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EQUIVARIANT COHOMOLOGY WITH LOCAL COEFFICIENTS

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ABSTRACT. For a topological category $\mathcal{D}$ and a $\mathcal{D}$-space we construct $\mathcal{D}$-cohomology groups with local coefficients. The main result (Theorem 3.3) presents some equivariant conditions describing any weak $\mathcal{D}$-homotopy equivalence. The representability result for $\mathcal{D}$-cohomology (of $\mathcal{D}$-complexes) is developed and applications to G-spaces, where G is a locally compact topological group, are given.

0. Introduction

Let G be a locally compact topological group. For a G-map $f: X \to Y$, an equivariant version of the Whitehead theorem has been proved ([4], [6]). By [8], a weak G-homotopy equivalence leads to an isomorphism of the Illman ([5], [7]) cohomology groups. In [3], by means of the Postnikov tower for nilpotent G-connected spaces a partial inverse of this is established. The aim of this paper is to extend that result to arbitrary G-spaces and to give some equivalent algebraic conditions describing any weak G-homotopy equivalence.

Let $\mathcal{D}$ be a topological small category and CGH the category of compactly generated Hausdorff spaces and continuous maps. The key example of such a category is the category $\mathcal{O}_G$ of orbit types determined by a topological group G in CGH. Put $\mathcal{F}(\mathcal{D})$ for the functor category whose objects are continuous contravariant functors and whose morphisms are natural transformations between them; its objects are called $\mathcal{D}$-spaces. By [13], the cellular approximation and J. H. C. Whitehead's theorems hold in that category. The obstruction theory (cf. [10]), as well as the ordinary homotopy theory, may be also developed in that category.

In Section 2, we construct $\mathcal{D}$-cohomology groups of a $\mathcal{D}$-space with local coefficient (cf. [9], [10]) and give some equivalent descriptions for $\mathcal{D}$-complexes.

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Then, in Section 3, we present the main result (Theorem 3.3) which gives some equivalent algebraic conditions for a weak $\mathcal{D}$-homotopy equivalence. By means of obstruction theory, we develop the representability result (Corollary 3.6) for $\mathcal{D}$-cohomology of $\mathcal{D}$-complexes. Then we apply our results to G-spaces.

1. Preliminaries

Throughout this note, we are concerned with the category CGH (resp. CGH_*) of compactly generated Hausdorff (resp. pointed) spaces. The product $S \times T$ and the function space T^S in CGH (resp. CGH_*) are given by the "k-ification" of the usual product topology and the compact-open topology.

DEFINITION 1.1.

(1) We call a category $\mathcal{D}$ a *topological category* if an arbitrary set of morphisms $\mathcal{D}(a,b)$ is a CGH-space and the composition $\mathcal{D}(a,b) \times \mathcal{D}(b,c) \to \mathcal{D}(a,c)$ is continuous for arbitrary objects a, b, c in $\mathcal{D}$. A contravariant functor $F\colon \mathcal{D} \to \mathcal{C}$ between topological categories is said to be *continuous* if the induced map $F_*\colon \mathcal{D}(a,b) \to \mathcal{C}(F(b), F(a))$ is continuous for $a, b \in \mathcal{D}$.

(2) We always consider CGH as a topological category with $CGH(S,T) = T^S$. Hereafter we fix a topological small category $\mathcal{D}$, and consider the functor category

$$\mathcal{F} = \mathcal{F}(\mathcal{D}) = \text{Cont Funct}(\mathcal{D}, CGH)$$

whose objects (called *functor spaces over $\mathcal{D}$* or simply *$\mathcal{D}$-spaces*) are continuous contravariant functors $X\colon \mathcal{D} \to CGH$ and whose morphisms (called *$\mathcal{D}$-morphisms*) are natural transformations between them.

(3) Any object c in $\mathcal{D}$ gives a continuous functor $D_c\colon \mathcal{D} \to CGH$ by $D_c(a) = \mathcal{D}(a,c)$ and $D_c(f) = f^*\colon \mathcal{D}(b,c) \to \mathcal{D}(a,c)$ for any a and $f\colon a \to b$ in $\mathcal{D}$; and we call this $\mathcal{D}$-space D_c the *$\mathcal{D}$-orbit* of c. We regard any CGH-space T as the $\mathcal{D}$-space T with $T(a) = T$ and $T(f) = \text{id}: T \to T$.

(4) Consider the category $[n]$ of ordered objects $0 < 1 < \cdots < n$ whose morphisms consist of the unique $i \to j$ for $i < j$.

