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ON AN ACCURACY OF CHANGE POINTS

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ABSTRACT. Many time processes are composed by several simple functions. A moment of a change of one function in another is frequently unknown or known only approximatively. There are several algorithms for the identification of change points. Sometimes not only their position but also the variance in a determination of their position is important. Similar problem occurs when an isobestic point must be determined. At this point several regression functions must cross. In general it is rather difficult problem, however in some situation a solution is relatively simple. Some comments to the solution are given in the paper.

Introduction

Models of many time processes are composed by several simple functions. Moments $T_1, \dots, T_s$, where one function is changed into another is sometimes known, e.g. when splines are used, sometimes unknown.

A typical situation where change points are given in advance can be described as follows.

Let an unknown function $f(\cdot)$ be characterized by a set of time points $t_1 < \cdots < t_n$, and by a set of values $y_i = f(t_i)$, $i = 1, \dots, n$. Let the interval $[t_1, t_n]$ be divided into $s-1$ ($\ll n$) subintervals $[T_1, T_2), [T_2, T_3), \dots, [T_{s-1}, T_s]$ and we want to approximate the time course of the function $f(\cdot)$ by $s-1$ polynomials $\phi_i(t) = \beta_{i,1} + \beta_{i,2}t + \beta_{i,3}t^2 + \beta_{i,4}t^3$ on the interval $[T_i, T_{i+1})$, $i = 1, \dots, s-1$. An approximation must be made in such a way that $\phi_i(T_{i+1}) = \phi_{i+1}(T_{i+1})$, $i = 1, \dots, s-1$, and also $\phi_i'(T_{i+1}) = \phi_{i+1}'(T_{i+1})$, $i = 1, \dots, s-1$ ($'$ denotes the derivative). The problem is to determine in some sense optimal values of the

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parameters $\beta_{i,j}$, $i = 1, \dots, s-1$, $j = 1, \dots, 4$. No problem of determination of the values $T_1, \dots, T_s$ occurs here.

Quite different situation occurs when a sequence of day's temperatures (measured with some errors) $\tau_1, \dots, \tau_n$ are given in the course of the last hundred years and the problem is to determine the moment T where the increasing tendency arises (very actual problem at present times), i.e to determine a change point of this sequence.

There are several algorithms which enable us to identify the moment T in the last example, or the moments $T_1, \dots, T_s$ in general (cf. [1]; here a large list of references on the change point problem is given).

Similar problems occur in chemistry when so called isobestic points ([3]) must be determined.

Let at known points $x_{i,1}, \dots, x_{i,n_i}$, $i = 1, \dots, s$, on the real line the values $f_i(x_{i,j}, \boldsymbol{\beta}_i)$, $j = 1, \dots, n_i$, and $i = 1, \dots, s$, be measured. An analytical form of the function $f_i(\cdot, \cdot\cdot)$ is assumed to be known but a k_i-dimensional parameter $\boldsymbol{\beta}_i$, $i = 1, \dots, s$, is unknown. Results of the measurement of the values $f_i(x_{i,j}, \boldsymbol{\beta}_i)$ are given by a set $\{y_{i,j} : j = 1, \dots, n_i, \ i = 1, \dots, s\}$. From a theory it is known that there exists a point (isobestic point) on the real line, where the equalities $f_1(T, \boldsymbol{\beta}_1) = f_2(T, \boldsymbol{\beta}_2) = \cdots = f_s(T, \boldsymbol{\beta}_s)$ must be satisfied. The problem is to estimate, except the parameters $\boldsymbol{\beta}_i$, $i = 1, \dots, s$, also the value T on the basis of the set $y_{i,j}$, $j = 1, \dots, n_i$, and $i = 1, \dots, s$, and to characterize an accuracy of the estimation.

Change point algorithms are focused on a determination of the position of a change point and a little less attention is given to a characterization of its accuracy. In some cases a knowledge on a regression models on the left hand and right hand sides of the change point can be used for a relatively adequate evaluation of the accuracy. In this case a determination of an accuracy of a change point and an isobestic point is a similar problem. A difference is that in the case of an isobestic point the regression function is estimated on both sides of the isobestic point, however, in the case of a change point a regression function can be estimated on one side of it only.

There are two typical situations here. In the first one, the change point is characterized by one condition only (e.g. at this point a polynomial of the second order is changed into a linear line), in the other case, there are several conditions (e.g. at the point also a derivative from the left must be the same as the derivative from the right). Solutions are different in different situations.

The aim of the paper is to make some comments to a problem how to characterize statistical properties of estimators of a position of mentioned points in some simple situations. The types of the regression models are assumed to be known.

1. Notation and auxiliary statements

Let $\boldsymbol{Y} \sim N_n(\boldsymbol{f}(\boldsymbol{\beta}_1), \Sigma)$, i.e. the n-dimensional observation vector $\boldsymbol{Y}$ is normally distributed, its mean value is $E(\boldsymbol{Y}) = \boldsymbol{f}(\boldsymbol{\beta})$, $\boldsymbol{f}(\cdot)$ is a known function however the k_1-dimensional vector parameter $\boldsymbol{\beta}_1$ is unknown and $\boldsymbol{\Sigma}$ is the covariance matrix of the vector $\boldsymbol{Y}$. It is either known, or its form is $\boldsymbol{\Sigma} = \sigma^2 \mathbf{V}$, where the $n \times n$ matrix $\mathbf{V}$ is positive definite and known and the parameter $\sigma^2 \in (0, \infty)$ is unknown. Also a situation $\boldsymbol{\Sigma} = \sigma_1^2 \mathbf{V}_1 + \sigma_2^2 \mathbf{V}_2$ may occur.

Let the parameter $\boldsymbol{\beta}_1$ satisfy the q conditions $\boldsymbol{h}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \mathbf{0}$. Here $\boldsymbol{\beta}_2$ is k_2-dimensional vector parameter which is also unknown. As an example let us consider a measurement of the values $f(x, \beta_1, \beta_2) = \beta_1 \exp(-\beta_2 x)$ at points $x_1 < \cdots < x_{n_1}$ and a measurement of the values $g(x, \beta_3, \beta_4) = \beta_3 + \beta_4 t$ at points $x_{n_1+1} < \cdots < x_{n_1+n_2}$, where $x_{n_1} < x_{n_1+1}$. Somewhere between x_{n_1} and x_{n_1+1} there is located a point T with the property $f(T, \beta_1, \beta_2) = g(T, \beta_3, \beta_4)$. Let the measurement of the values $f(x, \beta_1, \beta_2)$ be realized with the variance σ_1^2 and the measurement of the values $g(x, \beta_3, \beta_4)$ be realized with the variance σ_2^2 and measurements are stochastically independent. Thus

$$
\begin{aligned}
\boldsymbol{\beta}_1 &= (\beta_1, \beta_2, \beta_3, \beta_4)', \qquad \boldsymbol{\beta}_2 = T, \\
\boldsymbol{f}(\boldsymbol{\beta}_1) &= \big[f(x_1, \beta_1, \beta_2), \dots, f(x_{n_1}, \beta_1, \beta_2), g(x_{n_1+1}, \beta_3, \beta_4), \dots \\
&\qquad \dots, g(x_{n_1+n_2}, \beta_3, \beta_4)\big]', \\
\boldsymbol{h}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) &= \exp(-\beta_2 T) - (\beta_3 + \beta_4 T) = \mathbf{0}, \\
\boldsymbol{\Sigma} &= \sigma_1^2 \begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix} + \sigma_2^2 \begin{pmatrix} \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I} \end{pmatrix}.
\end{aligned}
$$

