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COMPRESSIBLE GROUPS

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ABSTRACT. We introduce and initiate a study of a new class of partially ordered abelian groups called *compressible groups*. The compressible groups generalize the order-unit space of self-adjoint operators on Hilbert space, the directed additive group of self-adjoint elements of a unital C*-algebra, lattice-ordered abelian groups with order unit, and interpolation groups with order unit. We identify elements called *projections* in a compressible group, show that the set P of projections forms an orthomodular poset, and give sufficient conditions, satisfied in a Rickart C*-algebra and in an interpolation group with order unit, for P to form an orthomodular lattice.

1. Introduction

In [7], the authors propose that a certain class of directed abelian groups with order units might serve as a unifying framework for the study of the mathematical and philosophical foundations of the experimental sciences.

The directed abelian group associated with orthodox quantum mechanics is the additive group $\mathbb{G}(\mathcal{H})$ of all bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, partially ordered as usual, and with the identity operator $\mathbf{1}$ as the order unit. The directed abelian group associated with a classical mechanical system is the additive group $\mathcal{F}(\Xi)$ of all bounded Borel-measurable $\mathbb{R}$-valued functions on the phase space Ξ of the system, partially ordered pointwise, and with the constant function 1 as the order unit.

Let G and H be directed abelian groups with order units u and v, respectively. A mapping $\Phi\colon G \to H$ is a *unital morphism* if and only if it is an order-preserving group homomorphism and $\Phi(u) = v$. If the system $\mathbb{R}$ of real numbers is regarded as a directed abelian group under addition with 1 as the order unit,

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then a *state* for G is a unital morphism $\omega\colon G \to \mathbb{R}$. An *observable* on G is a unital morphism $\Phi\colon \mathcal{F}(\mathbb{R}) \to G$. If ω is a state for G and Φ is an observable on G, then the composition $\omega \circ \Phi\colon \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ determines a *probability measure* μ on the field $\mathcal{M}$ of Borel subsets of $\mathbb{R}$ according to $\mu(M) := (\omega \circ \Phi)(\chi_M)$, where χ_M is the characteristic set function of $M \in \mathcal{M}$. A *symmetry* of G is a bijective unital morphism $\alpha\colon G \to G$ such that $\alpha^{-1}\colon G \to G$ is also a unital morphism. In this way, four of the fundamental ingredients of any viable approach to the foundations of experimental science — *states, observables, probability distributions*, and *symmetries* — have natural and compelling mathematical representations in the context of directed abelian groups with order units.

Suppose that G is a directed abelian group with order unit u and that the states, observables, probability distributions, and symmetries for a physical system $\mathcal{S}$ are represented on G. Then elements of the interval $E := \{e \in G : 0 \le e \le u\}$ can be regarded as logical entities, called *effects*, that are associated with measurements on the system $\mathcal{S}$. According to [12], an effect is supposed to correspond to "an elementary yes-no measurement that may be unsharp or imprecise". The set E, partially ordered by the restriction of the partial order on G, and with the partially defined binary operation $\oplus$ obtained by restriction of $+$ on G to E, forms a logical structure called an *effect algebra* ([2]).

For the directed abelian group $\mathbb{G}(\mathcal{H})$ over the Hilbert space $\mathcal{H}$, the effect algebra $\mathbb{E}(\mathcal{H})$, which consists of all self-adjoint operators between $\mathbf{0}$ and $\mathbf{1}$, contains the orthomodular lattice $\mathbb{P}(\mathcal{H})$ of projection operators on $\mathcal{H}$ as a sub-effect algebra. The projection operators can be regarded as effects that are "sharp" or "non fuzzy". Analogously, for the directed abelian group $\mathcal{F}(\Xi)$ over the phase space Ξ, the effect algebra $\mathcal{E}(\Xi)$, which consists of all Borel-measurable functions $f\colon \Xi \to \mathbb{R}$ such that $f(\Xi) \subseteq [0,1]$, contains as a sub-effect algebra the Boolean algebra $\mathcal{P}(\Xi)$ of all characteristic set functions χ_M of Borel-measurable subsets of Ξ. The functions χ_M can be regarded as effects that are "sharp" or "non-fuzzy".

For an arbitrary directed abelian group G with order unit u, one would like to be able to identify a sub-effect algebra P of the effect algebra $E := \{e \in G : 0 \le e \le u\}$ that could reasonably be regarded as consisting of the "non-fuzzy" effects. So far, in spite of some progress [11], [14], this goal has proved to be elusive, probably because the class of directed abelian groups with order unit is large and diverse, and there is no reason to expect that all members of this class are suitable for the representation of the states, observables, and symmetries of a physical system.

In this article, we introduce a class of directed abelian groups with order units for which certain sharp effects called *projections* are easily identified and behave as expected. We call these groups "compressible" because such a group admits a well-behaved set of so-called *compression operators* that are in bijective correspondence with the projections. The directed groups $\mathbb{G}(\mathcal{H})$ and $\mathcal{F}(\Xi)$ are compressible groups. So is every lattice-ordered group with order unit, and indeed, every interpolation group with order unit. We are not suggesting that the compressible groups are the only groups fit for the proper representation of states, observables, probabilities, and symmetries, or even that every compressible group satisfies this desideratum. We do propose that a better understanding of compressible groups will contribute to the identification of an appropriate class of directed groups that are pertinent to the study of the foundations of the experimental sciences.

2. Partially ordered abelian groups

In this section, we offer a brief review of some mathematical notions that we shall be using.

If G is a partially ordered abelian group, we denote the positive cone in G by

$$G^+ := \{g \in G : \ 0 \leq g\}.$$

A *directed abelian group* is a partially ordered abelian group G such that every element $g \in G$ can be written as $g = a - b$ with $a, b \in G^+$ ([9; p. 4]). Since we shall be dealing only with partially ordered abelian groups, we shall refer to a directed abelian group simply as a *directed group*.

Let G be a partially ordered abelian group and let H be a subgroup of G. The partial order on H obtained by restriction to H of the partial order on G is called the *induced partial order* on H. Under the induced partial order, H forms a partially ordered abelian group with the positive cone $H^+ = H \cap G^+$, but, even if G is directed, H need not be directed. If, whenever $h_1, h_2 \in H$ and $g \in G$, the condition $h_1 \leq g \leq h_2$ implies that $g \in H$, then H is said to be *order convex* in G. It is easy to see that H is order convex in G if and only if, for all $g \in G$, $0 \leq g \leq h \in H \implies g \in H$. If H is directed and order convex in G, then H is called an *ideal* in G ([9; p. 8]).

An element $u \in G^+$ is called an *order unit* if and only if each element in G is dominated by some positive-integer multiple of u ([9; p. 4]). If, for all $a, b \in G$, the condition that $na \leq b$ for all positive integers n implies that $-a \in G^+$, then G is called *archimedean* ([9; p. 20]). If $u \in G^+$, we define the *interval*

$$G^+[0, u] := \{e \in G : \ 0 \leq e \leq u\}$$

and regard $G^+[0, u]$ as a bounded partially ordered set under the restriction of the partial order $\leq$ on G. A subset $E \subseteq G^+$ is said to be *cone generating* if and only if every nonzero element in G^+ is the sum of a finite sequence of (not necessarily distinct) elements of E. If $u \in G^+$ and the interval $E := G^+[0, u]$ is cone generating, then u is called a *generative* element of G^+ ([2]).

A *unital group* is a directed group G with a specified generative element $u \in G^+$ called the *unit* of G ([6]). The unit u in a unital group G is automatically an order unit in G. If G is a unital group with unit u, then the bounded partially ordered set $E := G^+[0,u]$ is called the *unit interval* in G. The mapping $e \mapsto u-e$ is an order-reversing involution on the unit interval E. With the partially defined binary operation $\oplus$ on E obtained by restriction of $+$ on G to E, the unit interval E forms a so-called *effect algebra*. For the details, see [2], [4], [8], [10], [17].

If G is a unital group with unit interval E, then an element $p \in E$ is *principal* if and only if

$$(\forall e_1, e_2)\big((e_1, e_2, e_1{+}e_2 \in E \quad \& \quad e_1, e_2 \leq p) \implies e_1 + e_2 \leq p\big)$$

([8; Definition 3.2]). If both p and $u-p$ are principal elements of E, and every element $e \in E$ can be written in the form $e = e_1{+}e_2$ with $e_1 \leq p$ and $e_2 \leq u{-}p$, then p is called a *central* element of E ([8; Definition 5.1]). The set of all central elements of E forms a Boolean algebra ([8; Theorem 5.4]). An element $p \in E$ is *sharp* if and only if the only element $q \in E$ such that $q \leq p$ and $q \leq u-p$ is $q = 0$ ([10]). Every principal element of E is sharp ([8; Lemma 3.3]).