The category $\mathcal{F}_{n+1}$ of $(n+1)$-ples $X \leftarrow A_1 \leftarrow \cdots \leftarrow A_n$ of *$\mathcal{D}$-spaces* is considered as $\mathcal{F}_{n+1} = \text{Funct}([n], \mathcal{F}) = \mathcal{F}(\mathcal{D} \times [n])$ over the product category $\mathcal{D} \times [n]$ (with the usual topology). For $n = 1$, we call $\mathcal{F}_2$ the *category of pairs of $\mathcal{D}$-spaces.* We call $A \subset X$ a *relative pair* if $A(a)$ is a closed subspace of $X(a)$ for any a in $\mathcal{D}$.

The category $\mathcal{F}$ is closed under (co)limits and becomes a homotopy category by the cylinder and path functors obtained by composing those functions for CGH (resp. CGH_*). Moreover, for an arbitrary $\mathcal{D}$-space X and a constant functor C the natural isomorphisms

$$\mathcal{D}(D_a \times C, X) = CGH(C, X(a)), \qquad \mathcal{D}[D_a \times C, X] = CGH[C, X(a)]$$

are determined by the Yoneda Lemma ($[-,-]$ is the $\mathcal{D}$-homotopy set).

Following [11], [13], we now introduce the notion of functor complexes as follows, where $\emptyset$ is the empty space, B^n is the n-ball and $S^{n-1} = \dot{B}^n$ the $(n-1)$-sphere ($B^0 = *$, $S^{-1} = \emptyset$) in CGH.

DEFINITION 1.2.

(1) We call K (resp. (K, L)) a *functor complex over $\mathcal{D}$* or a *$\mathcal{D}$-complex* (resp. *relative one*) if $K = \varinjlim K^n$ in $\mathcal{D}$ for $\mathcal{D}$-spaces K^n with $K^{-1} = \emptyset$ (resp. L) and $j_n \colon K^{n-1} \to K^n$, which are constructed inductively by the pushout diagrams

$$\begin{array}{ccc} \coprod_i D_{a_i} \times S^{n-1} & \longrightarrow & \coprod_i D_{a_i} \times B^n \\ {\scriptstyle \coprod_i f_i} \downarrow & & \downarrow \\ K^{n-1} & \xrightarrow{j_n} & K^n \end{array}$$

in $\mathcal{F}$ for some object a_i in $\mathcal{D}$, $\mathcal{D}$-morphisms f_i and the upper $\mathcal{D}$-morphism induced by $S^{n-1} \subset B^n$. Here $\coprod$ stands for the direct sum in $\mathcal{F}$ and $D_c \times T$ is the product of the $\mathcal{D}$-orbit D_c and a CGH-space T, i.e., $(D_c \times T)(a) = \mathcal{D}(a, c) \times T$ and $(D_c \times T)(f) = \mathcal{D}(f, c) \times T$. We call K^n the *n-skeleton*, $D_{a_i} \times B^n$ the *n-cells attached by f_i*, and the maximum integer of n's (appearing in the above diagram) the *dimension* of K (resp. (K, L)).

(2) A *pair (K_0, K_1) of $\mathcal{D}$-complexes*, or a *$\mathcal{D}$-subcomplex* K_1 of a $\mathcal{D}$-complex K_0 is defined to be a functor complex K over $\mathcal{D} \times [1]$. Then K_1 consists of n-cells $D_{(a_i,1)} \times B^n$, and K_0 of these together with $D_{(a_i,0)} \times B^n$.

EXAMPLE 1.3. Let G be a topological group in CGH, and consider the category CGH_G of G-spaces. The quotient spaces of left cosets in CGH and G-maps between them form the category $\mathcal{O}_G$ of orbit types. Then $\mathcal{O}_G$ becomes a small topological category according to the natural isomorphism $\mathcal{O}_G(G/H, G/K) = (G/K)^H$. Any G-space X can be identified with the $\mathcal{F}(\mathcal{O}_G)$-space $\bar{X} \colon \mathcal{O}_G \to CGH$ given by $\bar{X}(G/H) = X^H$ (the invariant subspace of X). A G-CW complex X is defined by a colimit $\varinjlim X^n$ by taking $\coprod_j G/H_j \times S^n \to \coprod_j G/H_j \times B^n$ in place of the upper morphisms in our diagram. The above correspondences and definitions induce the one-to-one correspondence between $\mathcal{F}(\mathcal{O}_G)$-complexes and G-CW complexes.

We list some of the basic properties of $\mathcal{D}$-complexes. Remark that their proofs are completely the same as in the case of G-CW complexes ([7], [14]). Here (X, A) (resp. X) is *N-connected* if $\pi_n(X(a), A(a), x) = 0$ (resp. $\pi_n(X(a), x) = 0$) for any $a \in \mathcal{D}$, $x \in A(a)$ (resp. $X(a)$) and $0 \leq n \leq N$.