The linear version of the considered model is

$$
\boldsymbol{Y} - \boldsymbol{f}_0 \sim N_n(\mathbf{F}\Delta, \Sigma), \qquad \boldsymbol{h}_0 + \mathbf{H}_1 \Delta + \mathbf{H}_2 \delta = \mathbf{0} \tag{1}
$$

(the model with the constraints of the type II), where $\boldsymbol{f}_0 = \boldsymbol{f}(\boldsymbol{\beta}_{1,0})$, $\boldsymbol{\beta}_{1,0}$ is an approximative value of the vector $\boldsymbol{\beta}_1$, $\mathbf{F}$ is the matrix of the first derivatives at the point $\boldsymbol{\beta}_{1,0}$ of the function $\boldsymbol{f}(\cdot)$, $\boldsymbol{h}_0 = \boldsymbol{h}(\boldsymbol{\beta}_{1,0}, \boldsymbol{\beta}_{2,0})$, $\boldsymbol{\beta}_{2,0}$ is an approximative value of the parameter $\boldsymbol{\beta}_2$ (change point). The first derivatives at the point $\boldsymbol{\beta}_{1,0}$ and $\boldsymbol{\beta}_{2,0}$, respectively, of the vector function $\boldsymbol{h}(\cdot, \cdot\cdot)$ with respect to $\boldsymbol{\beta}_1$ create the matrix $\mathbf{H}_1$ and the first derivatives at the same point of the vector function $\boldsymbol{h}(\cdot, \cdot\cdot)$ with respect to $\boldsymbol{\beta}_2$ create the matrix $\mathbf{H}_2$. Further $\Delta = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_{1,0}$, $\delta = \boldsymbol{\beta}_2 - \boldsymbol{\beta}_{2,0}$.

The quadratic version of the considered model is

$$
\boldsymbol{Y} - \boldsymbol{f}_0 \sim N_n\Big(\mathbf{F}\Delta + \frac{1}{2}\kappa(\Delta), \Sigma\Big), \quad \boldsymbol{h}_0 + \mathbf{H}_1 \Delta + \mathbf{H}_2 \delta + \frac{1}{2}\omega(\Delta, \delta) = \mathbf{0}, \tag{2}
$$

where

$$\begin{aligned}\boldsymbol{\kappa}(\Delta) &= \big(\kappa_1(\Delta),\dots,\kappa_n(\Delta)\big)',\\ \kappa_i(\Delta) &= \Delta'\mathbf{D}_i\Delta, \qquad i=1,\dots,n,\\ \mathbf{D}_i &= \partial^2 f_i(\boldsymbol{u})/\partial\boldsymbol{u}\partial\boldsymbol{u}'\big|_{\boldsymbol{u}=\boldsymbol{\beta}_1}.\end{aligned}$$

An analogous meaning has the q-dimensional vector $\boldsymbol{\omega}(\Delta,\delta)$.

The notation $\hat{}$ means the estimators neglecting the constraints and $\hat{\hat{}}$ means the estimator respecting the constraints. The BLUE (best linear unbiased estimator) $\hat{\Delta}$ of the parameters Δ, in the model without constraints, means a linear estimator (linear vector function of the vector $\boldsymbol{Y}$) with the following properties. It is unbiased, i.e. $(\forall\,\Delta\in\mathbb{R}^{k_1})\big(E_\Delta(\hat{\Delta})=\Delta\big)$ and

$$(\forall\,\tilde{\Delta}\in\mathcal{D})(\forall\,\boldsymbol{p}\in\mathbb{R}^{k_1})\big(\mathrm{Var}(\boldsymbol{p}'\hat{\Delta})\le\mathrm{Var}(\boldsymbol{p}'\tilde{\Delta})\big),$$

where $\mathcal{D}$ is the class of all linear unbiased estimators of Δ in the model without constraints.

Analogously for the BLUE $\hat{\hat{\Delta}}$, in the model with constraints, it is valid

$$\left(\forall\,\Delta\in\left\{\begin{pmatrix}\Delta\\ \delta\end{pmatrix} : \mathbf{H}_1\Delta+\mathbf{H}_2\delta+\boldsymbol{h}=\mathbf{0}\right\}\right)\left(E_\Delta(\hat{\hat{\Delta}})=\Delta\right)$$

and

$$(\forall\,\tilde{\Delta}\in\mathcal{D}_{\mathrm{constr}})(\forall\,\boldsymbol{p}\in\mathbb{R}^{k_1})\big(\mathrm{Var}(\boldsymbol{p}'\hat{\Delta})\le\mathrm{Var}(\boldsymbol{p}'\tilde{\Delta})\big),$$

where $\mathcal{D}_{\mathrm{constr}}$ is the class of all linear unbiased estimators of Δ in the model with constraints.

The notations $\hat{\delta}$ and $\hat{\hat{\delta}}$ have analogous meaning.

LEMMA 1.1. *Let in* (1), $\boldsymbol{h}_0=\mathbf{0}$ *and the rank of the matrices* $\mathbf{F}$, $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *be* $\mathrm{r}(\mathbf{F})=k_1<n$, $\mathrm{r}(\mathbf{H}_1,\ \mathbf{H}_2)=q<k_1+k_2$ *and* $\mathrm{r}(\mathbf{H}_2)=k_2\le q$, *respectively. Let the matrix* $\boldsymbol{\Sigma}$ *be positive definite. Then the BLUE of the vector* $(\Delta',\delta')'$ *in* (1) *is*

$$\begin{aligned}\hat{\hat{\Delta}} &= \big(\mathbf{I}-\mathbf{C}^{-1}\mathbf{H}_1'\{(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1'+\mathbf{H}_2\mathbf{H}_2')^{-1}-(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1'+\mathbf{H}_2\mathbf{H}_2')^{-1}\mathbf{H}_2\times\\ &\qquad\times\big[\mathbf{H}_2'(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1'+\mathbf{H}_2\mathbf{H}_2')^{-1}\mathbf{H}_2\big]^{-1}\mathbf{H}_2'(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1'+\mathbf{H}_2\mathbf{H}_2')^{-1}\}\mathbf{H}_1\big)\hat{\Delta},\\ \hat{\Delta} &= \mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y}-\boldsymbol{f}_0), \qquad \mathbf{C}=\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F},\\ \hat{\hat{\delta}} &= -\big[\mathbf{H}_2'(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1'+\mathbf{H}_2\mathbf{H}_2')^{-1}\mathbf{H}_2\big]^{-1}\mathbf{H}_2'(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1'+\mathbf{H}_2\mathbf{H}_2')^{-1}\mathbf{H}_1\hat{\Delta}.\end{aligned}$$

P r o o f . See [4; p. 134]. □

COROLLARY 1.2. *If, in Lemma* 1.1, $k_2 = q$, *i.e. the matrix* $\mathbf{H}_2$ *is regular, then*

$$\hat{\hat{\Delta}} = \hat{\Delta},$$

$$\hat{\hat{\delta}} = -\mathbf{H}_2^{-1}\mathbf{H}_1\hat{\Delta}.$$

In this case the estimator of the vector parameter $\boldsymbol{\beta}_1$ *is not influenced by the constraints. Thus the coordinate vector* $\boldsymbol{\beta}_2$ *of a change point (isobestic point) is given by a crossing of regression function only. If the matrix* $\mathbf{H}_2$ *is not regular, i.e.* $k_2 < q$, *then the estimator of the vector* Δ *depends on the constraints.*

If $\mathrm{r}(\mathbf{H}_1) = q$, *i.e. the matrix* $\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1'$ *is regular, then*

$$\hat{\hat{\Delta}} = (\mathbf{I} - \mathbf{C}^{-1}\mathbf{H}_1'\{(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1')^{-1} - (\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_2 \times$$
$$\times \left[\mathbf{H}_2'(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_2\right]^{-1}\mathbf{H}_2'(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\}\mathbf{H}_1)\hat{\Delta},$$
$$\hat{\hat{\delta}} = -\left[\mathbf{H}_2'(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_2\right]^{-1}\mathbf{H}_2'(\mathbf{H}_1'\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_1\hat{\Delta}.$$

The assumption on the regularity of $\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1$ is always satisfied in the further consideration.