A mapping $\psi\colon E \to K$ from the unit interval E of a unital group G to an abelian group K is called a *K-valued measure* if and only if

$$(\forall e_1, e_2)\big(e_1, e_2, e_1{+}e_2 \in E \implies \psi(e_1 + e_2) = \psi(e_1) + \psi(e_2)\big).$$

If each K-valued measure $\psi\colon E \to K$ can be extended to a group homomorphism $\Psi\colon G \to K$, then G is called a *K-unital group.* If G is a K-unital group for every abelian group K, then G is called a *unigroup* ([7]).

Let G be a partially ordered abelian group. If, whenever $a, b, c, d \in G$ with $a, b \leq c, d$ (i.e., $a \leq c$, $a \leq d$, $b \leq c$, and $b \leq d$) there exists $t \in G$ with $a, b \leq t \leq c, d$, then G is said to have the *interpolation property* and G is called an *interpolation group* ([9]). In an interpolation group, every order unit is automatically generative. If, as a partially ordered set, G is a lattice (i.e., the greatest lower bound $g \wedge h$ and the least upper bound $g \vee h$ of g and h exist for every $g, h \in G$), then G is said to be *lattice ordered.* Every totally-ordered group is a lattice-ordered group, and every lattice-ordered group is an interpolation group. A unital interpolation group, i.e., a unital group with the interpolation property, is automatically a unigroup ([17]).

The ordered field $\mathbb{R}$ of real numbers with the standard positive cone $\mathbb{R}^+ = \{x^2 : x \in \mathbb{R}\}$, regarded as a totally ordered group under addition, is an archimedean unigroup with unit 1, and the corresponding unit interval is the standard unit interval $[0,1] \subseteq \mathbb{R}^+$. The ordered integral domain $\mathbb{Z}$ of integers with the standard positive cone $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$, regarded as a totally ordered group under addition, is an archimedean unigroup with unit 1, and the corresponding unit interval is the two-element set $\{0, 1\}$.

The following example will motivate and illustrate much of the subsequent development.

2.1. EXAMPLE. Let A be a C*-algebra with unit 1 and let $G(A)$ be the additive abelian group of self-adjoint elements of A. Then $G(A)$ is organized into a directed group with positive cone $G(A)^+ := \{aa^* : a \in A\}$ and, with 1 as the unit, $G(A)$ forms an archimedean unigroup. Elements e in the unit interval $E(A) := G(A)^+[0,1]$ are the *effects* and idempotent elements $p = p^2 \in G^+$ are the *projections* in A. The set $P(A)$ of all projections in A is a sub-effect algebra of $E(A)$ and, as an effect algebra in its own right, $P(A)$ forms an orthomodular poset. An element of $E(A)$ is principal if and only if it is sharp if and only if it is a projection [8; Theorem 6.8].

Each projection $p \in P(A)$ gives rise to a mapping $a \mapsto pap$ from A to A called the *compression* determined by p. If $g \in G(A)$, then pgp is self-adjoint, so the compression determined by p maps the unital group $G(A)$ into itself. Denote by $J_p\colon G(A) \to G(A)$ the restriction to $G(A)$ of the compression determined by p, so that $J_p(g) = pgp$ for all $g \in G(A)$. As $1 \in G(A)$ and $J_p(1) = p$, the compression $a \mapsto pap$ on A is determined by J_p. We shall refer to J_p as the *compression on* $G(A)$ determined by $p \in P(A)$.

In Example 2.1, the compression J_p has the following properties:

(1) $J_p\colon G(A) \to G(A)$ is an order-preserving group endomorphism on $G(A)$,
(2) $p = J_p(1) \leq 1$,
(3) for all $e \in G(A)^+$, $e \leq p \implies e = J_p(e)$.

Properties (1), (2), and (3) make sense in any unital group, thus suggesting a generalized notion of compression called a *retraction* ([5]).

2.2. DEFINITION. Let G be a unital group with unit u and unit interval E. Then a mapping $J\colon G \to G$ is called a *retraction* on G if and only if the following conditions hold:

(i) $J\colon G \to G$ is an order-preserving group homomorphism.
(ii) $J(u) \leq u$.
(iii) If $e \in G^+$, then $e \leq J(u) \implies e = J(e)$.

If J is a retraction on G, then $J(u)$ is called the *focus* of J.

The following basic properties of a retraction were proved in [5].

2.3. LEMMA. *Let G be a unital group with unit u and unit interval E, and let J be a retraction on G with focus p. Then:*

(i) *J is idempotent, i.e., $J(J(g)) = J(g)$ for all $g \in G$.*

(ii) *If $e \in E$, then $e \leq u - p \implies J(e) = 0$.*

(iii) *The focus p of J is a principal element of E, hence p is a sharp element of E.*

2.4. LEMMA. *Let J be a retraction on the unital group G with unit u, let $p := J(u)$ be the focus of J, and let $H := J(G)$ be the image of J. Then, $H = \{h \in G : J(h) = h\}$ and, with the induced partial order, H is a unital group with unit p. Furthermore, if I is a retraction on the unital group H, then the composition $I \circ J$ is a retraction on G with focus $I(p)$.*

Proof. That $H = \{h \in G : J(h) = h\}$ follows from Lemma 2.3(i). Suppose $h \in H$. Then there exist $g_1, g_2 \in G^+$ such that $h = g_1 - g_2$, and we have $h = J(h) = J(g_1) - J(g_2)$ with $J(g_1), J(g_2) \in H \cap G^+ = H^+$, so H is directed. Evidently, $p = J(u) \in H^+$. Suppose $h \in H^+$. Then $h \in G^+$, so there exist $e_i \in G$ with $0 \leq e_i \leq u$ for $i = 1, 2, \ldots, n$ such that $h = \sum_{i=1}^{n} e_i$, whence $h = J(h) = \sum_{i=1}^{n} J(e_i)$. But, $0 \leq e_i \leq u$ implies that $0 \leq J(e_i) \leq J(u) = p$ for $i = 1, 2, \ldots, n$, and it follows that p is a generative element of H. Therefore, H is a unital group with unit p.

Let I be a retraction on H with focus $q := I(p)$. Then, regarded as a mapping from G to G, $I \circ J$ is an order-preserving group endomorphism. Also, $(I \circ J)(u) = I(p) = q \leq p \leq u$. If $e \in G$ with $0 \leq e \leq q$, then $0 \leq e \leq p$, so $J(e) = e$ and $e \in H$. Therefore, $(I \circ J)(e) = I(J(e)) = I(e) = e$. □

Let J be a retraction on the unital group G, let $H := J(G)$ be the image of J, and let $K := \ker(J) = J^{-1}(0)$ be the kernel of J. Define $\Phi\colon H \times K \to G$ by $\Phi(h, k) := h + k$ for all $h \in H$, $k \in K$. Because J is idempotent, Φ is a group isomorphism and $\Phi^{-1}(g) = (J(g), g - J(g))$ for all $g \in G$. Thus, *as an abelian group*, G is the direct product of H and K. We regard the subgroups $H = J(G)$ and $K = \ker(J)$ of G as partially ordered abelian groups with the induced partial orders. Then H is directed (Lemma 2.4), but it need not be order convex in G. On the other hand, K (being the kernel of an order-preserving group homomorphism) is order convex in G, but it need not be directed. If $H \times K$ is organized into a partially ordered abelian group with coordinatewise partial order, then $\Phi\colon H \times K \to G$ is order preserving, but $\Phi^{-1}\colon G \to H \times K$ need not be order preserving, i.e., G is not necessarily the direct product of H and K *in the category of partially ordered abelian groups*. A necessary and sufficient condition for Φ^{-1} to be order preserving is that the retraction J is *direct* in the sense of part (i) of the following definition.

2.5. DEFINITION. Let G be a unital group with unit interval E, let J be a retraction on G with $K := \ker(J)$, and organize K into a partially ordered abelian group with positive cone $K^+ := K \cap G^+$. Then:

(i) J is a *direct* retraction if and only if $g \in G^+ \implies J(g) \leq g$.
(ii) J is a *compression* if and only if $e \in E \cap K \implies e + J(u) \in E$.

If a retraction J on G is direct, then it is a compression and both $H = J(G)$ and $K = \ker(J)$ are ideals in G ([5]). Even if both H and K are ideals in G, the retraction J need not be a compression and it need not be direct.

2.6. LEMMA. *Let G be a unital group with unit interval E and let J be a retraction on G. Then J is direct if and only if $e \in E \implies J(e) \leq e$.*

Proof. Suppose $e \in E \implies J(e) \leq e$ and let $g \in G^+$. Then we can write $g = \sum_{i=1}^{n} e_i$ with $e_i \in E$ for $i = 1, 2, \ldots, n$, and it follows that $J(g) = \sum_{i=1}^{n} J(e_i) \leq \sum_{i=1}^{n} e_i = g$, so J is direct. As $E \subseteq G^+$, the converse is obvious. □

The following theorem pertaining to Example 2.1 is a consequence of [5].