PROPOSITION 1.4. (J. H. C. WHITEHEAD THEOREM) *Let $f\colon (X,A) \to (Y,B)$ be a $\mathcal{D}$-morphism between pairs of $\mathcal{D}$-complexes. Suppose that f is a weak $\mathcal{D}$-homotopy equivalence, that is, the induced maps*

$$f_*\colon \pi_n\big(X(a),x\big) \to \pi_n\big(Y(a),f(x)\big) \quad \text{and} \quad f_*\colon \pi_n\big(A(a),x\big) \to \pi_n\big(B(a),f(x)\big)$$

are bijective for $n \le \max(\dim X, \dim Y)$, any $a \in \mathcal{D}$ and $x \in A(a)$, then f is a homotopy equivalence in $\mathcal{F}_2$.

PROPOSITION 1.5. ($\mathcal{D}$-CW APPROXIMATION THEOREM) *Let (X,A) be a pair of $\mathcal{D}$-spaces. Then there is a pair of $\mathcal{D}$-complexes $(\Gamma X, \Gamma A)$ and a $\mathcal{D}$-map $\rho_{(X,A)}\colon (\Gamma X, \Gamma A) \to (X,A)$ satisfying the following properties.*

(1) *$\rho_{(X,A)}$ is a weak $\mathcal{D}$-homotopy equivalence.*
(2) *For an arbitrary $\mathcal{D}$-map $f\colon (X,A) \to (Y,B)$, a canonical $\mathcal{D}$-cellular map $\Gamma(f)\colon (\Gamma X, \Gamma A) \to (\Gamma Y, \Gamma B)$ is defined and $f \circ \rho_{(X,A)} = \rho_{(Y,B)} \circ K(f)$. That is, Γ is a functor from the category of $\mathcal{D}$-spaces and $\mathcal{D}$-maps to the category of $\mathcal{D}$-complexes and $\mathcal{D}$-cellular maps, and ρ is a natural transformation.*

Our cohomology theory requires a certain amount of categorical machinery which we now introduce. Let X be a $\mathcal{D}$-space.

The fundamental category (cf. [1]) $\pi(\mathcal{D},X)$ of X is defined by:

(i) Objects are $\mathcal{D}$-maps $x_a\colon D_a \to X$ for all $a \in \mathcal{D}$. Sometimes we write x for brevity.
(ii) A morphism $(\sigma,[\omega])\colon x_a \to y_b$ consists of a map $\sigma\colon a \to b$ in $\mathcal{D}$ and a homotopy class $[\omega]$ relative to $D_a \times \partial I$ of $\mathcal{D}$-maps $\omega\colon D_a \times I \to X$ with $\omega_1 = x_a$ and $\omega_0 = y_b \circ D_\sigma$. We often abbreviate $(\sigma,[\omega])$ by (σ,ω).

For an object $x\colon D_a \to X$, define $X_x(a)$ to be the *component* of $X(a)$ determined by x. Assume that the components $X_x(a)$ have suitable local properties ensuring that the universal coverings $p_x(a)\colon \widetilde{X_x}(a) \to X_x(a)$ can be defined in the standard manner using equivalence classes of paths.

The universal covering of X is the contravariant functor

$$\widetilde{X}\colon \pi(\mathcal{D},X) \to CGH$$

sending x_a to $X_x(a)$ and $(\sigma,\omega)\colon x_a \to y_b$ to $\widetilde{X}(\sigma,\omega)\colon \widetilde{X_x}(a) \to \widetilde{X_y}(b)$.

Objects of the *discrete fundamental category* $\pi/(\mathcal{D},X)$ are again $\mathcal{D}$-maps $x\colon D_a \to X$. In contrast to the previously defined category, we now identify suitably homotopic maps $\sigma\colon a \to b$. Let $(\sigma_i,\omega_i)\colon x \to y$ be two morphisms in $\pi(\mathcal{D},X)$ $(i = 0,1)$. They are called equivalent (i.e., $(\sigma_0,\omega_0) \sim (\sigma_1,\omega_1)$) if there exists a path $\tau\colon I \to \mathcal{D}(a,b)$ between σ_0 and σ_1 and a $\mathcal{D}$-homotopy

$\Lambda\colon D_a \times I \times I \to X$ such that:

$$\begin{aligned} \Lambda(-,0,t) &= x(-)\,, \\ \Lambda(-,1,t) &= y \circ D_{\tau(t)}(-)\,, \\ \Lambda(-,s,i) &= \omega_i(-,s)\,, \quad i = 0,1\,. \end{aligned}$$

A local contravariant coefficient system on a $\mathcal{D}$-space X is a contravariant functor

$$M\colon \pi/(\mathcal{D},X) \to \mathbf{Ab}$$

to the category $\mathbf{Ab}$ of abelian groups. A $\mathcal{D}$-map $f\colon X \to Y$ induces a functor $\pi/(\mathcal{D},f)\colon \pi/(\mathcal{D},X) \to \pi/(\mathcal{D},Y)$ between the discrete fundamental groupoids, and for a local coefficient system M on Y the local coefficient system $f^*M = M \circ \pi/(\mathcal{D},f)$ on X is called the *pull-back* of M. Put $\mathbf{Z}\pi/(\mathcal{D},X)$ for the local contravariant coefficient (or covariant) system determined by the free category on $\pi/(\mathcal{D},X)$.