Remark 1.3. The vector $\boldsymbol{h}_0$ is assumed to be $\mathbf{0}$ in the following, since approximate values $\boldsymbol{\beta}_{1,0}$, $\boldsymbol{\beta}_{2,0}$ of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, respectively, should be chosen in such a way that $\boldsymbol{h}(\boldsymbol{\beta}_{1,0}, \boldsymbol{\beta}_{2,0}) = \mathbf{0}$. In the case of another choice the bias of the linear estimator would be larger (in detail cf. [4]).

In the following, the model will be considered in the form

$$\begin{pmatrix} \boldsymbol{Y}_1 - \boldsymbol{f}_{1,0} \\ \boldsymbol{Y}_2 - \boldsymbol{f}_{2,0} \end{pmatrix} \sim N_{n+m}\left(\begin{pmatrix} \mathbf{F}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{F}_2 \end{pmatrix}\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{2,2} \end{pmatrix}\right),$$
$$\mathbf{H}_1\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} + \mathbf{H}_2\delta = \mathbf{0} \tag{3}$$

(the linearized model) and in the form

$$\begin{pmatrix} \boldsymbol{Y}_1 - \boldsymbol{f}_{1,0} \\ \boldsymbol{Y}_2 - \boldsymbol{f}_{2,0} \end{pmatrix} \sim N_{n+m}\left(\begin{pmatrix} \mathbf{F}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{F}_2 \end{pmatrix}\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \kappa_1(\Delta_1) \\ \kappa_2(\Delta_2) \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{2,2} \end{pmatrix}\right), \tag{4}$$

$$\mathbf{H}_1\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} + \mathbf{H}_2\delta + \frac{1}{2}\omega(\Delta_1, \Delta_2, \delta) = \mathbf{0} \tag{5}$$

(the quadratized form of the model).

The notations

$$\boldsymbol{Y} - \boldsymbol{f}_0 = \begin{pmatrix} \boldsymbol{Y}_1 - \boldsymbol{f}_{1,0} \\ \boldsymbol{Y}_2 - \boldsymbol{f}_{2,0} \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} \mathbf{F}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{F}_2 \end{pmatrix},$$
$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{2,2} \end{pmatrix}, \qquad \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}, \qquad \kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}$$

will be used in the following. Here $\delta = T - T_0$, T is the actual coordinate of the change point and T_0 is an approximate value. The symbol δ means also a vector if several coordinates of change points are taken into account. The vector $\boldsymbol{Y}_i$ is linked with the ith regression functions and analogously it is valid for $\mathbf{F}_i$, $\Sigma_{i,i}$, Δ_i, etc.

2. A linear case

In this section a simple situation characterized by the regular models

$$\boldsymbol{Y}_1 \sim N_n(\mathbf{F}_1\boldsymbol{\beta}_1, \Sigma_{1,1}), \quad \boldsymbol{\beta}_1 \in \mathbb{R}^k, \qquad \boldsymbol{Y}_2 \sim N_m(\mathbf{F}_2\boldsymbol{\beta}_2, \Sigma_{2,2}), \quad \boldsymbol{\beta}_2 \in \mathbb{R}^l,$$

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix} \sim N_{n+m}(\mathbf{F}\boldsymbol{\beta}, \Sigma),$$

$$\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{F}_1, & 0 \\ 0, & \mathbf{F}_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{1,1}, & 0 \\ 0, & \Sigma_{2,2} \end{pmatrix},$$

$$\mathbf{C}_1 = \mathbf{F}_1'\Sigma_{1,1}^{-1}\mathbf{F}_1, \quad \mathbf{C}_2 = \mathbf{F}_2'\Sigma_{2,2}^{-1}\mathbf{F}_2, \quad \mathbf{C} = \begin{pmatrix} \mathbf{C}_1, & 0 \\ 0, & \mathbf{C}_2 \end{pmatrix}$$

is considered.

The coordinate T of a change point is given by an equality $(\boldsymbol{h}_1' - \hat{T}\boldsymbol{h}_2')\hat{\boldsymbol{\beta}} = 0$, where $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are known vectors. If, for instance, $\{\boldsymbol{Y}_1\}_i = \beta_{1,1} + \beta_{1,2}t_i$, $i = 1,\dots,n$, and $\{\boldsymbol{Y}_2\}_j = \beta_{2,1} + \beta_{2,2}t_j$, $j = 1,\dots,m$, then $\boldsymbol{h}_1' = (1,0,-1,0)$ and $\boldsymbol{h}_2' = (0,1,0,-1)$.

LEMMA 2.1. *The $(1-\alpha)$-confidence interval for the value T is $[A-B, A+B]$ (or its complement), where*

$$A = \frac{\boldsymbol{h}_1'\hat{\boldsymbol{\beta}}\boldsymbol{h}_2'\hat{\boldsymbol{\beta}} - v_{1,2}\chi_1^2(1-\alpha)}{(\boldsymbol{h}_2'\hat{\boldsymbol{\beta}})^2 - \chi_1^2(1-\alpha)v_{2,2}},$$

$$B = \frac{\sqrt{\chi_1^2(1-\alpha)\det(\mathbf{V})\left[(\boldsymbol{h}_1'\hat{\boldsymbol{\beta}}, \boldsymbol{h}_2'\hat{\boldsymbol{\beta}})\mathbf{V}^{-1}\begin{pmatrix} \boldsymbol{h}_1'\hat{\boldsymbol{\beta}} \\ \boldsymbol{h}_2'\hat{\boldsymbol{\beta}}_2 \end{pmatrix} - \chi_1^2(1-\alpha)\right]}}{(\boldsymbol{h}_2'\hat{\boldsymbol{\beta}})^2 - \chi_1^2(1-\alpha)v_{2,2}},$$

$$\mathbf{V} = \begin{pmatrix} \boldsymbol{h}_1' \\ \boldsymbol{h}_2' \end{pmatrix}\mathbf{C}^{-1}(\boldsymbol{h}_1, \boldsymbol{h}_2) = \begin{pmatrix} v_{1,1}, & v_{1,2} \\ v_{2,1}, & v_{2,2} \end{pmatrix}.$$

P r o o f . The well-known idea (cf. e.g. [6; p. 199]) is used here. Let $\hat{u}_1 = \boldsymbol{h}_1'\hat{\boldsymbol{\beta}}$, $\hat{u}_2 = \boldsymbol{h}_2'\hat{\boldsymbol{\beta}}$. Since $\hat{u}_1 - T\hat{u}_2 \sim N_1\left(0, (1,-T)\mathbf{V}\begin{pmatrix} 1 \\ -T \end{pmatrix}\right)$,

$$P\left\{(\hat{u}_1 - T\hat{u}_2)^2 \le (1,-T)\mathbf{V}\begin{pmatrix} 1 \\ -T \end{pmatrix}\chi_1^2(1-\alpha)\right\} = 1-\alpha.$$