2.7. THEOREM. *If A is a C*-algebra with unit 1, then every retraction on $G(A)$ is a compression and has the form $g \mapsto J_p(g) = pgp$ for $p \in P(A)$, $g \in G$. Thus, there is a bijective correspondence $p \leftrightarrow J_p$ between projections $p \in P(A)$ and compressions J_p on $G(A)$.*

3. Compressible groups

In Theorem 2.7, if $p \in P(A)$, then $u-p \in P(A)$ and the compressions J_p and J_{u-p} are *quasicomplementary* in the sense of the following definition (cf. [1]).

3.1. DEFINITION. Let J, I be retractions on the unital group G. Then J and I are *quasicomplementary* if and only if, for all $g \in G^+$, $J(g) = g \iff I(g) = 0$ and $J(g) = 0 \iff I(g) = g$.

If J and I are quasicomplementary retractions, we say that I is a *quasicomplement* of J and that J is a *quasicomplement* of I.

3.2. LEMMA. *Let G be a unital group with unit u and unit interval E. Suppose that J and I are quasicomplementary retractions on G. Then:*

(i) *If $p = J(u)$ is the focus of J, then the focus of I is $u - p$.*
(ii) *The images $J(G)$ and $I(G)$ of J and I are ideals in G.*
(iii) *J and I are compressions.*

Proof.

(i) We have $0 \leq u - p$ with $J(u) = p$ and $J(p) = p$, whence $J(u-p) = 0$, so $I(u) - I(p) = I(u-p) = u - p$. But, $0 \leq p$ with $J(p) = p$, so $I(p) = 0$, and it follows that $I(u) = u - p$.

(ii) Let $H := J(G)$. By Lemma 2.4, H is directed, so we have only to prove that H is order convex in G. Assume that $g \in G$ and $h \in H$ with $0 \leq g \leq h$. Since $0 \leq h$ and $J(h) = h$, we have $0 \leq I(g) \leq I(h) = 0$, whence $I(g) = 0$. Therefore, $g = J(g) \in H$ and H is order convex in G. By symmetry, $I(G)$ is order convex in G.

(iii) Let $p = J(u)$, so $u - p = I(u)$ by (i). Suppose $e \in E$ with $J(e) = 0$. Then $I(e) = e$ and since $0 \leq e \leq u$, we have $0 \leq e = I(e) \leq I(u) = u - p$. Thus, J is a compression, and by symmetry, so is I. □

The unital group $G(A)$ in Theorem 2.7 satisfies the conditions in the following definition.

3.3. Definition. A *compressible group* is a unital group G such that every retraction on G is uniquely determined by its focus and every retraction on G has a quasicomplement. Let G be a compressible group with unit u. An element $p \in G$ is called a *projection* if and only if it is the focus $p = J(u)$ of a retraction J on G. The set of all projections in G is denoted by $P(G)$. If $p \in P(G)$, we denote by J_p the unique retraction on G with focus p, so that $J_p(u) = p$.

3.4. Lemma. *Let G be a compressible group with unit u and let $p \in P(G)$. Then every retraction on G is a compression and the compression J_p has a unique quasicomplement, namely J_{u-p}.*

Proof. Assume the hypotheses. By Lemma 3.2(iii), every retraction on G is a compression. Let I be a compression on G, and suppose that I is a quasicomplement of J_p. Then $0 \leq p, u-p \leq u$ and by Lemma 3.2(i), I has focus $u - p$, so $u - p \in P(G)$ and $I = J_{u-p}$. □

The terminology "compressible group" is suggested by the notion that the compressions on such a group are particularly well-behaved. The conditions in Definition 3.3 are fairly strong. For instance, if G is a compressible group and $p \in P(G)$, then a unital automorphism of G fixes p if and only if it commutes with J_p. Also, if $p \in P(G)$ and p is an atom in the unit interval E of G, then there is one and only one $\mathbb{Z}$-valued state ω on G with $\omega(p) = 1$.

The archimedean unigroups affiliated with unital C*-algebras (Example 2.1 and Theorem 2.7) provide prototypic examples of compressible groups. Another important class of compressible groups is afforded by the following theorem.

3.5. THEOREM. *Let G be an interpolation group and let u be an order unit in G. Then G is a unigroup with unit u and, as such, G is a compressible group. If E is the unit interval in G, then $P(G)$ is the set of sharp elements of E, $P(G)$ is a sub-effect algebra of E, and $P(G)$ forms a Boolean algebra in such a way that, for $p \in P(G)$, $u-p$ is the Boolean complement of p. Furthermore, if $g \in G$ and $p \in P(G)$, then $g = J_p(g) + J_{u-p}(g)$.*

Proof. By [17; Theorem 3], G is a unigroup with unit u. Following [9; p. 127], we say that an element $p \in E$ is *characteristic* if and only if the greatest lower bound $p \wedge (u-p)$ of p and $u-p$ exists in G and $p \wedge (u-p) = 0$. Evidently, a characteristic element is sharp. Conversely, suppose p is a sharp element of E, i.e., 0 is the greatest lower bound of p and $u-p$ as calculated in E. Suppose $g \in G$ and $g \le p, u-p$. Then $g, 0 \le p, u-p$ and, as G is an interpolation group, there exists $t \in G$ with $g, 0 \le t \le p, u-p$, whence $t = 0$ and $g \le t \le 0$. Therefore 0 is the greatest lower bound of p and $u-p$ as calculated in G. Consequently, the characteristic elements in E coincide with the sharp elements in E.

Let p be a characteristic element of E, let

$$H := \{h \in G : \ (\exists\, n \in \mathbb{Z}^+)(-np \le h \le np)\},$$

and let

$$K := \{k \in G : \ (\exists\, n \in \mathbb{Z}^+)\big(-n(u-p) \le k \le n(u-p)\big)\}.$$

By [9; Lemma 8.2], H and K are ideals in G, $G = H + K$, and $H \cap K = \{0\}$. Let J_p be the projection of G onto H with kernel K and let J_{u-p} be the projection of G onto K with kernel H. Then J_p and J_{u-p} are quasicomplementary retractions on G and $g = J_p(g) + J_{u-p}(g)$ for all $g \in G$ ([9; p. 128]).

Suppose J is a retraction on G and let $p := J(u)$. By Lemma 2.3(iii), p is a sharp element of E, hence p is characteristic and we can form the retraction J_p. Suppose $h \in H^+$. Then there exists $n \in \mathbb{Z}^+$ such that $0 \le h \le np$. By [9; Proposition 2.2(b)], there are elements $e_i \in E$ with $e_i \le p$ for $i = 1, 2, \ldots, n$ such that $h = \sum_{i=1}^{n} e_i$, whence $J(h) = \sum_{i=1}^{n} J(e_i) = \sum_{i=1}^{n} e_i = h$. Therefore, J coincides with J_p on H^+, hence on H. A similar argument using the fact that $J(e) = 0$ for all $e \in E$ with $e \le u-p$ shows that $J(k) = 0$ for all $k \in K$, hence that $J = J_p$. Consequently, each retraction J on G is uniquely determined by its focus, and G is a compressible group. Furthermore, $P(G)$ is exactly the set of all sharp elements, i.e., characteristic elements, of G. By [9; Theorem 8.7] and its proof, $P(G)$ is a Boolean algebra and for $p \in P(G)$, $u-p$ is the Boolean complement of p. □

If G is a lattice-ordered abelian group with order unit u, then G is an interpolation group, so by Theorem 3.5, G is a compressible unigroup with unit u. Various rings of bounded $\mathbb{R}$-valued functions, partially ordered pointwise, and regarded as additive abelian groups, form lattice-ordered archimedean unital groups with the constant function 1 as unit. An example is the ring $\mathcal{F}(\Xi)$ of bounded Borel-measurable functions on a phase space Ξ. Here are two more examples.

3.6. EXAMPLE. If X is a compact Hausdorff space, then the ring $C(X, \mathbb{R})$ of continuous $\mathbb{R}$-valued functions on X with the usual positive cone $C(X, \mathbb{R})^+ := \{f \in C(X, \mathbb{R}) : f(X) \subseteq \mathbb{R}^+\}$ forms a lattice-ordered archimedean unigroup under addition with the constant function 1 as unit. Thus, $C(X, \mathbb{R})$ is a compressible unigroup. The corresponding effect algebra $E(C(X, \mathbb{R}))$ is an MV-algebra ([4], [16]). The projections in $C(X, \mathbb{R})$ are the characteristic set functions of compact open subsets of X, hence $P(C(X, \mathbb{R}))$ is a Boolean algebra.

