheadrightarrow (B \twoheadrightarrow C)$, we get a natural order-isomorphism between $\mathrm{Bil}(A{\times}B,C)$ and $A \twoheadrightarrow (B \twoheadrightarrow C)$.

LEMMA 5.9. *Let A, B and C be objects in $\mathcal{P}re\mathcal{F}rm$. Then the restrictions of curry and uncurry are mutually inverse order-isomorphisms between $\mathrm{Bil}(A{\times}B,C)$ and $A \twoheadrightarrow (B \twoheadrightarrow C)$. In particular, $\mathrm{Bil}(A{\times}B,C)$ is an object in $\mathcal{P}re\mathcal{F}rm$ and the restrictions curry and uncurry are isomorphisms in $\mathcal{P}re\mathcal{F}rm$.*

Proof. Given *curry* and *uncurry* as defined on all set-theoretic functions $C^{A\times B}$ and $(C^B)^A$, we know that they are mutually inverse functions on these sets; but, from the definition of bilinearity, we easily get that the curried version of $f \in \mathrm{Bil}(A\times B, C)$ is in $A \multimap (B \multimap C)$, and that the uncurried version of some $g \in A \multimap (B \multimap C)$ is bilinear. Since *curry* and *uncurry* are monotone, they preserve all suprema and infima which exist in $\mathrm{Bil}(A\times B, C)$, $A \multimap (B \multimap C)$ respectively. □

Therefore, we obtain the natural isomorphism $A \otimes B \multimap C \cong A \multimap (B \multimap C)$ by showing

$$\mathrm{Bil}(A\times B, C) \cong A \otimes B \multimap C\,. \tag{4}$$

For that, it is sufficient to have a preframe $A\otimes B$ and a bilinear map $\otimes\colon A\times B \to A \otimes B$ which is universal among all bilinear maps of type $f\colon A \times B \to C$. For all such f, there exists a unique preframe morphism $\bar{f}\colon A \otimes B \to C$ such that $\bar{f} \circ \otimes = f$. The isomorphism is then verified by sending f to $\bar{f}$.

Now, let us first construct, for meet-semilattices A and B, their meet-semilattice tensor product $A \otimes_m B$ as

$$\begin{aligned} A \otimes_m B = \wedge - \mathcal{S}emi\mathcal{L}at\big\langle a \otimes_m b \ \ (a \in A, b \in B) \mid \\ \textstyle\bigwedge S \otimes_m b = \bigwedge\{a \otimes_m b:\ a \in S\}\ (S \subseteq\subseteq A) \\ \textstyle\bigwedge a \otimes_m T = \bigwedge\{a \otimes_m b:\ b \in T\}\ (T \subseteq\subseteq B)\big\rangle\,. \end{aligned}$$

Recall that, if A, B and C are meet-semilattices, and $f\colon A \times B \to C$ is a bimorphism with respect to finite infima, then there exists a meet-semilattice morphism $\bar{f}\colon A \otimes_m B \to C$ such that $\bar{f}(a \otimes_m b) = f(a, b)$.

Now, let us construct the preframe tensor product $A \otimes B$ of preframes A and B (see [11]). We take their meet-semilattice tensor product $A \otimes_m B$ and then equip it with the coverage R generated by all pairs $(X \otimes_m b, a \otimes_m b)$ and $(a \otimes_m Y, a \otimes_m b)$, where X and Y are monotone nets in A and B, with joins a and b, $X \otimes_m b$ denotes the monotone net $(x \otimes_m b \mid x \in X)$, and $a \otimes_m Y$ denotes the monotone net $(a \otimes_m y \mid y \in Y)$. Then the preframe $A \otimes B$ is presented as $\langle A \otimes_m B \mid R\rangle$.

Recall the following proposition (see [11]).

PROPOSITION 5.10. *$\mathcal{P}re\mathcal{F}rm$ has a symmetric monoidal structure* $(\otimes, \mathbf{I})$, *where* $\mathbf{I} = \mathbf{2}$, *and* $- \otimes A$ *is a left adjoint to* $A \multimap -$.

DEFINITION 5.11. A filter $F \subseteq A$, A a preframe, is called *Scott-open* if $\bigsqcup^{\uparrow} M \in F$ implies $F \cap M \neq \emptyset$ for all directed $M \subseteq A$.