2. Cohomology for $\mathcal{D}$-spaces

We may now proceed to describe the equivariant form of the local cohomology of Steenrod (cf. [5]). Let Δ_n be the standard n-simplex. A $\mathcal{D}$-map $T\colon \Delta_n \times D_a \to X$ is called a *$\mathcal{D}$-singular n-simplex* in a $\mathcal{D}$-space X. Put $T_{e_0}\colon D_a \to X$ for the induced $\mathcal{D}$-map determined by the first vertex e_0 of Δ_n. For a $\mathcal{D}$-space X with a local contravariant coefficient system M we define

$$\hat{C}^n_{\mathcal{D}}(X,M) = \prod_{T\colon \Delta_n \times D_a \to X} M(T_{e_0}) \qquad \text{for} \quad n \geq 0\,.$$

The coboundary homomorphism

$$\hat{\delta}^n\colon \hat{C}^n_{\mathcal{D}}(X,M) \to \hat{C}^{n+1}_{\mathcal{D}}(X,M)$$

is defined in the usual way. Thus we get the cochain complex

$$\hat{S}_{\mathcal{D}}(X,M) = \{\hat{C}^n_{\mathcal{D}}(X,M),\, \hat{\delta}^n\}\,.$$

Our main interest is not in the cochain complex $\hat{S}_{\mathcal{D}}(X,M)$, but in a suitable subcomplex of it.

Let

$$h\colon \Delta_n \times D_a \to \Delta_n \times D_{a'}$$

be a $\mathcal{D}$-map which covers $\mathrm{id}\colon \Delta_n \to \Delta_n$ and $T\colon \Delta_n \times D_a \to X$, $T'\colon \Delta_n \times D_{a'} \to X$ $\mathcal{D}$-singular n-simplexes in a $\mathcal{D}$-space X such that $T = T' \circ h$. Every $t \in \Delta_n$ gives rise to $\mathcal{D}$-maps $h_t\colon D_a \to D_{a'}$ and $T_t\colon D_a \to X$, where $h_t(-) = \mathrm{pr}_2 \circ h(t,-)$, and

$\mathrm{pr}_2\colon \Delta_n \times D_{a'} \to D_{a'}$ is the projection onto the second factor. Put $\sigma_t\colon a \to a'$ for the map in $\mathcal{D}$ induced by h_t and $\omega_t\colon D_a \times I \to X$ for the constant $\mathcal{D}$-homotopy determined by $T_t = T'_t \circ h_t$. If $t, t' \in \Delta_n$, then, by [4], the maps $h_t, h_{t'}\colon D_a \to D_{a'}$ are homotopic. Therefore $(\sigma_t, \omega_t) \sim (\sigma_{t'}, \omega_{t'})$, and we may consider the induced map (σ_*, ω_*) in $\pi/(\mathcal{D}, X)$.

Define $C^n_{\mathcal{D}}(X, M)$ to be the subcomplex of $\hat{C}^n_{\mathcal{D}}(X, M)$ consisting of all $c \in \hat{C}^n_{\mathcal{D}}(X, M)$ which satisfy the following condition. If $T\colon \Delta_n \times D_a \to X$ and $T'\colon \Delta_n \times D_{a'} \to X$ are $\mathcal{D}$-singular n-simplexes in X, and $h\colon \Delta_n \times D_a \to \Delta_n \times D_{a'}$ is a $\mathcal{D}$-map which covers $\mathrm{id}\colon \Delta_n \to \Delta_n$ and $T = T' \circ h$, then

$$c_T = M(\sigma_*, \omega_*) c_{T'} \in M(T_{e_0}).$$

The coboundary $\hat{\delta}^{n-1}\colon \hat{C}^{n-1}_{\mathcal{D}}(X, M) \to \hat{C}^n_{\mathcal{D}}(X, M)$ restricts to

$$\delta^{n-1}\colon C^{n-1}_{\mathcal{D}}(X, M) \to C^n_{\mathcal{D}}(X, M).$$

Thus we have the cochain complex

$$S_{\mathcal{D}}(X, M) = \{C^n_{\mathcal{D}}(X, M), \delta^n\}$$

that gives the *$\mathcal{D}$-singular cohomology groups* $H^*_{\mathcal{D}}(X, M)$ of X with coefficients M.