3.7. EXAMPLE. Let X be a nonempty set, let $\mathcal{M}$ be a field of subsets of X, and define $F(X, \mathcal{M})$ to be the commutative ring under pointwise operations and with pointwise partial order of all bounded functions $f \colon X \to \mathbb{Z}$ that are measurable in the sense that $f^{-1}(z) \in \mathcal{M}$ for all $z \in \mathbb{Z}$. Then, under addition, and with the constant function 1 as unit, $F(X, \mathcal{M})$ forms a lattice-ordered archimedean unigroup, whence it is a compressible unigroup. Every element of the unit interval $E(F(X, \mathcal{M}))$ is a projection, i.e., $P(F(X, \mathcal{M})) = E(F(X, \mathcal{M}))$, and the projections are the characteristic set functions χ_M of sets $M \in \mathcal{M}$. The field of sets $\mathcal{M}$ is a Boolean algebra under set inclusion, and $E(F(X, \mathcal{M}))$ forms a Boolean algebra isomorphic to $\mathcal{M}$ under the correspondence $\chi_M \leftrightarrow M$.

By the Stone representation theorem, every Boolean algebra is isomorphic to the Boolean algebra formed by a field of subsets of a set, whence by Example 3.7, every Boolean algebra can be realized as the unit interval in a lattice-ordered archimedean compressible unigroup in which every effect is a projection (cf. Theorem 6.5 below).

Some unital groups G are compressible groups "by default" in that the only retractions on G are the zero compression J_0 and the identity compression J_u. For instance, any totally ordered unital group is compressible by default. So is the unital group G in the following example.

3.8. EXAMPLE. Let $\mathbb{Z}_2 = \{0, 1\}$ be the additive group of integers modulo 2 and let $G := \mathbb{Z} \times \mathbb{Z}_2$ with coordinatewise addition. With the usual order on $\mathbb{Z}$ and the trivial order on $\mathbb{Z}_2$ (i.e., $\mathbb{Z}_2^+ = \{0\}$), give G the lexicographic order ([9; p. 18]). Then G is a compressible unigroup with unit $u := (2, 0)$ and $P(G) = \{0, u\}$.

The "general comparability property", introduced in the next section (Definition 4.6), rules out compressible groups such as the unigroup G in Example 3.8 that admit only the zero and unit elements as projections, but are not totally ordered. It also rules out compressible groups that are not torsion free (Lemma 4.8).

4. Compatibility in a compressible group

For the remainder of this article, *we assume that G is a compressible group with unit u, that $E = E(G)$ is the unit interval in G, and that $P = P(G)$ is the set of projections in G. Elements $e \in E$ are called effects.*

If A is a unital C*-algebra (Example 2.1 and Theorem 2.7), $G(A)$ is the compressible archimedean unigroup of self-adjoint elements in A, and $P(A)$ is the orthomodular poset of projections in A, then for $g \in G(A)$ and $p \in P(A)$, $g = J_p(g)+J_{1-p}(g)$ if and only if g commutes with p. This suggests the following definition.

4.1. Definition. If $g \in G$ and $p \in P$, we say that g is *compatible with p* if and only if $g = J_p(g) + J_{u-p}(g)$. The set of all elements g in G that are compatible with the projection $p \in P$ is denoted by $C(p)$.

4.2. Lemma. *Let $p \in P$, let $H := J_p(G)$ be the image of the compression J_p, and let $K := \ker(J_p)$ be its kernel. Then, with $H^+ := H \cap G^+$, $K^+ := K \cap G^+$, and $C(p)^+ := C(p) \cap G^+$, we have:*

(i) *$C(p)$ is a subgroup of G and $u \in C(p)^+$.*
(ii) *$H^+ \cup K^+ \subseteq C(p)^+ = H^+ + K^+$.*
(iii) *$H \subseteq C(p)$.*
(iv) *With the induced partial order, $C(p)$ is a unital group with unit u and with unit interval $E(C(p)) = \{e+d:\ e, d \in E,\ \ e \le p,\ \ d \le u-p\}$.*

Proof.

(i) Evidently, $0 \in C(p)$. Suppose $a, b \in C(p)$. Then $J_p(a-b)+J_{u-p}(a-b) = J_p(a)+J_{u-p}(a)-\big(J_p(b)+J_{u-p}(b)\big) = a-b$. As $J_p(u)+J_{u-p}(u) = p+(u-p) = u$, we have $u \in C(p)^+$.

(ii) If $h \in H^+$, $k \in K^+$, and $g = h + k$, then $J_p(h) = h$, $J_p(k) = 0$, $J_{u-p}(k) = k$, and $J_{u-p}(h) = 0$, whence $g = h + k = J_p(g) + J_{u-p}(g) \in C(p)^+$. Conversely, if $g \in G^+$ with $g = J_p(g) + J_{u-p}(g)$, then $h := J_p(g) \in H^+$ and $k := J_{u-p}(g) \in K^+$.

(iii) By (ii), $H^+ \subseteq C(p)$, so by (i), $H = H^+ - H^+ \subseteq C(p)$.

(iv) As u is an order unit in G and $u \in C(p)^+$, it follows that u is an order unit in $C(p)$, hence that $C(p)$ is directed. Let $g \in C(p)^+$, so that $g = h+k$ with $h := J_p(g) \in H^+$ and $k := J_{u-p}(g) \in K^+$. As $h \in G^+$, we can write $h = \sum_{i=1}^{n} e_i$ with $e_i \in E$, whence $h = \sum_{i=1}^{n} a_i$ with $a_i := J_p(e_i) \in E \cap H^+$ for $i = 1, 2, \ldots, n$. Thus, $0 \le a_i \le u$ and $a_i = J_p(a_i) \in H^+ \subseteq C(p)$ for $i = 1, 2, \ldots, n$. By a similar argument, $k = \sum_{j=1}^{m} b_j$ with $0 \le b_j \le u$ and $b_j \in K^+ \subseteq C(p)$ for

$j = 1, 2, \ldots, m$. Consequently, $g = \sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j$ where each summand belongs to $\{c \in C(p) :\ 0 \leq c \leq u\}$. Therefore, u is a generative element of $C(p)$ and $C(p)$ is a unital group with unit u.

If $c \in C(p)$ with $0 \leq c \leq u$, then $c = e + d$ with $0 \leq e := J_p(c) \leq p$ and $0 \leq d := J_{u-p}(c) \leq u - p$. Conversely, if $c = e + d$ with $0 \leq e \leq p$ and $0 \leq d \leq u - p$, then $e \in H^+ \subseteq C(p)$ and $d \in K^+ \subseteq C(p)$, so $c \in H^+ + K^+ = C(p)^+$ with $c \leq p + (u - p) = u$. $\square$

4.3. THEOREM. *Let G be a compressible group with unit interval E and let $p \in P$. Then the following conditions are mutually equivalent:*

(i) $\ker(J_p)$ *is an ideal in* G.
(ii) $\ker(J_p) \subseteq C(p)$.
(iii) $G = C(p)$.
(iv) p *belongs to the center of the effect algebra* E.
(v) J_p *is a direct compression.*

Proof. Let $H := J_p(G)$, $H^+ := H \cap G^+$, $K := \ker(J_p)$, and $K^+ := K \cap G^+$.

(i) $\Longrightarrow$ (ii). Assume (i). Then $K = K^+ - K^+$, whence $K \subseteq C(p)$ by Lemma 4.2(i) and (ii).

(ii) $\Longrightarrow$ (iii). Assume (ii). By Lemma 4.2(iii), $H \subseteq C(p)$, whence $G = H + K \subseteq C(p)$ by Lemma 4.2(i).

(iii) $\Longrightarrow$ (iv). Assume (iii). By Lemma 2.3(iii), both p and $u - p$ are principal elements of E. If $e \in E$, then $e = J_p(e) + J_{u-p}(e)$ with $0 \leq J_p(e) \leq p$ and $0 \leq J_{u-p}(e) \leq u - p$, hence p belongs to the center of E.

(iv) $\Longrightarrow$ (v). Assume (iv). If $e \in E$, we can write $e = e_1 + e_2$ with $0 \leq e_1 \leq p$ and $0 \leq e_2 \leq u - p$, whence $J_p(e) = J_p(e_1) + J_p(e_2) = e_1 \leq e$. By Lemma 2.6, J is direct.