PROPOSITION 5.12. *Let A be a preframe, and let F be a subset of L. The following are equivalent:*

1. *F is a Scott-open filter.*
2. *χ_F is a preframe homomorphism from A to* $\mathbf{2}$, *here*

$$\chi_F(a) = \begin{cases} 1 & \text{if } a \in F\,, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let F be a Scott-open filter. Then χ_F preserves finite infima and directed suprema. Conversely, for a preframe morphism χ_F, $F = \chi_F^{-1}(\{1\})$ is a Scott-open filter. $\square$

DEFINITION 5.13. Let A be a preframe. The *points* of A are the Scott-open filters of A. The collection $\mathrm{pt}(A)$ of all points is turned into a preframe $\Theta(A)$ by requiring all those subsets of $\mathrm{pt}(A)$ to be in $\Theta(A)$ which are of the form

$$\mathcal{F}_x = \{F \in \mathrm{pt}(A) \mid x \in F\}, \qquad x \in A.$$

PROPOSITION 5.14. *The sets $\mathcal{F}_x$, $x \in A$, form a subpreframe of $\mathcal{P}(\mathrm{pt}(A))$. Moreover, any $\mathcal{F}_x$ is a Scott-open filter on $\mathrm{pt}(A)$.*

Proof. We have $\bigcap\limits_{m\in M} \mathcal{F}_{x_m} = \mathcal{F}_{\bigwedge\limits_{m\in M} x_m}$, M finite, because points are filters and $\bigcup\limits_{\imath\in I} \mathcal{F}_{x_\imath} = \mathcal{F}_{\bigvee\limits_{\imath\in I} x_\imath}$ because they are Scott-open. $\square$

We may assign to a preframe A the preframe $\mathrm{pt}(A) \cong A \multimap \mathbf{2}$ of all points of A, and, to a preframe morphism $h\colon B \to A$, the map $\mathrm{pt}(h)\colon \mathrm{pt}(A) \to \mathrm{pt}(B)$ which assigns to a point F the point $h^{-1}(F)$, we get a contravariant functor, also denoted by pt, from $\mathcal{PreFrm}$ to $\mathcal{PreFrm}$. Applying pt twice, we get a covariant functor Σ, from $\mathcal{PreFrm}$ to $\mathcal{PreFrm}$, i.e., a preframe A can be mapped into the preframe of points of $\mathrm{pt}(A)$. We map $a \in A$ to the Scott-open filter $\mathcal{F}_a$ of all Scott-open filters containing a. This assignment, which we denote by $\eta_A\colon A \to \mathrm{pt}(\mathrm{pt}(A))$, is a preframe morphism: Let $a \in A$, F be a Scott-open filter of A. Then we have: $\mathcal{F}_a \in \mathcal{F}_F \iff F \in \mathcal{F}_a \iff a \in F$. It also commutes with preframe morphisms $f\colon A \to B$:

$$\begin{aligned}
\mathrm{pt}(\mathrm{pt}(f))(\eta_A(a)) &= \mathrm{pt}(f^{-1})(\mathcal{F}_a) = (f^{-1})^{-1}(\{F \in \mathrm{pt}(A) :\ a \in F\}) \\
&= \{F' \in \mathrm{pt}(B) :\ a \in f^{-1}(F')\} = \{F' \in \mathrm{pt}(B) :\ f(a) \in F'\} \\
&= \mathcal{F}_{f(a)} = \eta_B \circ f(a).
\end{aligned}$$

So the family of all η_A constitutes a natural transformation from the identity functor to $\mathrm{pt} \circ \mathrm{pt}$.

We can now formulate the preframe version of the Stone Duality Theorem:

THEOREM 5.15. *The functor* $\mathrm{pt}\colon \mathcal{PreFrm} \to \mathcal{PreFrm}$ *is selfadjoint. The unit and counit is η.*

Proof. It remains to check the triangle equality

$$\begin{tikzcd}
\mathrm{pt}(A) \arrow[r, "\eta_{\mathrm{pt}(A)}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{pt}^3(A) \arrow[d, "\mathrm{pt}(\eta_A)"] \\
 & \mathrm{pt}(A)
\end{tikzcd}$$

