Janina Ewert; Stanislav P. Ponomarev

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ON THE EXISTENCE OF ω-PRIMITIVES ON ARBITRARY METRIC SPACES

JANINA EWERT — STANISLAV P. PONOMAREV

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ABSTRACT. In this paper a final solution to the problem of the existence of ω-primitives on an arbitrary metric space (X, d) is given. Namely, it is shown that if $f\colon X \to [0, \infty]$ is an upper semicontinuous function, vanishing at each isolated point of X, then there exists a function $F\colon X \to \mathbb{R}$ whose oscillation equals f at each point of X. We call such a function F an ω-primitive for f. Moreover, an ω-primitive can always be found in at most Baire class 2.

1. Preliminaries and basic definitions

In the present paper we give a final solution to the problem on the existence of ω-primitives on arbitrary metric spaces. That problem was solved earlier in some particular cases (see e.g. [1], [3], [4], [5]). We will use the basic notations and definitions from the mentioned papers. But for completeness and reader's convenience, we will recall them, along with citing some results that will be needed.

Let $X = (X, d)$ be a metric space. For each $A \subset X$ we use the standard notations A^d, $\operatorname{Int} A$, $\operatorname{Fr} A$ to denote the derived set, interior and boundary of A respectively. We will also use the well-known notations $B(x, r) = \{z \in X : d(z, x) < r\}$, $\operatorname{dist}(x, E) = \inf\{d(x, y) : y \in E\}$ for an open ball and the distance from a point to a set.

It is convenient to define the oscillation of a real valued function via its *upper* and *lower Baire functions.* Namely, given a function $F\colon X \to \mathbb{R}$, we let for each $x \in X$

$$M(F, x) = \lim_{r \to 0}\Bigl(\sup\{F(z) : z \in B(x, r)\}\Bigr),$$

$$m(F, x) = \lim_{r \to 0}\Bigl(\inf\{F(z) : z \in B(x, r)\}\Bigr).$$

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Then the *oscillation* of F at the point x is defined as

$$\omega(F,x) = M(F,x) - m(F,x)$$

(we put $\infty - \infty = 0$ ([6])).

We abbreviate USC (resp. LSC) for an *upper semicontinuous* (resp. *lower semicontinuous*) function. Instead of (X,d), we will often write X for brevity.

Given a metric space (X,d), we let, for each set $A \subset X$ containing more than one point,

$$\Delta A = \inf\{d(x,y) : \ x,y \in A, \ \ x \neq y\},$$

for a singleton $\{x\}$ we let $\Delta\{x\} = +\infty$ and $\Delta\emptyset = +\infty$.

Then, as a particular case of [3; Theorem 1.2], we have the following result.

LEMMA 1. *Let (X,d) be a metric space.*

(a) *If $A \subset X$ and $\Delta A > 0$, then $A^d = \emptyset$ (so that A is closed and discrete). If, in addition, $X = X^d$, then A is nowhere dense.*

(b) *The space X contains a dense set D of the form $D = \bigcup_{n=1}^{\infty} D_n$, where the sets D_n are pairwise disjoint and $\Delta D_n > 0$ for $n \in \mathbb{N}$.*

(c) *Each σ-discrete set $A \subset X$ can be represented in the form $A = \bigcup_{n=1}^{\infty} A_n$, where A_n are pairwise disjoint and $\Delta A_n > 0$ for $n \in \mathbb{N}$.*

A metric space X is said to be ([1]):

- *σ-discrete at a point $x \in X$* if there is an open neighborhood V of x which is a σ-discrete set, i.e. $V = \bigcup_{n=1}^{\infty} E_n$ where E_n are discrete sets; empty set is considered as discrete;
- *massive* if it is not σ-discrete at any of its points.

Given any metric space X, we let

$$\Sigma(X) = \{x \in X : \ X \text{ is } \sigma\text{-discrete at } x\}.$$

It follows immediately from these definitions that

(i) The set $\Sigma(X)$ is open and $X \setminus X^d \subset \Sigma(X)$.

(ii) Each massive space is dense in itself.

THEOREM 1. *Let (X,d) be a metric space. Then*

(a) *$\Sigma(X)$ is a σ-discrete set;*

(b) *if $X \setminus \Sigma(X) \neq \emptyset$, then it is a massive subspace of X.*

Proof.

(a) Let $\mathcal{B}$ be a σ-discrete base of $\Sigma(X)$ ([2; p. 352]). Therefore $\mathcal{B}$ can be written in the form

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n ,$$

where $\mathcal{B}_n = \{G_{nt} : t \in T_n\}$, the sets G_{nt} being open, σ-discrete, and $G_{nt'} \cap G_{nt''} = \emptyset$ whenever $t' \neq t''$. For each $t \in T_n$ we have $G_{nt} = \bigcup_{i=1}^{\infty} E_{nti}$, where E_{nti} are discrete. It follows that the set $\bigcup_{t\in T_n} E_{nti}$ is discrete since the open sets G_{nt} are disjoint (with respect to $t \in T_n$).