(v) $\Longrightarrow$ (i). Assume (v) and let $k \in K$. There exist $g_1, g_2 \in G^+$ with $k = g_1 - g_2$. Let $k_i := g_i - J_p(g_i)$ for $i = 1, 2$, so that $k = k_1 - k_2 + J_p(g_1 - g_2) = k_1 - k_2 + J_p(k) = k_1 - k_2$. By (v), $k_i \in G^+$, and it is clear that $k_i \in K$ for $i = 1, 2$, whence $k \in K^+ - K^+$. $\square$

If p is an effect in the center of E, then $p \in P$ and J_p satisfies the conditions in Theorem 4.3. Thus, there is a bijective correspondence $p \leftrightarrow J_p$ between effects p in the center of E and direct compressions J_p on G. If G is an interpolation group (Theorem 3.5), then every compression on G is direct and P is the center of the effect algebra E.

4.4. COROLLARY. *If A is a unital* C*-*algebra, and $p \in P(A)$, then $C(p) = G(A)$ if and only if p belongs to the center of A.*

4.5. DEFINITION. If $g \in G$ and $p \in P$, we write $g \in C^{\pm}(p)$ if and only if the following four conditions hold:

(i) $g \in C(p)$,
(ii) for all $q \in P$, $g \in C(q) \implies q \in C(p)$,
(iii) $0 \le J_p(g)$,
(iv) $J_{u-p}(g) \le 0$.

Suppose $g \in C^{\pm}(p)$. Although the projection p is not necessarily uniquely determined by g, it can be shown that $g^+ := J_p(g) \ge 0$ and $g^- := -J_{u-p}(g) \ge 0$ are uniquely determined, whence that $|g| := g^+ + g^- \ge 0$ is well defined. Furthermore, $g \le |g| = |-g|$ and $g = g^+ + (-g^-)$ is split into a "positive part" g^+ and a "negative part" $-g^-$, cf. [18; Section 108].

4.6. DEFINITION.

(i) G has the *general comparability property* if and only if, for every $g \in G$, there exists $p \in P$ such that $g \in C^{\pm}(p)$.

(ii) G has the *central comparability property* if and only if, for every $g \in G$, there exists $p \in P$ such that $G = C(p)$ and $g \in C^{\pm}(p)$.

For operator algebras, there is a close relationship between the existence of spectral measures and the general comparability property.

4.7. EXAMPLE. Let A be a von Neumann algebra with unit 1 and let $G(A)$ be the compressible unigroup of self-adjoint elements of A. If $g \in G(A)$, then by the spectral theorem, there is a projection $p \in A$ such that p commutes with every element of A that commutes with g, $0 \le pa = ap$, and $(1-p)a = a(1-p) \le 0$. Therefore, $G(A)$ has the general comparability property.

4.8. LEMMA. *Let G have the general comparability property. Then:*

(i) *G is torsion free, i.e., 0 is the only element in G with finite order.*

(ii) *G is unperforated, i.e., if n is a positive integer, $g \in G$, and $0 \le ng$, then $0 \le g$.*

Proof.

(i) Suppose $g \in G$, n is a positive integer, and $ng = 0$. By general comparability, there is a projection $p \in P$ such that $0 \le J_p(g)$, $0 \le J_{u-p}(-g)$, and $g \in C(p)$. But then $nJ_p(g) = J_p(ng) = 0$ and, owing to the fact that $J_p(g) \in G^+$, it follows that $J_p(g) = 0$. Likewise, as $n(-g) = 0$, we have $J_{u-p}(-g) = 0$, whence $J_{u-p}(g) = 0$. Therefore, $g = J_p(g) + J_{u-p}(g) = 0$.

(ii) Suppose $g \in G$, n is a positive integer, and $0 \le ng$. Then there exists $p \in P$ such that $0 \le J_p(g)$, $J_{u-p}(g) \le 0$, and $g = J_p(g) + J_{u-p}(g)$. Then $0 \le J_{u-p}(ng) = nJ_{u-p}(g) \le 0$, so $nJ_{u-p}(g) = 0$, and it follows from (i) that $J_{u-p}(g) = 0$. Hence, $0 \le J_p(g) = g$. □

4.9. THEOREM. *If the compressible group G has the central comparability property, then G is lattice ordered.*

Proof. Assume the hypotheses, let u be the unit in G, and let $g, h \in G$. Then there exists $p \in P$ such that $G = C(p)$, $0 \le J_p(g-h)$, and $J_{u-p}(g-h) \le 0$, whence $J_p(h) \le J_p(g)$ and $J_{u-p}(g) \le J_{u-p}(h)$. Let $a := J_p(h) + J_{u-p}(g)$. Then $a \le J_p(g) + J_{u-p}(g) = g$ and $a \le J_p(h) + J_{u-p}(h) = h$, so a is a lower bound in G for g and h. Suppose $b \in G$ and $b \le g, h$. Then $J_p(b) \le J_p(h)$ and $J_{u-p}(b) \le J_{u-p}(g)$, so $b = J_p(b) + J_{u-p}(b) \le a$. □

Let G be an interpolation group with order unit. By Theorem 3.5, G forms a compressible unigroup and $G = C(p)$ for every projection $p \in P$. Therefore, G has the general comparability property if and only if it has the central comparability property. Furthermore, the general comparability property is equivalent to the property of the same name in [9; p. 131]. Thus, Theorem 4.9 extends [9; Proposition 8.9] to cases in which G fails to have the interpolation property. A compressible group G without the interpolation property can have the general comparability property without being lattice ordered. For instance, the compressible unigroup $\mathbb{G}(\mathcal{H})$ of self-adjoint operators on a Hilbert space $\mathcal{H}$ has the general comparability property, yet it forms an *anti-lattice*, i.e., two self-adjoint operators have a greatest lower bound in $\mathbb{G}(\mathcal{H})$ if and only if they are comparable ([15]).

If G is a lattice-ordered unital group, then the unit interval E in G is an MV-algebra. If G is not only lattice ordered, but has the general comparability property, then E is not only an MV-algebra, but a Heyting algebra as well ([4]). If G is a Dedekind σ-complete lattice-ordered abelian group with order unit u, then G is a compressible unigroup with unit u, and by [9; Theorem 9.9], it has the central comparability property.

5. Projections in a compressible group

In this section, we shall be studying the structure of the system P of projections in the compressible group G and some of the basic calculus of compressions on G.

5.1. THEOREM. *If $p, q \in P$, then the following conditions are mutually equivalent:*

(i) $J_p(q) = 0$.
(ii) $J_{u-p} \circ J_q = J_q$.
(iii) $p + q \le u$.
(iv) $J_q(p) = 0$.
(v) $u - (p+q) \in P$ *and* $J_{u-(p+q)} = J_{u-p} \circ J_{u-q} = J_{u-q} \circ J_{u-p}$.
(vi) $p + q \in P$.

Proof. Let $p, q \in P$.

(i) $\Longrightarrow$ (ii). Suppose that $J_p(q) = 0$. Then $J_{u-p}(q) = q$, whence $J_{u-p} \circ J_q$ is an order-preserving endomorphism on G with $(J_{u-p} \circ J_q)(u) = J_{u-p}(J_q(u)) = J_{u-p}(q) = q$. Let $e \in G$ with $0 \le e \le q$. Then $J_q(e) = e$. Also, $0 \le J_p(e) \le J_p(q) = 0$, so $J_p(e) = 0$, whence $J_{u-p}(e) = e$. Therefore, $(J_{u-p} \circ J_q)(e) = e$, and it follows that $J_{u-p} \circ J_q$ is a retraction on G with focus q. Consequently $J_{u-p} \circ J_q = J_q$.

(ii) $\Longrightarrow$ (iii). Assume (ii). Then $q = J_q(q) = J_{u-p}(J_q(q)) = J_{u-p}(q) \le u-p$, whence $p + q \le u$.

(iii) $\Longrightarrow$ (iv). Assume (iii). Then $J_q(p) + q = J_q(p+q) \le J_q(u) = q$. But $0 \le J_q(p)$, and it follows that $J_q(p) = 0$.

(iv) $\Longleftrightarrow$ (i). By the arguments already made, (i) $\Longrightarrow$ (iv), so by symmetry (iv) $\Longrightarrow$ (i).

(iv) $\Longrightarrow$ (v). Assume (iv). Then, since (iv) $\Longleftrightarrow$ (i), we have both $J_p(q) = 0$ and $J_q(p) = 0$, whence $J_{u-p}(q) = q$ and $J_{u-q}(p) = p$. Also, since (i) $\Longrightarrow$ (iii), we have $p + q \le u$. The composition $J := J_{u-p} \circ J_{u-q}$ is an order-preserving group endomorphism on G and $J(u) = J_{u-p}(J_{u-q}(u)) = J_{u-p}(u-q) = u - p - J_{u-p}(q) = u-p-q = u-(p+q) \le u$. Suppose that $0 \le e \le u-(p+q)$. Then $0 \le J_q(e) \le J_q(u-q) - J_q(p) = 0+0 = 0$. Therefore, $J_q(e) = 0$, whence $J_{u-q}(e) = e$. By symmetry, $J_{u-p}(e) = e$, so $J(e) = J_{u-p}(J_{u-q}(e)) = e$, and it follows that J is a retraction with focus $J(u) = u - (p+q)$. Therefore, $u - (p+q) \in P$ and $J_{u-p} \circ J_{u-q} = J = J_{u-(p+q)}$. By symmetry, $J_{u-q} \circ J_{u-p} = J_{u-(p+q)}$.