Let (X, A) be a $\mathcal{D}$-pair. The inclusion $i\colon A \to X$ induces the short exact sequence

$$0 \to S_{\mathcal{D}}(X, A; M) \to S_{\mathcal{D}}(X, M) \xrightarrow{i^\#} S_{\mathcal{D}}(A, i^*M) \to 0,$$

where $S_{\mathcal{D}}(X, A; M) = \ker i^\#$. We now define

$$H^n_{\mathcal{D}}(X, A; M)$$

to be the nth cohomology module of the cochain complex $S_{\mathcal{D}}(X, A; M)$ for $n \geq 0$.

It is standard to check that the cohomology theory $H^*_{\mathcal{D}}(-\,; M)$ satisfies all seven Eilenberg-Steenrod axioms (cf. [16]). We omit the obvious details which may be seen without any extra effort. Among other obvious facts, we also mention that cohomology theory is additive, and we have the Mayer-Vietoris exact sequence in it. The notion of cup product generalizes to our cohomology with coefficients in a commutative ring coefficient system, making it a graded ring (cf. [5], [16]). Dually, we may construct the nth homology module $H^{\mathcal{D}}_n(X, A; M)$ for a covariant coefficient system M and $n \geq 0$.

Let $\rho = \rho_{(X,A)}\colon (\Gamma X, \Gamma A) \to (X, A)$ be the weak homotopy equivalence given by Proposition 1.5, and M a local coefficient system on X. Then, from the proof of Theorem 2* in [8], we may deduce the following:

PROPOSITION 2.1. *The induced homomorphism*

$$\rho^*: H^*_{\mathcal{D}}(X,A;M) \to H^*_{\mathcal{D}}(\Gamma X, \Gamma A; \rho^* M)$$

on the singular $\mathcal{D}$-cohomology is an isomorphism.

We first give a cellular description of the cohomology. Let (X,A) be a relative $\mathcal{D}$-complex, and M a local contravariant coefficient system on X. Then we can translate all the results listed in [16; pp. 286–287] into our context. Thus, if X^n is the n-skeleton of (X,A), we have

$$H^q_{\mathcal{D}}(X^n, X^{n-1}; M) = 0 \qquad \text{if} \quad q \neq n.$$

Next the exact sequences of the pairs (X^p, X^{p-1}) give rise to a spectral sequence whose limit term is the bigraded module associated with the filtration

$$0 = J^{n+1,-1} \subseteq \cdots \subseteq J^{p,n-p} \subseteq \cdots \subseteq J^{0,n} = H^n_{\mathcal{D}}(X,A;M),$$

where $J^{p,n-p} = \ker\left[H^n_{\mathcal{D}}(X,A;M) \to H^n_{\mathcal{D}}(X^{p-1},A;M)\right]$. It follows by a standard argument that if $\mathcal{C} = \{C^p, \delta^p\}$ is the cochain complex whose $C^p = H^p_{\mathcal{D}}(X^p, X^{p-1}; M)$ and $\delta^p: C^p \to C^{p+1}$ is the coboundary operator of the triple (X^{p+1}, X^p, X^{p-1}), then

$$H^*_{\mathcal{D}}(X,A;M) = H^*(\mathcal{C}).$$

Then we may consider the following alternative description of our cohomology on the category of relative $\mathcal{D}$-complexes. For a relative $\mathcal{D}$-complex (X,A) and a local coefficient system M on X let $\widetilde{M}$ denote the induced functor on the fundamental category of X. Then any map $\sigma: a \to b$ in the category $\mathcal{D}$ determines a natural transformation $\widetilde{M}(\sigma): \widetilde{M}(b) \to \sigma^* \widetilde{M}(a)$. For every $a \in \mathcal{D}$, the pair $\big(X(a), A(a)\big)$ is an ordinary CW-pair, and $\widetilde{M}(a)$ is the restriction of $\widetilde{M}$ to the fundamental groupoid of $X(a)$. We have a complex $\mathcal{C}_{\mathcal{D}}(X,A;M)$ with

$$C^n_{\mathcal{D}}(X,A;M) = \prod_{a \in \mathcal{D}} H^n\big(X^n(a), X^{n-1}(a); \widetilde{M}(a)\big), \qquad \delta^n = \prod_{a \in \mathcal{D}} \delta^n(a),$$