Let F be a Scott-open filter in A. Then

$$\begin{aligned}\mathrm{pt}(\eta_A)\big(\eta_{\mathrm{pt}(A)}(F)\big) = \eta_A^{-1}(\mathcal{F}_F) &= \{x \in A \mid \eta_A(x) \in \mathcal{F}_F\} \\ &= \{x \in A \mid \mathcal{F}_x \in \mathcal{F}_F\} \\ &= \{x \in A \mid F \in \mathcal{F}_x\} \\ &= \{x \in A \mid x \in F\} = F\,.\end{aligned}$$

□

LEMMA 5.16. **(PIT)** *Let A, B be continuous complete preframes, $f\colon A \to B$ a preframe morphism. Then f is a continuous map with respect to the Lawson topology. Moreover, f preserves arbitrary infima.*

Proof. It is an easy corollary of ([12; p. 301, Corollary]). □

6. Linear FS-preframes

LEMMA 6.1. *Let A and B be LFS-preframes. Then:*

(i) *the poset $A \multimap B$ is an LFS-preframe,*
(ii) *$A \oplus B$ is an LFS-preframe,*
(iii) *A is stably continuous, i.e., A is compact, and $a \ll b$, $a \ll c$ implies $a \ll b \wedge c$.*

Proof.
(i): The proof follows from the proof of Lemma 5 in [8] and from 5.1.
(ii), (iii): The proof copies the proof of 3.8. □

THEOREM 6.2. **(STONE DUALITY FOR LFS-PREFRAMES)** *The category of LFS-preframes is selfdual.*

Proof. Apply 5.15 and 6.1. □

PROPOSITION 6.3.
(STONE DUALITY FOR ALGEBRAIC LFS-PREFRAMES) *The category of algebraic LFS-preframes is selfdual.*

Proof. Apply 6.2 and the fact that a preframe is algebraic if and only if its lattice of Scott-open filters is algebraic. □

PROPOSITION 6.4. *Let A be a LFS-preframe. Then A is a complete stably continuous lattice.*

Proof. By 3.16, we know that A has arbitrary infima, i.e., it is a complete lattice. □

Recall that, for any preframe A, we have a preframe map $i_A\colon A \to \Upsilon(A)$, here $\Upsilon(A)$ is the lattice of all *Scott closed subsets* of A (lower sets closed under directed joins), defined by $i_A(a) = \{x \in A : x \le a\}$. Moreover, this map is (see [3; Proposition 1]) the universal preframe homomorphism from A to frames.

PROPOSITION 6.5. *Let A be an LFS-preframe. Then the frame $\Upsilon(A)$ of all Scott-closed subsets of A is an LFS-frame in which the subset* $\mathrm{Sp}(\Upsilon(A))$ *is closed under finite infima. Moreover, the directed set of finitely separated frame morphisms may be chosen such that those preserve $\vee$-prime elements.*

P r o o f. Let $\mathcal{D} \subseteq A \multimap A$ be a directed subset such that $\mathrm{id}_A = \bigsqcup^{\uparrow} \mathcal{D}$ and for all $d \in \mathcal{D}$ there is a finite set $M_d \subseteq\subseteq A$ such that for all $a \in A$ there is an element $m \in M_d$ such that $d(a) \le m \le a$. Evidently, for all $d \in \mathcal{D}$ the composition $i_A \circ d\colon A \to \Upsilon(A)$ is a preframe homomorphism. Then, by the universality of i_A, there is exactly one frame homomorphism $\widetilde{d}\colon \Upsilon(A) \to \Upsilon(A)$ such that $\widetilde{d} \circ i_A = i_A \circ d$. This gives us, for all $a \in A$, that $\widetilde{d}(i_A(a)) = i_A(d(a)) \le i_A(m) \le i_A(a)$. Since any element of $\Upsilon(A)$ is a join of elements of the form $i_A(a)$, $a \in A$, we have, putting $\widetilde{M_{\tilde{d}}}$ to be the join-subsemilattice generated by the set $i_A(M_d)$, that $\widetilde{d}$ is finitely separated from the identity and evidently $\mathrm{id}_{\Upsilon(A)} = \bigsqcup^{\uparrow} \widetilde{\mathcal{D}}$, here $\widetilde{\mathcal{D}} = \{\widetilde{d} : d \in \mathcal{D}\}$. □

Recall that, in [12], one defines a map $f\colon X \to Y$ between continuous posets to be a *Lawson map* if it is continuous, and, in addition, the map $f^{-1}\colon \sigma(Y) \to \sigma(X)$ preserves open filters, i.e., $\vee$-prime elements of $\sigma(X)$. So we shall say that a frame A is *Lawson* if $\mathrm{Sp}(A)$ is a meet-subsemilattice of A. A frame morphism between Lawson frames is said to be *Lawson* if it preserves $\vee$-prime elements. We shall say that a frame is a *Lawson LFS-frame* if it is an LFS-object in the subcategory of Lawson frames and Lawson frame morphisms.