The random event $\left\{(\hat{u}_1 - T\hat{u}_2)^2 \leq (1, -T)\mathbf{V}\begin{pmatrix}1\\-T\end{pmatrix}\chi_1^2(1-\alpha)\right\}$ can be rewritten in the form

$$\left\{(1,-T)\begin{pmatrix}A_{1,1}, & A_{1,2}\\ A_{2,1}, & A_{2,2}\end{pmatrix}\begin{pmatrix}1\\-T\end{pmatrix} \leq 0\right\},$$

where

$$\begin{pmatrix}A_{1,1}, & A_{1,2}\\ A_{2,1}, & A_{2,2}\end{pmatrix} = \begin{pmatrix}\boldsymbol{h}_1'\\ \boldsymbol{h}_2'\end{pmatrix}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}'(\boldsymbol{h}_1, \boldsymbol{h}_2) - \chi_1^2(1-\alpha)\mathbf{V},$$

i.e. $\{T^2 A_{2,2} - 2TA_{1,2} + A_{1,1} \leq 0\}$. Thus T must be covered by the random interval

$$\left[\frac{A_{1,2}}{A_{2,2}} - \frac{\sqrt{A_{1,2}^2 - A_{1,1}A_{2,2}}}{A_{2,2}},\ \frac{A_{1,2}}{A_{2,2}} + \frac{\sqrt{A_{1,2}^2 - A_{1,1}A_{2,2}}}{A_{2,2}}\right],$$

with probability $1-\alpha$. It can be easily verified that this interval is the interval from the statement of the lemma. □

Another characterization of an accuracy in a determination of a change point can be given by the values of $E(\hat{\hat{T}}) - T$ (here $\hat{\hat{T}} = \boldsymbol{h}_1'\hat{\boldsymbol{\beta}}/(\boldsymbol{h}_2'\hat{\boldsymbol{\beta}})$) and $\mathrm{Var}(\hat{\hat{T}})$. The first approximation of them, given by the Taylor series, is

$$E(\hat{\hat{T}}) - T = T\left(\frac{\sigma_{2,2}}{(\boldsymbol{h}_2'\boldsymbol{\beta})^2} - \frac{\sigma_{1,2}}{\boldsymbol{h}_1'\boldsymbol{\beta}\boldsymbol{h}_2'\boldsymbol{\beta}}\right),$$

$$\mathrm{Var}_1(\hat{\hat{T}}) = T^2\left(\frac{\sigma_{1,1}}{(\boldsymbol{h}_1'\boldsymbol{\beta})^2} - 2\frac{\sigma_{1,2}}{\boldsymbol{h}_1'\boldsymbol{\beta}\boldsymbol{h}_2'\boldsymbol{\beta}} + \frac{\sigma_{2,2}}{(\boldsymbol{h}_2'\boldsymbol{\beta})^2}\right).$$

Here $\boldsymbol{h}_1'\mathbf{C}^{-1}\boldsymbol{h}_1 = \sigma_{1,1}$, $\boldsymbol{h}_1'\mathbf{C}^{-1}\boldsymbol{h}_2 = \sigma_{1,2} = \varrho\sqrt{\sigma_{1,1}\sigma_{2,2}}$, $\boldsymbol{h}_2'\mathbf{C}^{-1}\boldsymbol{h}_2 = \sigma_{2,2}$.

If further terms of the Taylor series is taken into account, then

$$E(\hat{\hat{T}}) - T = T\left(\frac{\sigma_{2,2}}{(\boldsymbol{h}_2'\boldsymbol{\beta})^2} - \frac{\sigma_{1,2}}{\boldsymbol{h}_1'\boldsymbol{\beta}\boldsymbol{h}_2'\boldsymbol{\beta}} + \frac{\sigma_{2,2}^2}{(\boldsymbol{h}_2'\boldsymbol{\beta})^4} - \frac{\sigma_{1,2}\sigma_{2,2}}{\boldsymbol{h}_1'\boldsymbol{\beta}(\boldsymbol{h}_2'\boldsymbol{\beta})^3} + \cdots\right),$$

$$\begin{aligned}\mathrm{Var}_2(\hat{\hat{T}}) &= \\ &= T^2\left\{\frac{\sigma_{1,1}}{(\boldsymbol{h}_1'\boldsymbol{\beta})^2} - 2\frac{\sigma_{1,2}}{\boldsymbol{h}_1'\boldsymbol{\beta}\boldsymbol{h}_2'\boldsymbol{\beta}} + \frac{\sigma_{2,2}}{(\boldsymbol{h}_2'\boldsymbol{\beta})^2} + \frac{2}{(\boldsymbol{h}_2'\boldsymbol{\beta})^2}\left(\frac{\sigma_{2,2}}{\boldsymbol{h}_2'\boldsymbol{\beta}} - \varrho\frac{\sqrt{\sigma_{1,1}\sigma_{2,2}}}{\boldsymbol{h}_1'\boldsymbol{\beta}}\right)^2\right. \\ &\quad \left. - \frac{2\sigma_{1,1}\sigma_{2,2}(1-\varrho^2)}{(\boldsymbol{h}_1'\boldsymbol{\beta})^2(\boldsymbol{h}_2'\boldsymbol{\beta})^2} + 2\left(\frac{\sigma_{2,2}}{(\boldsymbol{h}'\boldsymbol{\beta})^2} - \frac{\sigma_{1,2}}{\boldsymbol{h}_1'\boldsymbol{\beta}\boldsymbol{h}_2'\boldsymbol{\beta}}\right)^2\right\} + \cdots.\end{aligned}$$

A confrontation of the $(1-\alpha)$-confidence interval and interval $\left[\hat{T} - u(1-\alpha/2)\sqrt{\mathrm{Var}_1(\hat{T})},\ \hat{T} + u(1-\alpha/2)\sqrt{\mathrm{Var}_1(\hat{T})}\right]$, where $u(1-\alpha/2)$ is the $(1-\alpha/2)$-quantile of normal distribution $N(0,1)$, is a relatively good check of a numerical calculation.

If $\mathbf{C}_1^{-1} = \sigma_1^2\mathbf{C}_{1,0}^{-1}$ and $\mathbf{C}_2^{-1} = \sigma_2^2\mathbf{C}_{2,0}^{-1}$ and estimators $\hat{\sigma}_1^2 \sim \sigma_1^2\chi^2_{n-k}/(n-k)$ and $\hat{\sigma}_2^2 \sim \sigma_2^2\chi^2_{m-l}/(m-l)$ are at our disposal, then it is necessary to use the ideas given in [5] and [8] in order to determine the confidence interval.

In the next section a criterion whether a nonlinear problem can be linearized is given.

3. Measures of nonlinearity

LEMMA 3.1.