where $\delta^n(a)$ is the coboundary operator of the triple $\big(X^{n+1}(a), X^n(a), X^{n-1}(a)\big)$. Now we define a *cellular cochain complex* as a subcomplex $\mathcal{C}_{\mathcal{D},c}(X,A;M)$ of $\mathcal{C}_{\mathcal{D}}(X,A;M)$. First recall that $H^n\big(X^n(a), X^{n-1}(a); \widetilde{M}(a)\big)$ can be identified with the group ([16; p. 287]) of all functions c which to each n-cell $\big(B^n \xrightarrow{\eta} X_n(a)\big)$ assigns an element $c(\eta) \in \widetilde{M}(a)(z_\eta)$, where $z_\eta = \eta(e_0)$ is the image of base point $e_0 \in B^n$. Then define $c \in \mathcal{C}_{\mathcal{D},c}(X,A;M)$ if and only if for every map $\sigma: a \to b$ in $\mathcal{D}$ and every n-cell $\eta: B^n \to X(b)$ we have

$$M(\sigma)\big(\eta(e_0)\big)\big(c(\sigma^*\eta)\big) = c(\eta).$$

The cochain complex $\mathcal{C}_{\mathcal{D},c}(X,A;M)$ gives rise to the *cellular cohomology* $H^*_{\mathcal{D},c}(X,A;M) = H^*(\mathcal{C}_{\mathcal{D},c}(X,A;M))$.

Remark 2.2. The cochain complex $\mathcal{C}_{\mathcal{D},c}(X,A;M)$ can be related to the universal covering $p\colon \widetilde{X} \to X$ (cf. [16]). Put $\widetilde{A} = p^{-1}(A)$ and consider a contravariant functor $\mathcal{C}(\widetilde{X},\widetilde{A})\colon \pi(\mathcal{D},X) \to \mathbf{Ab}$ such that $\mathcal{C}(\widetilde{X},\widetilde{A})(a) = \mathcal{C}(\widetilde{X}(a),\widetilde{A}(a))$. Then for any contravariant coefficient system on X, there is an isomorphism of cochain complexes

$$\operatorname{Hom}(\mathcal{C}(\widetilde{X},\widetilde{A}),M) \xrightarrow{\cong} \mathcal{C}_{\mathcal{D},c}(X,A;M).$$

3. Applications

Let (X,A) be a relative $\mathcal{D}$-pair and M a contravariant functor on fundamental category $\pi(\mathcal{D},X)$ of X. For an object $x\colon D_a \to X$ of $\pi(\mathcal{D},X)$ consider the functor M_x on $\pi(\mathcal{D},D_a)$ such that $M_x(\phi) = M(\phi \circ x)$ for an object $\phi\colon D_b \to D_a$ of $\pi(\mathcal{D},D_a)$. Then for any $q \geq 0$ we define a functor $\mathcal{H}^q$ on $\pi(\mathcal{D},X)$ such that $\mathcal{H}^q(x) = H^q_{\mathcal{D}}(D_a,M_x)$ for an object $x\colon D_a \to X$ in $\pi(\mathcal{D},X)$. Let $\rho_{(X,A)}\colon (\Gamma X,\Gamma A) \to (X,A)$ be the weak $\mathcal{D}$-equivalence given by Proposition 1.5. Now we can formulate the following proposition.

PROPOSITION 3.1. *For a relative $\mathcal{D}$-complex (X,A) the singular $\mathcal{D}$-cohomology of (X,A) coincides with the cellular $\mathcal{D}$-cohomology of $(\Gamma X,\Gamma A)$.*

The proof follows from Proposition 2.1 and the following generalization of a result in [8].

LEMMA 3.2. *Let (X,A) be a pair of $\mathcal{D}$-complexes and M a local coefficient system on X. Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^p_{\mathcal{D},c}(X,A;\widetilde{\mathcal{H}}^q) \implies H^{p+q}_{\mathcal{D}}(X,A;M),$$

where $\widetilde{\mathcal{H}}^q$ is the functor on $\pi(\mathcal{D},X)$ defined above and determined by $\widetilde{M}$.

Now we can state the following main theorem (cf. [12]).