(v) $\Longrightarrow$ (vi). If $u - (p+q) \in P$, then $p + q = u - (u - (p+q)) \in P$.

(vi) $\Longrightarrow$ (i). Suppose $p + q \in P$. Then $0 \le p + q \le u$, and it follows that $0 \le J_p(p) + J_p(q) \le J_p(u)$. Thus, $0 \le p + J_p(q) \le p$, from which it follows that $J_p(q) = 0$. □

5.2. COROLLARY.

(i) *If A is a finite nonempty subset of P and $\sum_{p \in A} p \le u$, then $\sum_{p \in B} p \in P$ for every nonempty subset $B \subseteq A$.*
(ii) *P is a sub-effect algebra of E.*
(iii) *P is an orthomodular poset.*
(iv) *If $p, q \in P$, then $p \le q \Longleftrightarrow q - p \in P$.*

Proof.

(i) Let $p, q, r \in P$ with $p+q+r \leq u$. Then $p+q \leq u$, so $p+q \in P$ by Theorem 5.1. Again by Theorem 5.1, $(p+q)+r \leq u$ implies that $(p+q)+r \in P$. Continuing in this way, we obtain the more general result by mathematical induction.

(ii) We have $0, u \in P$ and P is closed under $p \mapsto u-p$. By Theorem 5.1, if $p, q \in P$ and $p+q \leq u$, then $p+q \in P$, so P is a sub-effect algebra of E.

(iii) Follows from (i) and (ii).

(iv) Let $r := u-q$. Then $r \in P$, and $p \leq q \implies p+r \leq u \implies q-p = u-(p+r) \in P$ by Theorem 5.1. Conversely, if $q-p \in P$, then $0 \leq q-p$, so $p \leq q$. □

5.3. COROLLARY. *Let $p, q, p+q \in P$. Then, for all $g \in G^+$, $J_p(g) = J_q(g) = 0 \implies J_{p+q}(g) = 0$.*

Proof. If $J_p(g) = J_q(g) = 0$, then $J_{u-p}(g) = J_{u-q}(g) = g$, and it follows from Theorem 5.1(v) that $J_{u-(p+q)}(g) = g$, whence $J_{p+q}(g) = 0$. □

5.4. THEOREM. *Let $p, q \in P$. Then the following conditions are mutually equivalent:*

(i) $J_p \circ J_q = J_q \circ J_p$.
(ii) $J_p \circ J_q$ *is a compression.*
(iii) $J_p(q) \in P$ *and* $J_p(q) \leq q$.
(iv) *There exist* $r, s, t \in P$ *such that* $r+s+t \in P$, $p = r+s$, *and* $q = r+t$.
(v) *There exist* $r, s \in E$ *such that* $p = r+s$, $r \leq q$, *and* $s \leq u-q$.
(vi) $p \in C(q)$.

Proof.

(i) $\implies$ (ii). Assume (i), let $J := J_p \circ J_q = J_q \circ J_p$, and let $r := J(u) = J_p(J_q(u)) = J_p(q)$. Then $r = J_q(J_p(u)) = J_q(p)$. Evidently, $0 \leq r \leq p, q$. Also, J is an order-preserving endomorphism on G, and if $e \in E$ with $e \leq r$, then $e \leq p, q$ and $J(e) = J_p(J_q(e)) = J_p(e) = e$. Therefore, J is a retraction, hence a compression on G with focus r.

(ii) $\implies$ (iii). Assume (ii). Then there exists $r \in P$ with $J_p \circ J_q = J_r$. Thus $J_r(u-q) = J_p(J_q(u-q)) = J_p(0) = 0$, whence $u-q \leq u-r$, i.e., $r \leq q$. Therefore $J_p(q) = J_p(J_q(u)) = J_r(u) = r \in P$ and $J_p(q) = r \leq q$.

(iii) $\implies$ (iv). Assume (iii) and let $r := J_p(q)$. Then $r \in P$ with $r \leq p, q$. By Corollary 5.2(iv), $s := p-r \in P$ and $t := q-r \in P$. Also, $J_p(t) = J_p(q) - J_p(r) = r-r = 0$, so by Theorem 5.1, $r+s+t = p+t \in P$.

(iv) $\implies$ (v). Assume (iv). Then $s+q = r+s+t \leq u$, so $s \leq u-q$. Also, $r \leq r+t = q$, and $p = r+s$.

(v) $\implies$ (vi). Follows directly from Lemma 4.2(ii).

(vi) $\Longrightarrow$ (i). Suppose $p \in C(q)$. By Lemma 4.2(ii), there exist $r, s \in E$ with $p = r + s$, $r \leq q$, and $s \leq u - q$. Then $t := q - r \in E$ and we have $t + p = q - r + r + s = q + s \leq u$, so $t \leq u - p$. As $r \leq p$ and $t \leq u - p$, we have $J_p(r) = r$ and $J_p(t) = 0$, whence $J_p(q) = J_p(r + t) = r$. Define $J := J_p \circ J_q$. Then J is an order-preserving endomorphism on G and $J(u) = J_p(q) = r$. Suppose $e \in E$ with $e \leq r$. Then $e \leq r + s = p$ and $e \leq r \leq q$, so $J(e) = J_p(J_q(e)) = J_p(e) = e$, and it follows that J is a retraction with focus $r \in P$, i.e., $J_p \circ J_q = J = J_r$. We also have $q = r + t$ with $r, t \in E$, $r \leq p$, and $t \leq u - p$, so by symmetry, $J_q \circ J_p = J_r = J_p \circ J_q$. □

5.5. COROLLARY. *If $p, q \in P$, then the following conditions are mutually equivalent:*

(i) $q \leq p$.
(ii) $J_p(q) = q$.
(iii) $J_p \circ J_q = J_q \circ J_p = J_q$.
(iv) $J_p \circ J_q = J_q$.
(v) $J_q \circ J_p = J_q$.

Proof.

(i) $\Longrightarrow$ (ii). If $q \leq p$, then $0 \leq q \leq p \leq u$, and it follows that $J_p(q) = q$.

(ii) $\Longrightarrow$ (iii). If $J_p(q) = q$, then by Theorem 5.4(iii), $J_p \circ J_q = J_q \circ J_p = J_r$ with $r = J_p(q) = q$.

(iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i). Obviously (iii) $\Longrightarrow$ (iv). If (iv) holds, then $q = J_q(q) = J_p(J_q(q)) \leq p$.

(v) $\Longleftrightarrow$ (i). If (v) holds, then $J_q(u - p) = 0$, so $u - p \leq u - q$, and it follows that $q \leq p$. Conversely, if (i) holds, then by the arguments above, (iii) holds, and (v) follows. □

Condition (iv) in Theorem 5.4 is equivalent to the well-known condition that the elements p and q of the orthomodular poset P are *Mackey compatible*, i.e., that they *commute* ([16]).

5.6. COROLLARY. *If $p, q \in P$, then the following conditions are mutually equivalent:*

(i) $p \in C(q)$.
(ii) $q \in C(p)$.
(iii) $J_p(q) \in P$.
(iv) $J_q(p) \in P$.

Furthermore, if any one, hence all, of these conditions hold, then $p \wedge q = J_p(q) = J_q(p)$ is the greatest lower bound of p and q both in E and in P.

Proof.

(i) $\iff$ (ii). The condition in Theorem 5.4(i) is symmetric in p and q, so (i) $\iff$ (ii) by Theorem 5.4(vi).

(i) $\iff$ (iii). By parts (vi) and (iii) of Theorem 5.4, (i) $\implies$ (iii). Conversely, suppose that $r := J_p(q) \in P$. By Theorem 5.4(iii), it will be sufficient to prove that $r \leq q$. As $r \leq p$, Corollary 5.5 implies that $J_p(r - J_r(q)) = J_p(r) - J_r(J_p(q)) = r - J_r(r) = 0$. As $J_r(q) \leq r$, it follows that $0 \leq r - J_r(q) \leq u - p \leq u - r$, whence $r - J_r(q) = J_r(r - J_r(q)) = 0$, i.e., $r = J_r(q)$. Therefore, $J_r(u - q) = J_r(u) - J_r(q) = r - r = 0$, so $u - q \leq u - r$, i.e., $r \leq q$.

By symmetry, (ii) $\iff$ (iv).

