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FUNCTIONAL REPRESENTATION OF PREITERATIVE/COMBINATORY FORMALISM

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ABSTRACT. Both formalisms model systems of multi-argument selfmaps closed under composition as well as argument permutation and identification by using abstract algebraic operations for these transformations and substitution. Developed are the additional requirements, for each system, to be representable as a system of concrete selfmaps on some set in which these operations act in the expected way.

Introduction

The formal elements of a preiterative algebra are function symbols each of a fixed finite number of arguments: $f(x_1,\dots,x_n)$. On these act three operators of "mutation" $(\zeta f)(x_1,\dots,x_n) = f(x_2,\dots,x_n,x_1)$, $(\tau f)(x_1,x_2,\dots,x_n) = f(x_2,x_1,\dots,x_n)$, $(\triangle f)(x_1,x_2,\dots,x_n) = f(x_1,x_1,x_2,\dots,x_{n-1})$ and a binary operator of substitution (only for the first argument): $f \star g = f\bigl(g(x_1,\dots,x_m), x_{m+1},\dots,x_{m+n-1}\bigr)$. It will turn out that compositions of the three index transformations exhaust the full semigroup of transformations which move only finitely many symbols. Thus, if one has a representation of the formal functions *cum* indexed arguments on which the mutations operate in the indicated manner, then one automatically has an action by the full transformation semigroup; if instead one construes the mutations as abstract unary operators which with the binary substitution equip the "function" elements with a universal algebraic structure, then one must impose axioms to obtain this full semigroup action, since the representing function system will have it. In either event, it will become possible to extend the single substitution operation to a simul-

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taneous substitution at all arguments; and for the so augmented formalism, representability is easy to characterize.

The combinatory formalism can be handled similarly. We choose to operate with a subset of the combinators, Curry's B, C, W. The latter two have an action similar to τ and Δ, but carried out differently, with the result that ζ becomes derivable.

Both representations repose on the result which follows.

The basic (Cayley) representation

Three aspects of multi-argument formalism need to be considered: the substitutional (or superpositional), the transformational (or mutational) and compatibility requirements between them which ensure that these act in a mutually coherent manner.

In its most primitive form, substitution is simultaneous without identification of arguments. Suppose given a system F of finite argument formal functions closed under (formal, multiple) composition — thus each element $f \in F$ comes equipped with a finite number n of arguments (which one could imagine labelled with variables[1] and presented as) $f(x_1, \ldots, x_n)$; and for each n-tuple of elements $f_1, \ldots, f_n \in F$, one can form the result of simultaneously substituting for the n arguments (i.e. of replacing x_i with f_i), to obtain a specific element $f(f_1, \ldots, f_n) \in F$. The arguments of this composite are construed as the disjoint union of those of the f_i — this is to be substitution without identification of arguments.[2]

Every f is thus (assigned) an n-argument selfmap $F^n \to F$. The condition that the assignment preserve (i.e. convert formal to functional) substitution is the multi-argument "super" associative law: $f\big(f_1(g_1, \ldots, g_{n_1}), f_2(g_{n_1+1}, \ldots$ $\ldots, g_{n_1+n_2}), \ldots\big)$ is to be the same whether one first substitutes the f_i into f and then substitutes the g_j into the composite or first substitutes the g_j into f_i and then substitutes these composites into f.[3] This result is straightforward and essentially well known.

[1] In the abstract formalisms the displayed variables serve only to indicate for which arguments (at most) substitution might move the element: i.e. the formal function remains fixed under any replacement, by another variable or function, of a variable not in this list.

[2] Subsequent identification, as well as arbitrary permutation, of arguments will be attained below by postulating the presence of "mutations": for every selfmap σ of the variable indices, $\sigma f(x_1, \ldots, x_n) := f(x_{\sigma 1}, \ldots, x_{\sigma n})$ is to be in F.

[3] Formally, let $()$ take precedence over $[]$. The ordinary associative law is $f[g(h)] = [f(g)](h)$. The superassociative law is $f[f_1(g_1, \ldots, g_{n_1}), f_2(g_{n_1+1}, \ldots, g_{n_1+n_2}), \ldots] = [f(f_1, f_2, \ldots)](g_1, \ldots, g_{n_1}, \ldots, g_{n_1+n_2}, \ldots)$.