THEOREM 3.3. *For a $\mathcal{D}$-map $f\colon X \to Y$ the following conditions are equivalent:*

(i) *the $\mathcal{D}$-map $f\colon X \to Y$ is a weak $\mathcal{D}$-homotopy equivalence;*
(ii) *the induced map of $\mathcal{D}$-homotopy classes $f_{\#}\colon [W,X] \to [W,Y]$ is a bijection for an arbitrary $\mathcal{D}$-complex W;*
(iii) *the induced map $\pi/(\mathcal{D},f)\colon \pi/(\mathcal{D},X) \to \pi/(\mathcal{D},Y)$ is an equivalence of categories, and for any local coefficient system M on Y the induced map $f^*\colon H^*_{\mathcal{D}}(Y,M) \to H^*_{\mathcal{D}}(X,f^*M)$ is an isomorphism;*

(iv) *the induced map* $\pi/(\mathcal{D}, f)\colon \pi/(\mathcal{D}, X) \to \pi/(\mathcal{D}, Y)$ *is an equivalence of categories, and the induced map*

$$f_*\colon H_*^{\mathcal{D}}(X, f_*\mathbf{Z}\pi/(\mathcal{D}, Y)) \to H_*^{\mathcal{D}}(Y, \mathbf{Z}\pi/(\mathcal{D}, Y))$$

is an isomorphism;

(v) *the map* $\pi(f(a))\colon \pi(X(a)) \to \pi(Y(a))$ *of the fundamental groupoids is an equivalence, and the induced map*

$$f(a)^*\colon H^*(Y(a), M(a)) \to H^*(X(a), f(a)^*M(a))$$

is an isomorphism for any $a \in \mathcal{D}$ *and for a local coefficient system* $M(a)$ *on* $Y(a)$*;*

(vi) *the map* $\pi(f(a))\colon \pi(X(a)) \to \pi(Y(a))$ *of fundamental groupoids is an equivalence, and the induced map*

$$f(a)_*\colon H_*(X(a), f(a)_*\mathbf{Z}\pi Y(a)) \to H_*(Y(a), \mathbf{Z}\pi Y(a))$$

is an isomorphism for any $a \in \mathcal{D}$.

Proof. The equivalence (i) $\iff$ (ii) follows from [4], [6], (i) $\iff$ (v) $\iff$ (vi) is the well-known classical fact, and the implications (i) $\implies$ (iii) and (i) $\implies$ (iv) are determined by Proposition 2.1 and its dual, respectively.

To show the implication (iv) $\implies$ (iii), we follow [15] to construct a first quadrant spectral sequence (being a version of the Universal Coefficient Theorem) for which

$$E_2^{p,q} = \operatorname{Ext}^p(H_q^{\mathcal{D}}(X, \mathbf{Z}\pi/(\mathcal{D}, X), M) \implies H_{\mathcal{D}}^{p+q}(X, M).$$

For the proof of the implications (iii) $\implies$ (v) and (iv) $\implies$ (vi), by Propositions 2.1 and 3.1, we may assume that X and Y are $\mathcal{D}$-complexes, and then we use the spectral sequence considered in [9]. □

The general reference to obstruction theory is [16]. Let a $\mathcal{D}$-map $p\colon E \to B$ be a $\mathcal{D}$-*fibration* (i.e., for each $a \in \mathcal{D}$ the map $p(a)\colon E(a) \to B(a)$ is a Serre fibration) and (X, A) a relative $\mathcal{D}$-complex. Suppose $\phi\colon X \to B$ and $f\colon A \to E$ be $\mathcal{D}$-maps, so that $p \circ f = \phi|_A$, that is, f is a $\mathcal{D}$-partial lifting of ϕ. Then the lifting problem is to find a $\mathcal{D}$-map $\psi\colon X \to E$ such that $p \circ \psi = \phi$ and $\psi|_A = f$.

Let X^n be the n-skeleton of (X, A), and $\psi\colon X^n \to E$ a partial $\mathcal{D}$-lifting of ϕ for $n \geq 1$. Then it is routine to define an obstruction to extending ψ as

$$c^{n+1}(\psi) \in C_{\mathcal{D},c}^{n+1}(X, A; \psi^*\pi_n(p)),$$

where $\pi_n(p)$ is a functor on the fundamental category of B determined by the fibration $p\colon E \to B$.

Next let $\psi_0, \psi_1\colon X^n \to E$ be $\mathcal{D}$-liftings of ϕ, and let $\lambda\colon I \times X^{n-1} \to E$ be a vertical $\mathcal{F}$-homotopy rel A between $\psi_0|_{X^{n-1}}$ and $\psi_1|_{X^{n-1}}$. These maps fit together to give a partial lifting $\mu\colon \dot{I} \times X^n \cup I \times X^{n-1} \to E$ of $\phi \circ \pi$, where $\pi\colon I \times X \to X$ is the projection. Then the primary cohomology difference $\delta^n(\psi_0, \psi_1) \in H^n_{\mathcal{D},c}(X, A; \phi^*\pi_n(p))$ can be defined for the partial liftings $\psi_0, \psi_1\colon X \to E$ of ϕ.

All properties of classical obstruction theory ([16]) transform to the $\mathcal{D}$-case in a natural way leading to the following classification theorem.