Finally suppose conditions (i)–(iv) hold and let $r := J_p(q)$. Then $r \in P$ with $r \leq p$. By Theorem 5.4, $J_p \circ J_q = J_q \circ J_p = J_r$ and $r = J_q(p) \leq p, q$. If $e \in E$ with $e \leq p, q$, then $e = J_p(J_q(e)) = J_r(e) \leq r$, so r is the greatest lower bound of p and q in E, hence also in P. □

5.7. COROLLARY. *P is a Boolean algebra if and only if $p \in C(q)$ for all $p, q \in P$.*

There are orthomodular posets in which the pairwise compatibility of finitely many elements does not imply their joint compatibility, i.e., the elements need not lie in the same Boolean subalgebra. As a consequence of Theorem 5.4, it is not difficult to see that this anomaly cannot occur in the orthomodular poset P of projections in a compressible group.

5.8. LEMMA. *Let $p, q \in P$ with $q \leq p$, and let $h \in J_p(G)$ with $0 \leq h$. Then $J_q(h) = h \iff J_{p-q}(h) = 0$.*

Proof. As $q \leq p$, we have $p - q \in P$ by Corollary 5.2(iv). Also, $p - q \leq u - q$, so Corollary 5.5 implies that $J_{p-q} = J_{p-q} \circ J_{u-q}$. Therefore, if $J_q(h) = h$, then $J_{p-q}(h) = J_{p-q}(J_{u-q}(h)) = J_{p-q}(0) = 0$. Conversely, suppose that $J_{p-q}(h) = 0$. As $h \in J_p(G)$, we have $J_{u-p}(h) = 0$. Thus, with $s := p - q$ and $t := u - p$, we have $s, t, s+t=u-q \in P$ with $J_s(h) = J_t(h) = 0$, and it follows from Corollary 5.3 that $J_{u-q}(h) = 0$, so $J_q(h) = h$. □

5.9. THEOREM. *Let G be a compressible group, let $p \in P$, and let $H := J_p(G)$. Then, with the induced partial order and with p as a unit, H is a compressible group, and the set of projections in H is $P(H) = \{q \in P : q \leq p\}$. If $q \in P(H)$, then the compression on H with focus q is the restriction $J_q|_H$ to H of the compression J_q on G, and $J_q|_H \circ J_p = J_q$.*

Proof. By Lemma 2.4, H is a unital group with unit p. Suppose $q \in P$ with $q \leq p$. Then Corollary 5.5 implies that $J_p \circ J_q = J_q \circ J_p = J_q$. Therefore, if $h \in H$, then $J_p(J_q(h)) = J_q(J_p(h)) = J_q(h)$, whence $J_q(h) \in H$. Therefore, the restriction $J_q|_H$ is a group endomorphism on H, and it is clearly order preserving. As $p = q + (p - q)$ and $p - q \le u - q$, we have $J_q(p) = J_q(q) + J_q(p - q) = q + 0 = q$, whence $J_q|_H(p) = q \le p$. If $e \in H$ with $0 \le e \le q$, then $J_q|_H(e) = J_q(e) = e$, and it follows that $J_q|_H$ is a retraction on H with focus q. Evidently, $J_q|_H \circ J_p = J_q$.

Suppose I is a retraction on H with $I(p) = q$. Then $q \in H$ with $0 \le q \le p$ and by Lemma 2.4, $I \circ J_p = J_q$, so $q \in P$. Furthermore, $I \circ J_p = J_q = J_q|_H \circ J_p$, and since $H = J_p(G)$, it follows that $I = J_q|_H$. In particular, each retraction on H is uniquely determined by its focus. By Corollary 5.2(iv), $p - q \in P$ with $p - q \le p$, whence $J_{p-q}|_H$ is a retraction on H. By Lemma 5.8, $J_q|_H$ and $J_{p-q}|_H$ are quasicomplementary retractions on H, so H is a compressible group with unit p. □

5.10. THEOREM. *Let G be a compressible group and let $p \in P$. Then, with the induced partial order, $C(p)$ is a compressible group with unit u. Furthermore, $P(C(p)) = C(p) \cap P$, and for $q \in C(p) \cap P$, the compression on $C(p)$ with focus q is the restriction $J_q|_{C(p)}$ of J_q to $C(p)$.*

Proof. By Lemma 4.2(iv), $C(p)$ is a unital group with unit u. Suppose that I is a retraction on $C(p)$ with focus $q := I(u) \in E(C(p))$. With $s := J_p(q) \le p$ and $t := J_{u-p}(q) \le u - p$, we have $q = s + t$. Let $h := p - s$ and $k := u - p - t$, so $s + t + h + k = u$ with $s, t, h, k \in C(p)$. Then $h + k = u - q$, so $I(h + k) = I(h) + I(k) = 0$, and since $0 \le I(h), I(k)$, we have $I(h) = I(k) = 0$. As $J_p(G) \subseteq C(p)$, we can form the composition $(I \circ J_p)\colon G \to G$. Evidently, $I \circ J_p$ is an order-preserving endomorphism on G and $(I \circ J_p)(u) = I(p) = I(s + h) = I(s) + I(h) = I(s) = s$, the last equality following from the fact that $s \in C(p)$ with $0 \le s \le q$. Suppose $e \in E$ with $e \le s$. As $s \le p, q$, we have $0 \le e \le p, q$, whence $(I \circ J_p)(e) = I(e) = e$, and it follows that $I \circ J_p$ is a retraction on G with focus s. Therefore, $s \in P$ and $I \circ J_p = J_s$. A similar calculation shows that $t \in P$ and $I \circ J_{u-p} = J_t$. As $p, s, t \in P$, we have $q = s + t \in P$, $h = p - s \in P$, and $k = (u - p) - t \in P$. If $g \in C(p)$, then $I(g) = I(J_p(g) + J_{u-p}(g)) = (I \circ J_p)(g) + (I \circ J_{u-p})(g) = J_s(g) + J_t(g)$. Consequently, I is uniquely determined by its focus q.

Since $J_p(q) = s \in P$ and $s \le q$, Theorem 5.4 implies that $J_p \circ J_q = J_q \circ J_p$. Likewise, since $J_{u-p}(q) = t \in P$ and $t \le q$, we have $J_{u-p} \circ J_q = J_q \circ J_{u-p}$. Consequently, if $g \in C(p)$, then $J_q(g) = J_q(J_p(g) + J_{u-p}(g)) = J_p(J_q(g)) + J_{u-p}(J_q(g))$, whence $J_q(g) \in C(p)$. Therefore, the restriction $J_q|_{C(p)}$ of J_q to $C(p)$ maps $C(p)$ into $C(p)$, and it is obviously a retraction on $C(p)$ with focus q.

By the uniqueness established above, $I = J_q|_{C(p)}$. Similarly, $J_{u-q}|_{C(p)}$ is a retraction on $C(p)$, and it is clear that I and $J_{u-q}|_{C(p)}$ are quasicomplements. $\square$

If G has the general comparability property and $p \in P$, it can be shown that the compressible groups $J_p(G)$ in Theorem 5.9 and $C(p)$ in Theorem 5.10 inherit this property.

6. The projection-cover property

A Rickart C*-algebra (respectively, an AW*-algebra) is a C*-algebra in which the right annihilator of each element (respectively, each subset) is a principal right ideal generated by a projection ([13]). If A is a Rickart C*-algebra, an AW*-algebra, or a von Neumann algebra with unit 1, then the compressible group $G(A)$ of self-adjoint elements in A has the *projection-cover property* as per the following definition.

6.1. Definition. If $e \in E$, then a projection $c \in P$ is called a *projection cover* of (or for) e if and only if, for all $p \in P$, $e \leq p \iff c \leq p$. If $e \in E$ has a projection cover c, it is uniquely determined by e and is denoted by $\gamma(e) := c$. A projection $p \in P$ is a *Sasaki projection* if and only if, for every $q \in P$, $J_p(q)$ has a projection cover denoted by $\phi_p(q) := \gamma\big(J_p(q)\big)$. If p is a Sasaki projection in P, then $\phi_p \colon P \to P$ is called the *Sasaki mapping* for p. If every projection $p \in P$ is a Sasaki projection, then the compressible group G has the *Sasaki property*. The compressible group G has the *projection-cover property* if and only if each $e \in E$ has a projection cover $\gamma(e)$.

The projection-cover property is closely related to the sharp domination property for effect algebras ([11]). In fact, for a compressible group in which every sharp element is a projection, the two properties are equivalent. It can be shown that, if G is archimedean, G has the general comparability property, and P satisfies the ascending chain condition, then G has the projection-cover property. Also, if G has the projection-cover property and $p \in P$, then the compressible groups $J_p(G)$ in Theorem 5.9 and $C(p)$ in Theorem 5.10 inherit this property.