(i) *Let* $\mathbf{H}_2$ *in* (4) *be regular. Then the bias of the estimator* $\hat{\hat{\delta}}$ *of the parameter* δ *is*

$$E(\hat{\hat{\delta}}) - \delta = \frac{1}{2}\mathbf{H}_2^{-1}\left[\omega(\Delta_1,\Delta_2,\delta) - \mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\kappa(\Delta_1,\Delta_2)\right].$$

(ii) *Let* $\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1'$ *be regular. Then the bias of the estimator* $\hat{\hat{\delta}}$ *of the parameter* δ *is*

$$E(\hat{\hat{\delta}}) - \delta = $$
$$= \frac{1}{2}\left[\mathbf{H}_2'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_2\right]^{-1}\mathbf{H}_2'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\left(\omega(\Delta,\delta) - \mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\kappa(\Delta)\right).$$

Proof. With respect to Corollary 1.2

$$\begin{aligned}
E(\hat{\hat{\delta}}) &= -\mathbf{H}_2^{-1}\mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}E(\boldsymbol{Y} - \boldsymbol{f}_0)\\
&= -\mathbf{H}_2^{-1}\mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\left(\mathbf{F}\Delta + \frac{1}{2}\kappa(\Delta_1,\Delta_2)\right)\\
&= \mathbf{H}_2^{-1}\left(\mathbf{H}_2\delta + \frac{1}{2}\omega(\Delta_1,\Delta_2,\delta) - \frac{1}{2}\mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\kappa(\Delta_1,\Delta_2)\right)\\
&= \delta + \frac{1}{2}\mathbf{H}_2^{-1}\left[\omega(\Delta_1,\Delta_2,\delta) - \mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\kappa(\Delta_1,\Delta_2)\right].
\end{aligned}$$

Analogously the statement (ii) can be proved. □

In the following, the symbol $\mathbf{P}_{\mathbf{H}_2}^{(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}}$ means the projection matrix on the subspace $\mathcal{M}(\mathbf{H}_2)$ in the norm given by the matrix $(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}$ ($\|\boldsymbol{x}\|_{(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}} = \sqrt{\boldsymbol{x}'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\boldsymbol{x}}$), i.e.

$$\mathbf{P}_{\mathbf{H}_2}^{(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}} = \mathbf{H}_2\left[\mathbf{H}_2'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_2\right]^{-1}\mathbf{H}_2'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}.$$

DEFINITION 3.1. Let the matrix $\mathbf{H}_1\Sigma^{-1}\mathbf{H}_1'$ in (4) be regular. Then the *measure of nonlinearity for the parameter* δ is

$$C_{II,\delta}^{(\mathrm{par})} = \sup\left\{\frac{\sqrt{\gamma'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{P}_{\mathbf{H}_2}^{(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}}\gamma}}{\Delta'\mathbf{C}\Delta} : \ \Delta \in \mathbb{R}^{k_1+k_2}\right\},$$

where

$$\gamma = \omega(\Delta_1, \Delta_2, \delta) - \mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\kappa(\Delta_1, \Delta_2).$$

THEOREM 3.1. *Let in* (4) *the matrix* $\mathbf{H}_1\Sigma^{-1}\mathbf{H}_1'$ *be regular. Then*

$$\Delta'\mathbf{C}\Delta \le \frac{2\varepsilon}{C_{II,\delta}^{(\mathrm{par})}} \implies \sqrt{\boldsymbol{b}'\left[\mathrm{Var}(\hat{\hat{\delta}})\right]^{-1}\boldsymbol{b}} \le \varepsilon,$$

where

$$\boldsymbol{b} = \begin{cases} \frac{1}{2}\mathbf{H}_2^{-1}\left[\omega(\Delta_1, \Delta_2, \delta) - \mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\kappa(\Delta_1, \Delta_2)\right] & \text{if } \mathbf{H}_2 \text{ is regular}, \\ \frac{1}{2}\left[\mathbf{H}_2'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_2\right]^{-1}\mathbf{H}_2'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1} \times \\ \qquad \times\left[\omega(\Delta, \delta) - \mathbf{H}_1\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\kappa(\Delta)\right] & \text{if } \mathbf{H}\mathbf{C}^{-1}\mathbf{H}_1' \text{ is regular}, \end{cases}$$

$$\mathrm{Var}(\hat{\hat{\delta}}) = \left[\mathbf{H}_2(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_2'\right]^{-1}.$$

(*If T is a one dimensional parameter, then a better formulation of the statement is* $|\boldsymbol{b}| \le \varepsilon\sqrt{\mathrm{Var}(\hat{\hat{\delta}})}$.)

Further, if T is a vector (k_2 (> 1) *change points are under consideration*), *then*

$$\Delta'\mathbf{C}\Delta \le \frac{2\varepsilon}{C_{II,\delta}^{(\mathrm{par})}} \implies (\forall \boldsymbol{h} \in \mathbb{R}^{k_2})\left(|\boldsymbol{h}'\boldsymbol{b}| \le \varepsilon\sqrt{\boldsymbol{h}'\,\mathrm{Var}(\hat{\hat{\delta}})\boldsymbol{h}}\right).$$

Proof. It is easy to prove the equality

$$2\sqrt{\boldsymbol{b}'\left[\mathrm{Var}(\hat{\hat{\delta}})\right]^{-1}\boldsymbol{b}} = \sqrt{\gamma'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{P}_{\mathbf{H}_2}^{(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}}\gamma}.$$

Then

$$\frac{\sqrt{\gamma'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{P}_{\mathbf{H}_2}^{(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}}\gamma}}{\Delta'\mathbf{C}\Delta} = 2\frac{\sqrt{\boldsymbol{b}'\left[\mathrm{Var}(\hat{\hat{\delta}})\right]^{-1}\boldsymbol{b}}}{\Delta'\mathbf{C}\Delta} \le C_{II,\delta}^{(\mathrm{par})}.$$

Thus the relationships

$$2\sqrt{\boldsymbol{b}'\left[\mathrm{Var}(\hat{\hat{\delta}})\right]^{-1}\boldsymbol{b}} \le C_{II,\delta}^{(\mathrm{par})}\Delta'\mathbf{C}\Delta \le 2\varepsilon$$

imply the first statement of the theorem. The other statement is a consequence of the Scheffé theorem ([7]). □

Remark 3.1. Definition 3.1 follows an idea of Bates and Watts [2]. The statement of Theorem 2.1 is of a practical use if the $(1-\alpha)$-confidence ellipsoid for a sufficiently small value α is included into the linearization region $\{\Delta : \Delta'\mathbf{C}\Delta \le 2\varepsilon/C_{II,\delta}^{(\mathrm{par})}\}$ given by Theorem 2.1. In this case, if ε is sufficiently small, the estimator $\hat{\hat{\delta}}_2$ can be considered as an unbiased best estimators of a position of a change point.

Remark 3.2. In the case $\mathbf{H}_2$ is not regular, $\hat{\Delta} \neq \hat{\hat{\Delta}}$ and thus, in Definition 3.1, a utilization of the denominator $\Delta'\mathbf{C}\Delta$ must be commented. If $\mathbf{H}_2$ is regular, then the Definition has a strict geometrical meaning. It characterizes a parametric curvature (in the Bates and Watts sense) of the model

$$-\mathbf{H}_1\hat{\Delta} \sim_{k_1} \left(\mathbf{H}_2\delta + \frac{1}{2}\left(\omega(\Delta,\delta) - \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\kappa(\Delta)\right),\ \mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1'\right).$$