Therefore the space

$$\Sigma(X) = \bigcup \mathcal{B} = \bigcup_{n=1}^{\infty} \bigcup \mathcal{B}_n = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{t\in T_n} E_{nti}$$

is σ-discrete too, what was to be shown.

(b) We will omit a trivial proof of the fact that if $X \setminus \Sigma(X) \neq \emptyset$, then this closed set is a massive subspace of X. □

Corollary 1. *A metric space X is σ-discrete if and only if it is σ-discrete at each of its points. In that case we have $\Sigma(X) = X$.*

Corollary 2. *The subspace $\Sigma(X)$ is dense in itself if and only if X is so.*

Corollary 3. *If a metric space X is not σ-discrete, then it can be written in the form*

$$X = \Sigma(X) \cup \mathcal{M}(X) ,$$

where $\Sigma(X) \cap \mathcal{M}(X) = \emptyset$ and $\Sigma(X)$, $\mathcal{M}(X)$ are respectively σ-discrete and massive subspaces of X. Such a decomposition of X is unique.

We will use, in the subsequent proofs, the following results concerning the existence of ω-primitives in the case of massive spaces and in the case of σ-discrete spaces dense in themselves.

Lemma 2. ([1; Theorem 1]) *Let (X,d) be a massive metric space and $f\colon X \to [0,\infty)$ an upper semicontinuous function. Then there exists an ω-primitive $F\colon X \to [0,\infty)$ for f such that*

(a) $0 \le F \le f$;

(b) $M(F,x) = f(x)$ *and* $m(F,x) = 0$ *for each* $x \in X$;

(c) F *is in at most Baire class* 2.

LEMMA 3. ([4; Theorem 1]) *Let (X,d) be a σ-discrete dense in itself metric space. If $f\colon X \to [0,\infty)$ is a USC function and $g\colon X \to (0,\infty)$ an LSC function, then there exist a dense set $S \subset X$ and a function $F\colon X \to \mathbb{R}$, at most in Baire class 1, such that $-g < F \le f$, $F|_S \le 0$, and $\omega(F,\cdot) = f$. The ω-primitive F is given by the formula*

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \setminus S\,, \\ -f(x) + \limsup\limits_{z \to x} f(z) & \text{if } x \in S\,. \end{cases}$$

2. Main results

In this section we will show that the problem of the existence of ω-primitives is always solvable for any metric space.

First we study the case when X is an arbitrary σ-discrete metric space, not necessarily dense in itself.

THEOREM 2. *Let (X,d) be a σ-discrete metric space, and $f\colon X \to [0,\infty)$ a USC function which vanishes on the set $X \setminus X^d$. Then given any LSC function $g\colon X \to (0,\infty)$, there exists a dense set $S \subset X$ and a function $F\colon X \to \mathbb{R}$, at most Baire 1, such that $-g < F \le f$, $F|_S \le 0$, and $\omega(F,\cdot) = f$.*

P r o o f . We will reduce the assumptions of the theorem to the assumptions of Lemma 3, by taking a cartesian product. We also assume that $X \setminus X^d \ne \emptyset$, for otherwise, there would be nothing to prove in view of Lemma 3. Define

$$\widehat{X} = X^d \times \{0\} \cup (X \setminus X^d) \times I_q\,,$$

where I_q denotes the set of all rational numbers in the interval $[0,1]$. We consider the set $\widehat{X}$ as the subspace of the metric space $X \times I_q$ with the metric $\widehat{d}\big((x',t'),(x'',t'')\big) = \max\big(d(x',x''),|t'-t''|\big)$. It is obvious that the cartesian product of two σ-discrete spaces is again σ-discrete. So $\widehat{X}$ is σ-discrete and, what is also important, dense in itself. Now define the functions $\widehat{f}\colon \widehat{X} \to [0,\infty)$ and $\widehat{g}\colon \widehat{X} \to (0,\infty)$ letting $\widehat{f}(x,t) = f(x)$, $\widehat{g}(x,t) = g(x)$. It is easily checked that $\widehat{f}$ is USC whereas $\widehat{g}$ is LSC. Thus $\widehat{X}$, $\widehat{f}$, $\widehat{g}$ satisfy all assumptions of Lemma 3. Let $\widehat{F}\colon \widehat{X} \to \mathbb{R}$ be an ω-primitive for $\widehat{f}$ constructed by that lemma. We have that $-\widehat{g} < \widehat{F} \le \widehat{f}$. There also exists a set $\widehat{S}$, dense in $\widehat{X}$, such that $\widehat{F}|_{\widehat{S}} \le 0$.