THEOREM 3.4. *With the same notations as above, suppose that*

(i) *the fibre of each $p(a)\colon E(a) \to B(a)$ is q-simple for $n+1 \le q < \dim(X, A)$;*
(ii) *$H^q_{\mathcal{D},c}(X, A; \phi^*\pi_n(p)) = 0$ for $n+1 \le q < 1 + \dim(X, A)$;*
(iii) *$H^{q+1}_{\mathcal{D},c}(X, A; \phi^*\pi_n(p)) = 0$ for $n+1 \le q < \dim(X, A)$, and the $\psi_0\colon X \to E$ is an $\mathcal{F}$-lifting of $\phi\colon X \to B$.*

Then the correspondence $\psi \mapsto \delta^n(\psi_0, \psi)$ is a bijection between the set of vertical $\mathcal{D}$-homotopy classes $(\operatorname{rel} A)$ of $\mathcal{D}$-liftings of ϕ which agree with ψ_0 on A and the group $H^n_{\mathcal{D},c}(X, A; \phi^*\pi_n(p))$.

Recall that for any topological space X, a local coefficient system M and an integer n there is a sectioned $\mathcal{D}$-fibration

$$p\colon L(M, n) \to B\pi(X)$$

(i.e., with a section $s\colon B\pi(X) \to L(M, n)$), where $B\pi(X)$ is the classifying space of the fundamental groupoid $\pi(X)$. Now let X be a $\mathcal{D}$-space and $K = B\pi(\mathcal{D}, X)$ the $\mathcal{D}$-space such that $K(a) = B\pi(X(a))$ for $a \in \mathcal{D}$. Then the above construction may be easily transformed to the $\mathcal{D}$-case.

LEMMA 3.5. *Let X be a $\mathcal{D}$-space, and M a functor on $\pi(\mathcal{D}, X)$ to the category of abelian groups. Then there is a sectioned $\mathcal{D}$-fibration*

$$p\colon L(M, n) \to K.$$

(with a $\mathcal{D}$-section $s\colon K \to L(M, n)$).

Now let (X, A) be a relative $\mathcal{D}$-complex, $i\colon A \to X$ the inclusion $\mathcal{D}$-map, and $\alpha\colon X \to K$ a $\mathcal{D}$-map inducing the identity on the fundamental categories. Put $[X, L(M, n)]^A_K$ for the set of vertical $\mathcal{D}$-homotopy classes of $\mathcal{D}$-liftings of α which agree with $\alpha_0 = s \circ \alpha\colon X \to L(M, n)$ on A, and $[-,-]_\alpha$, $[-,-]_{\alpha \circ i}$ for the appropriate sets of relative $\mathcal{D}$-homotopy classes. Then, from Theorem 3.4, we obtain the following representability result.

COROLLARY 3.6. *Let (X, A) be a relative $\mathcal{D}$-complex, and M a local contravariant coefficient system on X. Then there exist group isomorphisms making the diagram*

$$\begin{array}{ccccccc}
[A, \Omega_K L(\widetilde{M}, n)]_{\alpha \circ i} & \xrightarrow{\delta} & [X, L(\widetilde{M}, n)]_K^A & \xrightarrow{j} & [X, L(\widetilde{M}, n)]_\alpha & \xrightarrow{i^*} & [A, L(\widetilde{M}, n)]_{\alpha \circ i} \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
H_{\mathcal{D}}^{n-1}(A, M) & \xrightarrow{\delta} & H_{\mathcal{D}}^n(X, A; M) & \xrightarrow{j} & H_{\mathcal{D}}^n(X, M) & \xrightarrow{i^*} & H_{\mathcal{D}}^n(A, M)
\end{array}$$

commutative. The isomorphisms are defined using the primary cohomology difference by $\phi \mapsto \delta^n(\alpha_0, \phi)$, where $\Omega_K L(\widetilde{M}, n)$ is the relative loop $\mathcal{D}$-space over K.

Now let G be a topological group in CGH, and $\mathcal{O}_G$ the category of orbit types considered in Example 1.3. Then any G-space X determines the $\mathcal{D}$-space $\bar{X}\colon \mathcal{O}_G \to CGH$. By the Elmendorf's result [2], for any $\mathcal{D} = \mathcal{O}_G$-space there is also an associated G-space. For a contravariant local coefficient system on $\bar{X}$ and a pair of G-spaces (X, A) the group $H_{\mathcal{D}}^n(\bar{X}, \bar{A}; M)$ is called the nth *equivariant cohomology group* of the pair (X, A) and denoted by $H_G^n(X, A; M)$ (cf. [9]).

Remark 3.7. The above results may be reformulated also in the G-case, and, in particular from Theorem 3.3, we deduce equivalent descriptions of weak G-homotopy equivalence ([3], [12; Chap. II, 3.19]).

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