However, in the case $\mathbf{H}_2$ is not regular (e.g. a change point must satisfy several conditions), Δ can vary in the linear manifold $\{\Delta :\ \Delta^* + \mathcal{M}(\mathrm{Var}(\hat{\hat{\Delta}}))\}$ only; here Δ^* is the actual value of the vector Δ. Since

$$\begin{aligned}
\mathrm{r}(\mathrm{Var}(\hat{\hat{\Delta}})) &= \mathrm{r}\left(\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{H}_1'(\mathbf{M}_{\mathbf{H}_2}\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1'\mathbf{M}_{\mathbf{H}_2})^+\mathbf{H}_1\mathbf{C}^{-1}\right) \\
&= \mathrm{r}\left(\mathbf{M}^{\mathbf{C}}_{\mathbf{C}^{-1}\mathbf{H}_1'\mathbf{M}_{\mathbf{H}_2}}\mathbf{C}^{-1}\right) = k_1 - r\left(\mathbf{C}^{-1}\mathbf{H}_1'\mathbf{M}_{\mathbf{H}_2}\right) \\
&= k_1 - (q - k_2) < k_1,
\end{aligned}$$

the strict geometrical meaning is lost. (The symbol $\mathbf{M}_{\mathbf{H}_2}$ means the projection matrix on the orthogonal complement of the subspace $\mathcal{M}(\mathbf{H}_2)$, i.e. $\mathbf{M}_{\mathbf{H}_2} = \mathbf{I} - \mathbf{P}^{\mathbf{I}}_{\mathbf{H}_2}$; the symbol $\mathbf{M}^{\mathbf{C}}_{\mathbf{C}^{-1}\mathbf{H}_1'\mathbf{M}_{\mathbf{H}_2}}$ means $\mathbf{I} - \mathbf{P}^{\mathbf{C}}_{\mathbf{C}^{-1}\mathbf{H}_1'\mathbf{M}_{\mathbf{H}_2}}$.) However, from the viewpoint of practice it does not matter and the measure of nonlinearity in the sense of Definition 3.1 is still useful.

4. Estimation of both crossing point coordinates

Let the coordinate of the crossing point (isobestic point) be $\binom{T}{\eta} = \binom{T_0+\delta_1}{\eta_0+\delta_2}$, where T_0 and η_0 are approximative values. There are m regression functions under consideration, i.e. $\boldsymbol{Y}_i - \boldsymbol{f}_{i,0} \sim N_{n_i}\left(\boldsymbol{f}_i(\boldsymbol{\beta}_i), \Sigma_{i,i}\right)$, $i = 1,\dots,m$. The linear version of the model is

$$\boldsymbol{Y} - \boldsymbol{f}_0 \sim N_{\sum_{i=1}^m n_i}(\mathbf{F}\Delta, \Sigma),$$

$$\boldsymbol{Y} - \boldsymbol{f}_0 = \left((\boldsymbol{Y}_1 - \boldsymbol{f}_{1,0})', \dots, (\boldsymbol{Y}_m - \boldsymbol{f}_{m,0})'\right)', \qquad \Delta = (\Delta_1', \dots, \Delta_m')',$$

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}_1, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & \mathbf{F}_m \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{1,1}, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & \Sigma_{m,m} \end{pmatrix},$$

$$\mathbf{H}_1\Delta + \mathbf{H}_2\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = \mathbf{0}, \tag{6}$$

$$\mathbf{H}_1 = \begin{pmatrix} \boldsymbol{h}_1', & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & \boldsymbol{h}_m' \end{pmatrix}, \qquad \mathbf{H}_2 = \begin{pmatrix} h_1, & -1 \\ \dots & \dots \\ h_m, & -1 \end{pmatrix}.$$

If δ_1 is to be determined only, then the constraints are of the form

$$\mathbf{G}_1\Delta + \boldsymbol{g}\delta_1 = \mathbf{0}, \tag{7}$$

$$\mathbf{G}_1 = \begin{pmatrix} \boldsymbol{h}_1', & -\boldsymbol{h}_2', & 0, & \dots, & 0 \\ 0, & \boldsymbol{h}_2', & -\boldsymbol{h}_3', & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & \boldsymbol{h}_{m-1}', & -\boldsymbol{h}_m' \end{pmatrix},$$

$$\boldsymbol{g} = (h_1, -h_2, \dots, h_{m-1} - h_m)'.$$

Lemma 4.1. *Let*

$$\mathbf{T} = \begin{pmatrix} 1, & -1, & 0, & \dots, & 0, & 0 \\ 0, & 1, & -1, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 0, & 1, & -1 \end{pmatrix}.$$

If $\boldsymbol{g} = (h_1, -h_2, \dots, h_{m-1} - h_m)' \neq \mathbf{0}$, then there exists a vector $\boldsymbol{r} \in \mathbb{R}^m$ such that $\mathbf{R} = \begin{pmatrix}\mathbf{T}\\ \boldsymbol{r}'\end{pmatrix}$ is regular, $\boldsymbol{r}'\mathbf{1} = 1$ and $\boldsymbol{r}'(h_1, \dots, h_m)' = 0$.

P r o o f . The following statements are equivalent.

(i) The vector $\boldsymbol{g}$ is not zero vector.

(ii) The vector $(h_1, \dots, h_m)'$ is not perpendicular to the subspace

$$\mathcal{M}(\mathbf{T}') = \{\mathbf{T}'\boldsymbol{u} : \ \boldsymbol{u} \in \mathbb{R}^{m-1}\}.$$

(iii) There exists a vector $\boldsymbol{r} \in \mathbb{R}^m$ such that $\boldsymbol{r} \notin \mathcal{M}(\mathbf{T}')$ and at the same time $\boldsymbol{r}'(h_1, \dots, h_m)' = 0$.

Further $\mathrm{r}(\mathbf{H}_2) = 2 \iff (h_1, \dots, h_m)'$ and $\mathbf{1}$ are linearly independent. Thus $\boldsymbol{r}$ can be chosen in such a way that $\boldsymbol{r}'\mathbf{1} = 1$. □

The assumption $\boldsymbol{g} \neq \mathbf{0}$ is natural with respect to the solved problem.

In the following, the notation $\mathbf{A}^-_{m(\mathbf{W})}$ will be used. Here $\mathbf{A}$ is an arbitrary matrix of the type $m \times n$, $\mathbf{W}$ is an $n \times n$ p.d. matrix and $\mathbf{G} = \mathbf{A}^-_{m(\mathbf{W})}$ satisfies the equalities $\mathbf{AGA} = \mathbf{A}$ & $\mathbf{WGA} = \mathbf{A}'\mathbf{G}'\mathbf{W}$. The matrix $\mathbf{G}$ has the following property. If $\mathbf{A}\boldsymbol{x} = \boldsymbol{y}$ is a consistent system, then $\boldsymbol{x} = \mathbf{G}\boldsymbol{y}$ is a solution with the minimum $\mathbf{W}$-norm.

THEOREM 4.1. *The estimator $\hat{\hat{\delta}}_1$ in the model with the constraints* (6) *is the same as the estimator $\tilde{\tilde{\delta}}_1$ in the model with the constraints* (7).