PROPOSITION 6.6. *Let A be a Lawson LFS-frame. Then* $\mathrm{Sp}(A)$ *is an LFS-preframe and a subpreframe of A.*

P r o o f. Let A be a Lawson LFS-frame. Then evidently, for any Lawson map $f\colon A \to A$, we have that $g = f|_{\mathrm{Sp}(A)}\colon \mathrm{Sp}(A) \to \mathrm{Sp}(A)$ preserves directed suprema and finite infima. Moreover, for any finitely separated function d in A we can always choose the set $M_d \subseteq\subseteq A$ such that $M_d \cap \mathrm{Sp}(A)$ is a separating set in $\mathrm{Sp}(A)$. So we have that $\mathrm{Sp}(A)$ is an LFS-preframe. □

THEOREM 6.7. (**STONE DUALITY FOR LAWSON LFS-FRAMES**) *The category of LFS-preframes is equivalent to the category of Lawson LFS-frames.*

P r o o f. Apply 5.15 and 6.1. □

PROPOSITION 6.8.
(**STONE DUALITY FOR ALGEBRAIC LAWSON LFS-FRAMES**) *The category of algebraic LFS-preframes is equivalent to the category of algebraic Lawson LFS-frames.*

Proof. Apply 6.2 and the fact that a preframe is algebraic if and only if its lattice of Scott-open filters is algebraic. □

PROPOSITION 6.9. *Let A be an algebraic complete stably continuous preframe such that $F \subseteq \mathrm{K}(A)$ finite implies that the sublattice $\langle F\rangle_1 \subseteq \mathrm{K}(A)$ generated by $F \cup \{1\}$ is again finite. Then A is an algebraic LFS-preframe.*

Proof. Let $F \subseteq\subseteq \mathrm{K}(A)$. We define a map $d_F\colon A \to A$ as follows

$$d_F(a) = \bigsqcup\nolimits^{\uparrow}\{x \in \langle F\rangle_1 :\ x \le a\}$$

for all $a \in A$. Then evidently, $\bigsqcup\limits_{\alpha}^{\uparrow} d_F(a_\alpha) \le d_F\Big(\bigsqcup\limits_{\alpha}^{\uparrow} a_\alpha\Big)$. Now, let $x \in \langle F\rangle_1$, $x \le \bigsqcup\limits_{\alpha}^{\uparrow} a_\alpha$. Then there is α_0 such that $x \le a_{\alpha_0}$, i.e., $x \le \bigsqcup\limits_{\alpha}^{\uparrow} d_F(a_\alpha)$. So we have that d_F preserves directed suprema. We shall prove that d_F preserves finite infima. Evidently, $d_F(1) = 1$. We have $d_F(a) \wedge d_F(b) \ge d_F(a \wedge b)$. Assume that $x, y \in \langle F\rangle_1$, $x \le a$, $y \le b$. Then $x \wedge y \in \langle F\rangle_1$, $x \wedge y \le a \wedge b$, i.e., $x \wedge y \le d_F(a \wedge b)$. So we have $d_F(a) \wedge d_F(b) \le \bigsqcup^{\uparrow}\{x \in \langle F\rangle_1 :\ x \le a\} \wedge \bigsqcup^{\uparrow}\{y \in \langle F\rangle_1 :\ y \le b\} \le \bigsqcup^{\uparrow}\{x \wedge y \in \langle F\rangle_1 :\ x \wedge y \le a \wedge b\} \le d_F(a \wedge b)$.

Evidently, $d_F \cdot d_F = d_F$, $\operatorname{im} d_F = \langle F\rangle_1$ is finite, $d_F(a) \le a$ and $\bigsqcup\limits_{F \subseteq\subseteq \mathrm{K}(A)}^{\uparrow} d_F(a) = a$ by the algebraicity of A. So we have that A is an LFS-preframe. □

Similarly as in [8], we can prove an internal description of the algebraic LFS-preframes.