6.2. Lemma. *Let $p, q, r \in P$ and suppose that p is a Sasaki projection. Then:*

(i) $J_p(q) \leq \phi_p(q) \leq p$.
(ii) $q \leq r \implies \phi_p(q) \leq \phi_p(r)$.
(iii) $q \leq p \iff \phi_p(q) = q$.
(iv) $\phi_p\big(u - \phi_p(q)\big) = J_p\big(u - \phi_p(q)\big) = p - \phi_p(q) \leq u - q$.
(v) $r + J_p(q) \in E \iff q + J_p(r) \in E$.

(vi) $\phi_p(q) \le r \iff q \le u - \phi_p(u-r)$.

(vii) *The greatest lower bound of* p *and* q *in* P *exists and is given by* $p \wedge q = \phi_p(u - \phi_p(u-q)) = J_p(u - \phi_p(u-q)) = p - \phi_p(u-q)$.

Proof.

(i) As $J_p(q) \in E$ with $J_p(q) \le p \in P$, we have $J_p(q) \le \gamma(J_p(q)) \le p$.

(ii) If $q \le r$, then $J_p(q) \le J_p(r) \le \phi_p(r) \in P$, so $\phi_p(q) = \gamma(J_p(q)) \le \phi_p(r)$.

(iii) If $q \le p$, then $J_p(q) = q$, whence $\phi_p(q) = \gamma(q) = q$, and the converse follows from (i).

(iv) Define $t := u - \phi_p(q)$ and $s := \phi_p(t)$. Then $t, s \in P$, and by (i), $s \le p$. By (i) again, $\phi_p(q) \le p$, so $J_p(t) = p - \phi_p(q)$. Also, by Corollary 5.2(iv), $J_p(t) = p - \phi_p(q) \in P$, whence $s = \gamma(J_p(t)) = J_p(t) = p - \phi_p(q)$.

By (i), $J_p(q) \le \phi_p(q) = u - t$, whence $J_t(J_p(q)) = 0$. As $\phi_p(q) \le p$, we have $\phi_p(q) \in C(p)$, and it follows that $t = u - \phi_p(q) \in C(p)$. Therefore, by Theorem 5.4, $J_p \circ J_t = J_t \circ J_p$ is a compression with focus $J_p(J_t(u)) = J_p(t) = s$. Consequently, $J_s(q) = J_t(J_p(q)) = 0$, so $q \le u - s$, i.e., $s \le u - q$.

(v) $r + J_p(q) \in E \implies J_p(q) \le u - r \implies \phi_p(q) = \gamma(J_p(q)) \le u - r \implies r \le u - \phi_p(q)$. By (i), (ii), and (iv), $r \le u - \phi_p(q) \implies J_p(r) \le \phi_p(r) \le \phi_p(u - \phi_p(q)) \le u - q \implies q + J_p(r) \le u \implies q + J_p(r) \in E$. Thus, $r + J_p(q) \in E \implies q + J_p(r) \in E$, and the converse follows by symmetry.

(vi) By (v) with r replaced by $u - r$, we have $J_p(q) \le r \iff J_p(u-r) \le u - q$, whence $\phi_p(q) \le r \iff \phi_p(u-r) \le u - q \iff q \le u - \phi_p(u-r)$.

(vii) Let $s := \phi_p(u - \phi_p(u-q))$. By (iv), $s = J_p(u - \phi_p(u-q)) = p - \phi_p(u-q) \le u - (u-q) = q$, and it is clear that $s \le p$. Suppose $r \in P$ with $r \le p, q$. Then by (iii), $r = \phi_p(r) \le q$, so by (vi), $r \le u - \phi_p(u-q)$, whence $r = \phi_p(r) \le \phi_p(u - \phi_p(u-q)) = s$. Therefore, $s = p \wedge q$. □

The condition in Lemma 6.2(vi) implies that the Sasaki mapping ϕ_p is residuated ([3]).

6.3. THEOREM. *Let* G *have the Sasaki property. Then* P *is an orthomodular lattice and, for all* $p, q \in P$, $\phi_p(q) = p \wedge (q \vee (u-p))$.

Proof. By Corollary 5.2(iii), and Lemma 6.2(vii), P is an orthomodular poset in which every pair of elements has a greatest lower bound, hence P is an orthomodular lattice. We omit the straightforward proof that $\phi_p(q) = p \wedge (q \vee (u-p))$. □

If A is a unital von Neumann algebra and $e \in E(A)$, then the projection cover $\gamma(e)$ commutes with every element of A that commutes with e, hence it commutes with every projection that commutes with e. For a compressible group G, we have the following analogous result.

6.4. THEOREM. *Suppose that G has the projection-cover property, and let $e \in E$. Then, for every $p \in P$, $e \in C(p) \implies \gamma(e) \in C(p)$.*

Proof. Assume that $e \in E$, $p \in P$, and $e \in C(p)$. Then $e = e_1 + e_2$ with $e_1 := J_p(e) \le p$ and $e_2 := J_{u-p}(e) \le u - p$. Therefore, $e_1 \le \gamma(e_1) \le p$ and $e_2 \le \gamma(e_2) \le u - p$, so $e = e_1 + e_2 \le \gamma(e_1) + \gamma(e_2) \le u$. By Theorem 5.1, $\gamma(e_1) + \gamma(e_2) \in P$, and it follows that $\gamma(e) \le \gamma(e_1) + \gamma(e_2)$. Also, $e_1, e_2 \le e \le \gamma(e)$, whence $\gamma(e_1), \gamma(e_2) \le \gamma(e)$. But $\gamma(e)$ is a principal element of E, so $\gamma(e_1) + \gamma(e_2) \le \gamma(e)$, and therefore $\gamma(e) = \gamma(e_1) + \gamma(e_2)$. As $\gamma(e_1) \le p$ and $\gamma(e_2) \le u - p$, it follows that $\gamma(e_1), \gamma(e_2) \in C(p)$, and therefore $\gamma(e) \in C(p)$. □

If $\mathcal{M}$ is a field of subsets of the nonempty set X, then $F(X, \mathcal{M})$ in Example 3.7 is a compressible group in which every effect is a projection. Conversely, we have the following theorem:

6.5. THEOREM. *Suppose that G is a compressible group in which every effect is a projection. Then $E = P$ is a Boolean algebra. Furthermore, if X is the Stone space of P and $\mathcal{M}$ is the field of compact open subsets of X, then there is a unital isomorphism from G onto $F(X, \mathcal{M})$.*

Proof. Assume the hypotheses. Then $P = E$, G has the projection-cover property, every element in E being its own projection-cover. In particular, G has the Sasaki property, so by Theorem 6.3, P is an orthomodular lattice such that, for $p, q \in P$, $p \wedge q = J_p(u - J_p(u-q)) = p - p + J_p(q) = J_p(q) = p \wedge (q \vee (u-p))$. Consequently, $P = E$ is a Boolean algebra and $p \wedge q = J_p(q) = J_q(p)$. By Corollary 5.6, we have $p \in C(q)$ for all $p, q \in P = E$. Since every element $g \in G$ can be written as a finite linear combination of elements of $P = E$ with integer coefficients, it follows from Lemma 4.2(i) that $G = C(p)$ for every $p \in P$.

Let $g \in G$ and write g as $g = \sum_{j=1}^{m} \lambda_j e_j$ with $e_1, e_2, \ldots, e_m \in E = P$ and $\lambda_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, m$. Let B be the Boolean subalgebra of P generated by e_j, $j = 1, 2, \ldots, m$. Since a finitely generated Boolean algebra is finite, it follows that every element in B can be written uniquely as a least upper bound, hence as a sum, of the atoms $p_1, p_2, \ldots, p_n$ in B. In particular, each e_j can be so written, and it follows that there are integers $\alpha_i \in \mathbb{Z}$ such that $g = \sum_{i=1}^{n} \alpha_i p_i$. Let $I := \{i : 1 \le i \le n,\ 0 \le \alpha_i\}$ and $p := \sum_{i \in I} p_i$. Then, $p \in P$, $0 \le J_p(g)$, and $J_{u-p}(g) \le 0$, so G has the central comparability property. Consequently, by Theorem 4.9, G is lattice ordered, so G is a unigroup.

Let X be the Stone space of the Boolean algebra E, let $\mathcal{M}$ be the field of compact open subsets of X, and form the lattice-ordered archimedean unigroup $F(X,\mathcal{M})$ (Example 3.7). Then there is a Boolean isomorphism ϕ from E onto $E\big(F(X,\mathcal{M})\big)$ and, since both G and $F(X,\mathcal{M})$ are unigroups, ϕ can be extended to a unital isomorphism from G onto $F(X,\mathcal{M})$. $\square$

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