P r o o f. In the case of (6) the best estimator of $\binom{\delta_1}{\delta_2}$ based on $\boldsymbol{Y}$ is

$$\begin{pmatrix} \hat{\hat{\delta}}_1 \\ \hat{\hat{\delta}}_2 \end{pmatrix} = -\left[(\mathbf{H}_2')^-_{m(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')}\right]' \mathbf{H}_1\hat{\Delta}$$

and in the case (7) the best estimator of δ_1 is

$$\tilde{\tilde{\delta}}_1 = -\left[(\boldsymbol{g}')^-_{m(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')}\right]' \mathbf{G}_1\hat{\Delta}.$$

Let $\mathbf{R} = \binom{\mathbf{T}}{\boldsymbol{r}'}$ be such regular matrix that

$$\mathbf{T} = \begin{pmatrix} 1, & -1, & 0, & \dots, & 0, & 0 \\ 0, & 1, & -1, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 0, & 1, & -1 \end{pmatrix},$$

$\boldsymbol{r}'(h_1, h_2, \dots, h_m)' = 0$ and $\boldsymbol{r}'\mathbf{1} = 1$. Obviously

$$\mathbf{R}\mathbf{H}_1 = \begin{pmatrix} \mathbf{G}_1 \\ \boldsymbol{r}'\mathbf{H}_1 \end{pmatrix}, \qquad \mathbf{R}\mathbf{H}_2 = \begin{pmatrix} \boldsymbol{g}, & 0 \\ 0, & 1 \end{pmatrix}.$$

$$\begin{aligned}
\hat{\hat{\delta}}_1 = &-\left\{ \begin{pmatrix} \boldsymbol{u}'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\boldsymbol{u}, & \boldsymbol{u}'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{1} \\ \mathbf{1}'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\boldsymbol{u}, & \mathbf{1}'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{1} \end{pmatrix}^{-1} \times \right. \\
&\left. \times \begin{pmatrix} \boldsymbol{u}' \\ \mathbf{1}' \end{pmatrix} (\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1} \right\}_{1,\cdot} \mathbf{H}_1\hat{\Delta} \\
= &-\left\{ \{\mathbf{H}_2'\mathbf{R}'(\mathbf{R}\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1'\mathbf{R}')^{-1}\mathbf{R}\mathbf{H}_2\}^{-1} \right\}_{1,\cdot} \times \\
&\times \mathbf{H}_2'\mathbf{R}'(\mathbf{R}\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1'\mathbf{R}')^{-1}\mathbf{R}\mathbf{H}_1\hat{\Delta} \\
= &-\left\{ \left\{ \begin{pmatrix} \boldsymbol{g}', & 0 \\ 0, & 1 \end{pmatrix} \begin{pmatrix} \mathbf{A}_{1,1}, & \boldsymbol{a}_{1,2} \\ \boldsymbol{a}_{2,1}, & a_{2,2} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{g}, & 0 \\ 0, & 1 \end{pmatrix} \right\}^{-1} \right\}_{1,\cdot} \times \\
&\times \begin{pmatrix} \boldsymbol{g}', & 0 \\ 0, & 1 \end{pmatrix} \begin{pmatrix} \mathbf{A}_{1,1}, & \boldsymbol{a}_{1,2} \\ \boldsymbol{a}_{2,1}, & a_{2,2} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{G}_1 \\ \boldsymbol{r}'\mathbf{H}_1 \end{pmatrix} \hat{\Delta},
\end{aligned}$$

where

$$\begin{aligned}
\begin{pmatrix} \mathbf{A}_{1,1}, & \boldsymbol{a}_{1,2} \\ \boldsymbol{a}_{2,1}, & a_{2,2} \end{pmatrix}^{-1} &= \begin{pmatrix} \mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1', & \mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r} \\ \boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{G}_1', & \boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \mathbf{A}^{1,1}, & \boldsymbol{a}^{1,2} \\ \boldsymbol{a}^{2,1}, & a^{2,2} \end{pmatrix}.
\end{aligned}$$

Here

$$\begin{aligned}
\mathbf{A}^{1,1} &= \mathbf{A}_{1,1}^{-1} + \mathbf{A}_{1,1}^{-1}\boldsymbol{a}_{1,2}\left[a_{2,2} - \boldsymbol{a}_{2,1}\mathbf{A}^{-1}\boldsymbol{a}_{1,2}\right]^{-1}\boldsymbol{a}_{2,1}\mathbf{A}_{1,1}^{-1}\,, \\
&= (\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1} + (\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r} \times \\
&\qquad \times \left[\boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1' - \boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{G}_1'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}\right]^{-1} \times \\
&\qquad \times \boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{G}_1'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1} \\
&= (\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1} \\
&\qquad + (\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}a^{2,2}\boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{G}_1'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\,,
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{a}^{1,2} &= \left(\boldsymbol{a}^{2,1}\right)' = -\mathbf{A}^{1,1}\boldsymbol{a}_{1,2}(a_{2,2})^{-1}\,, \\
a^{2,2} &= \left(\boldsymbol{r}'\mathbf{H}_1\left(\mathbf{M}_{\mathbf{G}_1'}\mathbf{C}\mathbf{M}_{\mathbf{G}_1'}\right)^{+}\mathbf{H}_1'\boldsymbol{r}\right)^{-1}.
\end{aligned}$$

Thus

$$\hat{\hat{\delta}}_1 = -\left\{\begin{pmatrix} \boldsymbol{g}'\mathbf{A}^{1,1}\boldsymbol{g}, & \boldsymbol{g}'\boldsymbol{a}^{1,2} \\ \boldsymbol{a}^{2,1}\boldsymbol{g}, & a^{2,2}\end{pmatrix}^{-1}\right\}_{1,\cdot}\begin{pmatrix} \boldsymbol{g}'\mathbf{A}^{1,1}, & \boldsymbol{g}'\boldsymbol{a}^{1,2} \\ \boldsymbol{a}^{2,1}, & a^{2,2}\end{pmatrix}\mathbf{R}\mathbf{H}_1\hat{\Delta}\,.$$

Further

$$\begin{aligned}
\left\{\begin{pmatrix} \boldsymbol{g}'\mathbf{A}^{1,1}\boldsymbol{g}, & \boldsymbol{g}'\boldsymbol{a}^{1,2} \\ \boldsymbol{a}^{2,1}\boldsymbol{g}, & a^{2,2}\end{pmatrix}^{-1}\right\}_{1,1} &= \left[\boldsymbol{g}'\mathbf{A}^{1,1}\boldsymbol{g} - \boldsymbol{g}'\boldsymbol{a}^{1,2}(a^{2,2})^{-1}\boldsymbol{a}^{2,1}\boldsymbol{g}\right]^{-1} \\
&= \left\{\boldsymbol{g}'\left[\mathbf{A}^{1,1} - \boldsymbol{a}^{1,2}(a^{2,2})^{-1}\boldsymbol{a}^{2,1}\right]\boldsymbol{g}\right\}^{-1} \\
&= (\boldsymbol{g}'\mathbf{A}_{1,1}^{-1}\boldsymbol{g})^{-1} \\
&= \left[\boldsymbol{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\boldsymbol{g}\right]^{-1}
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\begin{pmatrix} \boldsymbol{g}'\mathbf{A}^{1,1}\boldsymbol{g}, & \boldsymbol{g}'\boldsymbol{a}^{1,2} \\ \boldsymbol{a}^{2,1}\boldsymbol{g}, & a^{2,2}\end{pmatrix}^{-1}\right\}_{1,2} \\
&\qquad = -\left[\boldsymbol{g}'\mathbf{A}^{1,1}\boldsymbol{g} - \boldsymbol{g}'\boldsymbol{a}^{1,2}(a^{2,2})^{-1}\boldsymbol{a}^{2,1}\boldsymbol{g}\right]^{-1}\boldsymbol{g}'\boldsymbol{a}^{1,2}(a^{2,2})^{-1} \\
&\qquad = \left[\boldsymbol{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\boldsymbol{g}\right]^{-1}\boldsymbol{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}\,.
\end{aligned}$$