PROPOSITION 6.10. *A preframe A is an algebraic LFS-preframe if and only if $\mathrm{id}\, A = \bigsqcup^{\uparrow} \mathcal{D}$ for some directed set $\mathcal{D}$ in $\mathcal{P}re\mathcal{F}rm(A, A)$ such that $d^2 = d$ and $\operatorname{im} d$ is finite for all $d \in \mathcal{D}$.*

Proof. Such a preframe A is an LFS-preframe, for each $d \in \mathcal{D}$ is finitely separated from $\mathrm{id}\, A$ by its image; it is also algebraic with $\mathrm{K}(A) = \bigcup\limits_{d \in \mathcal{D}} \operatorname{im} d$.

Conversely, if A is an algebraic LFS-preframe, we are done if $k \in \mathrm{K}(A)$ is in the image of some $d^2 = d \le \mathrm{id}\, A$ in $\mathcal{P}re\mathcal{F}rm(A, A)$ such that $\operatorname{im} d$ is finite. As A is a LFS-preframe, we have $k \le f(k) \le k$ for some f separated from $\mathrm{id}\, A'$ in $\mathcal{P}re\mathcal{F}rm(A, A)$ by some finite set $M = \{m_1, \ldots, m_l\}$. Then, $d_f := f^{l+1}$ is in $\mathcal{P}re\mathcal{F}rm(A, A)$, idempotent and below f. Its image, $\operatorname{fix} f \subseteq M$ is finite and clearly contains k. □

Applying 6.9 and 6.10, we have

THEOREM 6.11. *Let A be a preframe. Then the following are equivalent*

(i) *A is an algebraic LFS-preframe.*

(ii) *A is an algebraic complete stably continuous preframe such that any sublattice of A generated by a finite subset of compact elements of A is finite.*

(iii) $\operatorname{id} A = \bigsqcup^{\uparrow} \mathcal{D}$ *for some directed set $\mathcal{D}$ in $\mathcal{PreFrm}(A, A)$ such that $d^2 = d$ and* $\operatorname{im} d$ *is finite for all $d \in \mathcal{D}$.*

Proof.

(i) $\iff$ (iii): By 6.10.

(ii) $\implies$ (i): By 6.9.

(iii) $\implies$ (ii): Let $S \subseteq\subseteq \mathrm{K}(A)$. Then there is $d \in \mathcal{D}$ such that $d^2 = d$, $d(s) = s$ for all $s \in S$, i.e., $S \subseteq \operatorname{im} d$. Let $a, b \in \operatorname{im} d$. Then $a, b \in \mathrm{K}(A)$ and $d(a) = a$, $d(b) = b$. This gives us $d(a \vee b) \leq a \vee b \leq d(a) \vee d(b) \leq d(a \vee b)$, i.e., $a \vee b \in \operatorname{im} d$. Similarly, $d(a \wedge b) = d(a) \wedge d(b) = a \wedge b$, i.e., $a \wedge b \in \operatorname{im} d$, i.e., $\operatorname{im} d$ is a finite sublattice of A. □

PROPOSITION 6.12. *Let A be a complete stably continuous preframe such that $F \subseteq A$ finite implies that the sublattice $\langle F \rangle \subseteq A$ generated by F is again finite. Then A is an LFS-preframe. Moreover, A is a retract of an algebraic LFS-preframe.*

Proof. We define a pair of maps $e\colon A \to \mathrm{Id}(A)$, $p\colon \mathrm{Id}(A) \to A$, $\mathrm{Id}(A)$ being the lattice of all ideals of A, as follows

$$e(a) = \downarrow a\,, \qquad p(a) = \bigsqcup^{\uparrow} I$$

for all $a \in A$ and all $I \in \mathrm{Id}(A)$. Then evidently, by stable continuity of A, $e(a) \cap e(b) = e(a \wedge b)$, $e(0) = \{0\}$, $e(1) = A$, and, by the interpolation property of $\ll$, $\bigsqcup^{\uparrow}_{\alpha} e(a_\alpha) = e\Big(\bigsqcup^{\uparrow}_{\alpha} a_\alpha\Big)$. We easily see that p preserves directed suprema and finite infima. Moreover, $p \cdot e = \mathrm{id}_A$, i.e., A is a preframe retraction of $\mathrm{Id}(A)$. Since compact elements of $\mathrm{Id}(A)$ are principal ideals, we have that $\mathrm{Id}(A)$ satisfies the condition (ii) of 6.11, i.e., $\mathrm{Id}(A)$ is an algebraic LFS-preframe. So we have that A is an LFS-preframe. □

Stone duality for abstract $\wedge$-semilattice bases.