Next observe that for each $x \in X \setminus X^d$ the set $\{x\} \times I_q$ is an open subset of the space $\widehat{X}$. Moreover, since $\widehat{f}$ vanishes on $\{x\} \times I_q$, we have, by (1) of Lemma 3, that $\widehat{F}$ also vanishes on that set.

Define $F\colon X \to \mathbb{R}$ letting $F(x) = \widehat{F}(x,0)$. We claim that F is the required ω-primitive for f. Indeed, we have for each $r > 0$ and each $x \in X$

$$\widehat{F}\big(B((x,0),r)\big) = F\big(B(x,r)\big),$$

where $B\big((x,0),r\big)$ is the open ball in $\widehat{X}$, centered at $(x,0)$ and of radius r. It readily follows that for each $x \in X$

$$\omega(F,x) = \omega\big(\widehat{F},(x,0)\big) = \widehat{f}(x,0) = f(x).$$

Now let $p\colon X \times I_q \to X$ be the natural projection. Since $\widehat{S}$ is dense in $\widehat{X}$, the set $S = p(\widehat{S})$ is dense in X. It remains to show that $F|_S \le 0$. First note that $X \setminus X^d \subset S$. Indeed, $\widehat{S}$ is dense in $\widehat{X}$ whereas $\widehat{F}$ vanishes on each set $\{x\} \times I_q$, $x \in X \setminus X^d$, which, as it has been already observed above, is open in $\widehat{X}$. Hence F vanishes on $X \setminus X^d$. Now if $x \in S \cap X^d$ we obviously have that $(x,0) \in \widehat{S}$. Therefore $F(x) = \widehat{F}(x,0) \le 0$. Thus $F|_S \le 0$. As for the Baire class of F, it was shown in [4] that each function on a σ-discrete metric space is at most Baire 1. □

Remark. Note that given an arbitrary metric space X, the assumption that a USC function $f\colon X \to [0,\infty)$ vanishes at each isolated point of X is obviously necessary for the existence of an ω-primitive for f. Theorem 2 thus shows that the assumption is sufficient in the case of σ-discrete spaces. It will be proved in Theorem 3 and Theorem 4 that this assumption is also sufficient for any metric space.

THEOREM 3. *Let (X,d) be an arbitrary metric space and $f\colon X \to [0,\infty)$ a USC function which vanishes on $X \setminus X^d$. Then for each LSC function $g\colon X \to (0,\infty)$, there exists a function $F\colon X \to \mathbb{R}$, at most in Baire class 2, such that $\omega(F,\cdot) = f$ and $-g < F \le f$.*

Proof. By Corollary 3, we have the decomposition $X = \Sigma(X) \cup \mathcal{M}(X)$ where $\Sigma(X)$ is σ-discrete and $\mathcal{M}(X)$ is massive. We assume that neither of these sets is empty, for otherwise, the proof is immediate by Lemma 2 or Theorem 2. Define the function $g_1 : X \to [0,\infty)$ letting $g_1(x) = \min\big(g(x), \operatorname{dist}(x, \mathcal{M}(X))\big)$. Obviously g_1 is LSC and $g_1|_{\Sigma(X)} > 0$. By Theorem 2, there exist an ω-primitive $F_1\colon \Sigma(X) \to \mathbb{R}$ for $f|_{\Sigma(X)}$ and a set $S \subset \Sigma(X)$, dense in $\Sigma(X)$, such that $-g_1|_{\Sigma(X)} < F_1 \le f|_{\Sigma(X)}$ and $F_1|_S \le 0$.

By Lemma 2, there exists an ω-primitive $F_2\colon \mathcal{M}(X) \to [0,\infty)$ such that $F_2 \le f|_{\mathcal{M}(X)}$, $M(F_2,\cdot) = f|_{\mathcal{M}(X)}$ and $m(F_2,\cdot) = 0$ (note that here the Baire functions M and m are considered, of course, on the subspace $\mathcal{M}(X)$).

Now define the function $F\colon X \to \mathbb{R}$ letting

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in \Sigma(X)\,, \\ F_2(x) & \text{if } x \in \mathcal{M}(X)\,. \end{cases}$$

It is immediate that F is at most in Baire class 2 and $-g < F \le f$. Since $\Sigma(X)$ is open and disjoint from $\mathcal{M}(X)$, it is clear that to prove that F is an ω-primitive for f, it suffices only to show that $\omega(F,x) = f(x)$ holds for $x \in \operatorname{Fr}\mathcal{M}(X)$. Let $x_0 \in \operatorname{Fr}\mathcal{M}(X)$. Since $F \le f$ and $M(F_2,x_0) = f(x_0)$, we may write at once that $f(x_0) = M(F_2,x_0) \le M(F,x_0) \le M(f,x_0) = f(x_0)$, i.e. $M(F,x_0) = f(x_0)$. On the other hand, recalling the properties of F_2, we have for each $r > 0$ and each $x \in B(x_0,r) \cap S$ that

$$-r \le -\operatorname{dist}\big(x,\mathcal{M}(X)\big) \le -g_1(x) < F_1(x) = F(x) \le 0\,.$$