Since

$$\begin{pmatrix} \boldsymbol{g}'\mathbf{A}^{1,1}, & \boldsymbol{g}'\boldsymbol{a}^{1,2} \\ \boldsymbol{a}^{2,1}, & a^{2,2}\end{pmatrix}\mathbf{R}\mathbf{H}_1\hat{\Delta} = \begin{pmatrix} A \\ B \end{pmatrix},$$

where

$$\begin{aligned}
A = \boldsymbol{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\hat{\Delta} + \boldsymbol{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}a^{2,2} &\times \\
\times\, \boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{G}_1'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\hat{\Delta}& \\
- \boldsymbol{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}a^{2,2}\boldsymbol{r}'\mathbf{H}_1\hat{\Delta}&\,,
\end{aligned}$$

$$B = -a^{2,2}\boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{G}_1'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\hat{\Delta} + a^{2,2}\boldsymbol{r}'\mathbf{H}_1\hat{\Delta}\,,$$

the estimator $\hat{\hat{\delta}}_1$ can be rewritten in the form

$$\begin{aligned}\hat{\hat{\delta}}_1 = -\Big(&\big[\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1)^{-1}\mathbf{g}\big]^{-1}\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1)^{-1}\mathbf{G}_1\hat{\Delta} + \big[\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1)^{-1}\mathbf{g}\big]^{-1} \times \\ &\times \mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1)^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}a^{2,2}\boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{G}_1'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1)^{-1}\mathbf{G}_1\hat{\Delta} \\ &- \big[\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{g}\big]^{-1}\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}a^{2,2}\boldsymbol{r}'\mathbf{H}_1\hat{\Delta} \\ &- \big[\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{g}\big]^{-1}\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}a^{2,2}\boldsymbol{r}'\mathbf{H}_1\mathbf{C}^{-1}\mathbf{G}_1' \times \\ &\times (\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\hat{\Delta} \\ &+ \big[\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{g}\big]^{-1}\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\mathbf{C}^{-1}\mathbf{H}_1'\boldsymbol{r}a^{2,2}\boldsymbol{r}'\mathbf{H}_1\hat{\Delta}\Big) \\ = -&\big[\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{g}\big]^{-1}\mathbf{g}'(\mathbf{G}_1\mathbf{C}^{-1}\mathbf{G}_1')^{-1}\mathbf{G}_1\hat{\Delta} = \tilde{\tilde{\delta}}_1\,.\end{aligned}$$

□

If the model is considered in the quadratized form, then an analogous criterion as in Section 3 can be used. Instead of quantity δ_1 quantity $\binom{\delta_1}{\delta_2}$ must be used.

5. Example

Let two linear regression functions $f_1(t) = \beta_1 t$ and $f_2(t) = \beta_2 + \beta_3 t$ be considered. Let

$$\boldsymbol{Y}_1 \sim N_4\left(\begin{pmatrix}1\\2\\3\\4\end{pmatrix}0.5,\ 0.4^2\mathbf{I}\right), \qquad \boldsymbol{Y}_2 \sim N_4\left(\begin{pmatrix}1, & 7\\1, & 8\\1, & 9\\1, & 10\end{pmatrix}\begin{pmatrix}7\\-2/3\end{pmatrix},\ 0.2^2\mathbf{I}\right),$$

t	1	2	3	4
$f_1(t)$	0.5	1	1.5	2
y	0.7	0.6	1.3	2.4

t	7	8	9	10
$f_2(t)$	2.33	1.67	1	0.33
y	2.1	1.8	0.9	0.5

In this case $T = 6$. Cf. also the following Figure 5.1.

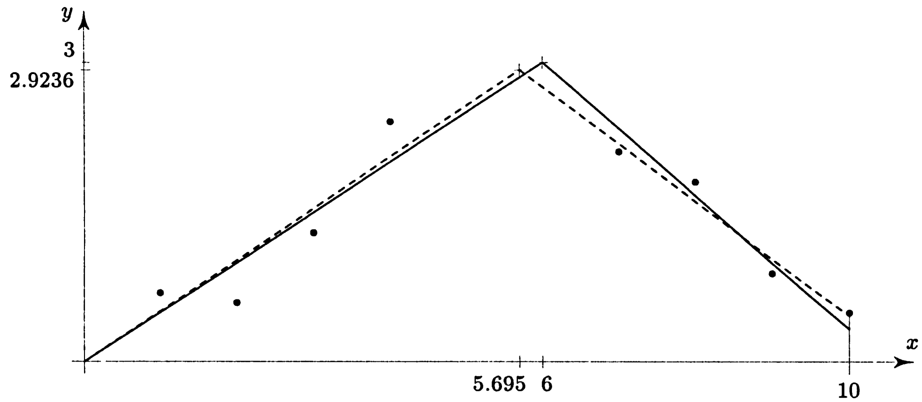

FIGURE 5.1.

With respect to the Section 2,

$$1-\alpha = 0.95\,, \quad A = 6.183\,, \quad B = 0.897\,, \quad [A{-}B, A{+}B] = [5.286, 7.080]$$

and

$$\hat{\hat{T}} = \frac{\hat{\beta}_2}{\hat{\beta}_1 - \hat{\beta}_3} = 5.695\,, \quad E(\hat{\hat{T}}) - T = 0.007\,, \quad \sqrt{\operatorname{Var}_1(\hat{\hat{T}})} = 0.43\,,$$

$$\left[\hat{\hat{T}} - 1.96\sqrt{\operatorname{Var}_1(\hat{\hat{T}})},\, \hat{\hat{T}} + 1.96\sqrt{\operatorname{Var}_1(\hat{\hat{T}})}\,\right] = [4.85, 6.54]$$

($\sqrt{\operatorname{Var}_2(\hat{\hat{T}})}$ is practically the same as $\sqrt{\operatorname{Var}_1(\hat{\hat{T}})}$).

Further (with respect to the Section 3)

$$\mathbf{H}_1 = (T_0, -1, -T_0) = (6, -1, -6)\,,$$
$$\mathbf{H}_2 = (\beta_{1,0} - \beta_{3,0}) = 0.5 - (-2/3) = 3.5/3\,,$$
$$\frac{1}{2}\omega(\Delta_1, \Delta_2, \delta) = (\Delta_1 - \Delta_3)\delta\,.$$

(For the sake of simplicity the approximate values are chosen as the actual ones.)

Thus $C^{(\mathrm{par})}_{II,\delta} = 0.116$ and the ellipsoid $\Delta'\mathbf{C}\Delta \leq 2\varepsilon/C^{(\mathrm{par})}_{II,\delta}$ ($\varepsilon = 0.5$) has the semiaxes equal to $A = 0.214$, $B = 0.033$, $C = 2.264$. Since semiaxes of the 0.95-confidence ellipsoid $\Delta'\mathbf{C}\Delta \leq 7.81$ ($= \chi^2_3(0.95)$) are $a = 0.204$, $b = 0.032$, $c = 2.157$, the linearization is possible, however, some caution is useful, since the confidence ellipsoid is almost so large as the linearization region. Nevertheless the values $\hat{\hat{T}} = T_0 - \mathbf{H}_2^{-1}\mathbf{H}_1\hat{\Delta} = 6 - 0.283 = 5.717$ and $\sqrt{\operatorname{Var}(\hat{\hat{\delta}})} = \sqrt{\left[\mathbf{H}_2'(\mathbf{H}_1\mathbf{C}^{-1}\mathbf{H}_1')^{-1}\mathbf{H}_2\right]^{-1}} = 0.43$ are in a good agreement with the values $\hat{\hat{T}} = 5.695$ and $\sqrt{\operatorname{Var}_1(\hat{\hat{T}})} = 0.43$.

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