DEFINITION 6.13. An *abstract $\wedge$-semilattice base* $(B, \prec, \wedge, 1)$ is given by a set B together with a $\wedge$-semilattice transitive relation $\prec$ on B and a $\wedge$-semilattice operation $\wedge$ on B and a largest element 1 with respect to the semilattice ordering, such that $\prec$ is an abstract base on B and the following conditions are satisfied:

$$\text{(SLAT1)} \qquad y \prec N \implies y \prec \bigwedge N\,,$$

(SLAT2) $\leq \circ \prec \circ \leq \subseteq \prec \subseteq \leq$ and $\downarrow\colon B \to 2^B$ is an injective mapping.
(SLAT3) $(B, \wedge, 1)$ is a $\wedge$-semilattice,

holds for all elements y and finite subsets N of B.

DEFINITION 6.14. An approximable relation R between abstract $\wedge$-semilattice bases B and C is called $\wedge$*-stable approximable relation* if the following condition is satisfied:

$$\forall y{\in}C\ \ \forall N{\subseteq\subseteq}B\ \ \big(\forall x{\in}N\ \ xRy \implies \text{there is an element } w \in B \text{ such that } \textstyle\bigwedge N \succ wRy\big)\,.$$

Evidently, any directed union of stable approximable relations is a stable approximable relation. Similarly as in [1], for domain prelocales, we shall show that stable prelocales and stable approximable relations form a category.

PROPOSITION 6.15. *$\wedge$-stable approximable relations are closed under composition.*

Proof. It is evident. □

Similarly as for stable prelocales we may define a category $\wedge - \mathcal{A}sb$ of abstract $\wedge$-semilattice bases and $\wedge$-stable approximable relations.

PROPOSITION 6.16. *Let B be an abstract $\wedge$-semilattice base. Then, if we put $\mathcal{I}d(B)$ to be the set of all directed lower Scott-closed subsets the Scott-closed ideals, $\mathcal{I}d(B)$ is a stably continuous preframe.*

Proof. Evidently, a finite intersection of Scott-closed ideals is a Scott-closed ideal, and a directed union of Scott-closed ideals is a Scott-closed ideal. Moreover, for a Scott-closed ideal I, $a \in I$ if and only if $\downarrow a \ll I$. This gives us that $\mathcal{I}d(B)$ is a stably continuous preframe. □

PROPOSITION 6.17. *Let L be a stable continuous preframe. Then, if we put $\mathcal{B}(L) = B$ to be the $\wedge$-semilattice L, $\mathcal{B}(L) = (B, \ll)$ is an abstract $\wedge$-semilattice.*

Proof. Similar to 4.13. □

THEOREM 6.18. *The category $\wedge - \mathcal{A}sb$ of abstract $\wedge$-semilattices and $\wedge$-stable approximable relations is equivalent to* **STCONTPREF**, *the category of stably continuous preframes and preframe morphisms.*

Proof. The idea of the proof follows the proof of 4.14. □

COROLLARY 6.19. *The category of reflexive abstract $\wedge$-semilattices and $\wedge$-stable approximable relations is equivalent to* **STALGPREF**, *the category of stably continuous algebraic preframes and preframe morphisms.*

DEFINITION 6.20. A $\wedge$-stable approximable relation $R \subseteq \succ_B$ on an abstract $\wedge$-semilattice B is said to be *finitely separated from the identity* if there is a finite subset $M \subseteq\subseteq B$ such that aRb implies that we can find an element $m \in M$ such that $a \succ m \succ b$.

An abstract $\wedge$-semilattice B is said to be an *LFS-$\wedge$-semilattice* if $\succ_B$ is a directed union of $\wedge$-stable approximable relations finitely separated from the identity.

THEOREM 6.21. *The category of LFS-$\wedge$-semilattices and $\wedge$-stable approximable relations is equivalent to the category of LFS-preframes and preframe morphisms.*

Proof. It follows immediately from the definitions and 6.18. □

COROLLARY 6.22. *The category of reflexive LFS-$\wedge$-semilattices and $\wedge$-stable approximable relations is equivalent to the category of algebraic LFS-preframes and preframe morphisms.*

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