It follows, since $F_2 \ge 0$, that $-r \le \inf F|_{B(x_0,r)} \le 0$. Whence, letting $r \to 0$, we get $m(F,x_0) = 0$. Therefore $\omega(F,x_0) = M(F,x_0) - m(F,x_0) = f(x_0)$, which completes the proof. □ □

Now we are in a position to prove our main result (Theorem 4). The only point that differs it from Theorem 3, is that now a USC function f may take the value $+\infty$. We will use the scheme much like the proof of [1; Theorem 1′] or [3; Theorem 2], modulo some modifications due to the fact that this time we deal with metric spaces which might contain isolated points.

THEOREM 4. *Let (X,d) be an arbitrary metric space and $f\colon X \to [0,\infty]$ a USC function which vanishes on $X \setminus X^d$. Then for each LSC function $g\colon X \to (0,\infty)$ there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$ and $-g < F \le f$. Such a function F can always be found in at most Baire class 2.*

P r o o f . Let $E = \{x \in X :\ f(x) = +\infty\}$. Clearly, E is closed. We have the decomposition of X into disjoint subspaces

$$X = (X \setminus E) \cup \operatorname{Int} E \cup \operatorname{Fr} E\,.$$

Without loss of generality we may assume, of course, that none of the terms in the above decomposition is empty. By Theorem 3, there exists an ω-primitive $F_1\colon X \setminus E \to \mathbb{R}$ for $f|_{(X \setminus E)}$, at most in Baire class 2, such that

$$-g|_{(X \setminus E)} < F_1 \le f|_{(X \setminus E)}\,.$$

Since $f|_{(X \setminus X^d)} = 0$, we have $E \subset X^d$. It follows that $(X \setminus X^d) \cap \operatorname{Int} E = \emptyset$. Therefore the open subset $\operatorname{Int} E$ is dense in itself. By Lemma 1(**b**), there exists a set $D = \bigcup_{k=1}^{\infty} D_k$, dense in $\operatorname{Int} E$, where D_k are closed and pairwise disjoint.

Moreover, by (**b**) of Lemma 1, we have that the set $\bigcup_{k=n}^{\infty} D_k$ is dense in $\operatorname{Int} E$ for each $n \in \mathbb{N}$.

Define the function $F_2\colon \operatorname{Int} E \to [0,\infty)$ letting

$$F_2(x) = \begin{cases} k & \text{if } x \in D_k\,, \quad k \in \mathbb{N}\,, \\ 0 & \text{if } x \in \operatorname{Int} E \setminus D\,. \end{cases}$$

So F_2 is a Baire 2 function, such that for each $x \in \operatorname{Int} E$ we have $\omega(F_2, x) = +\infty$.

Next let us consider the continuous function $T\colon X \setminus E \to (0,\infty)$ defined by $T(x) = \big(\operatorname{dist}(x,E)\big)^{-1}$.

Finally, define $F\colon X \to \mathbb{R}$ as follows.

$$F(x) = \begin{cases} F_1(x) + T(x) & \text{if } x \in X \setminus E\,, \\ F_2(x) & \text{if } x \in \operatorname{Int} E\,, \\ 0 & \text{if } x \in \operatorname{Fr} E\,. \end{cases}$$

In much the same way as in [1], [3] or [4], it could be easily checked that the function F is indeed an ω-primitive for f on X satisfying the required properties. □

REFERENCES

[1] DUSZYŃSKI, Z.—GRANDE, Z.—PONOMAREV, S. P.: *On the ω-primitive*, Math. Slovaca **51** (2001), 469–476.

[2] ENGELKING, R.: *General Topology*, Warszawa, PWN, 1977.

[3] EWERT, J.—PONOMAREV, S. P.: *Oscillation and ω-primitives*, Real Anal. Exchange **26** (2000/2001), 687–702.

[4] EWERT, J.—PONOMAREV, S. P.: *ω-Primitives on σ-discrete metric spaces*, Tatra Mt. Math. Publ. **24** (2002), 13–27.

[5] KOSTYRKO, P.: *Some properties of oscillation*, Math. Slovaca **30** (1980), 157–162.

[6] ŁOJASIEWICZ, S.: *An Introduction to the Theory of Real Functions*, Warszawa, PWN, 1973. (Polish)

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Institute of Mathematics
Pedagogical University
Arciszewskiego 22b
PL–76-200 Slupsk
POLAND

E-mail: stapon@o2.pl