The formalisms to be considered do not necessarily permit simultaneous substitution at different arguments: they might be restricted to single substitution at a designated argument. A simultaneous substitution at distinct arguments can of course be synthesized from single substitutions given at each of the arguments (in general dependent on the order in which they are made); alternatively, if F includes an (at least left) identity function — i.e. a unary e whose action is to reproduce the function substituted into it, $e(f) = f$ — then single substitutions could be construed as the multiple ones all but one of whose substituted terms is this e; more generally, partial substitutions as those whose complementary arguments are e.[4] The superassociative law would then yield the commutativity of single substitutions at different arguments (the substitution in either order is equal to the double substitution, which is what ensures that the deduced partial forms are unambiguous) as well as associativity at each argument i: i.e. equality of $f(f_1, \ldots, f_{i-1}, f_i(g_{n_1+\cdots+n_{i-1}+1}, \ldots), f_{i+1}, \ldots, f_n)$ with the other bracketing[5] $f(f_1, \ldots, f_i, \ldots, f_n)(e, \ldots, e, g, \ldots)$.

For the single substitution formalisms, both should be postulated; together they imply the superassociative law for the derived simultaneous substitution since one can rebracket one index at a time.

Associativity at each argument can be broken down further. In n-argument f_i, substitutions can be made one argument at a time: it suffices to have the (superassociative) equalities

$$\begin{aligned} &f(\ldots, e, f_i(\ldots, e, g_{j+1}, \ldots), e, \ldots)(\ldots, e, g_j, e, \ldots) \\ &\qquad = f(\ldots, e, f_i(\ldots, e, g_j, g_{j+1}, \ldots), e, \ldots) \end{aligned}$$

in order to be able to deduce (after substitution into the remaining places in f)

$$f(f_1, \ldots, f_i(g_1, \ldots, g_n), f_{i+1}, \ldots) = f(f_1, \ldots, f_i, f_{i+1}, \ldots)(e, \ldots, g_1, \ldots, g_n, \ldots).$$

The $f_{i'}$ substituted could have their argument places filled with g's or e. Substituting $g_{j'}$ into $f_{i'}$ on both sides with $f_{i'}$ filled with e, yields

$$\begin{aligned} &f(\ldots, f_{i'}(\ldots, g_{j'}, \ldots), \ldots, f_i(\ldots, g_j, \ldots) \ldots) \\ &\qquad = f(\ldots, f_{i'}, \ldots, f_i, \ldots)(\ldots, g_j, \ldots)(\ldots, g_{j'}, \ldots) \end{aligned}$$

and continuing for all the indices, the full superassociative law.

[4] Substitution into $f(e, \ldots, e)$ has the same effect as substitution into f. The map $f \to f(e, \ldots, e)$ is a retraction onto a set of representatives for the equivalence of inducing the same selfmap, on which e acts as a right identity, hence on which the representation is faithful.

[5] Of course this follows from the case $f_{i'}$, $i' \neq i = e$, and entails the same identity with the $f_{i'}$ possibly (partially) applied to g's.

To summarize: the superassociative law is necessary and sufficient for an abstractly given simultaneous substitution to be representable as a functional substitution. In the presence of a left identity function e, partial substitutions become available and compose to furnish the simultaneous substitution just when they commute. This commutativity follows from superassociativity and in conjunction with associativity at each argument becomes equivalent to it.

Turning next to the transformational, the replacement of the variables in a multi-argument function by others may be regarded as a transformation $f \to \sigma f(x_i, \ldots, x_n) := f(x_{\sigma 1}, \ldots, x_{\sigma n})$ where σ is a selfmap of the variable indices, made to act in this manner on the functions. On actual functions, this is a semigroup action — i.e., it is multiplicative; inasmuch as only finite argument functions are being considered, the relevant semigroup is that of the selfmaps moving only finitely many indices.

The index selfmaps which move only finitely many indices are generated by composition from the "transpositions" (m, n) which exchange n with m, and the "replacements" (n/m) which send n on m, leaving all other indices fixed. It suffices to have a single replacement and a subset of transpositions which generate them all — e.g. the $(n, n+1)$. Indeed, one gets all the transpositions from these by conjugation — e.g. $(1,3) = (2,3)(1,2)(2,3) = (1,2)(2,3)(1,2)$ — as well as all the replacements from $(2/1)$ — e.g. $(2,4)(1,3)(2/1)(1,3)(2,4) = (4/3)$. Every cycle is a composite of transpositions, unique up to cyclic permutation. An equational axiomatization of this semigroup may be found in Jónsson. (The denominator on the left of his (iv) should be z not y.)

That an abstract selfmap σ on the formal functions go over to this variable replacement on the represented functions comes to its commutativity with substitution:

$$(\sigma f)(f_1, \ldots, f_n) = f(f_{\sigma 1}, \ldots, f_{\sigma n}) \qquad \text{for every sequence} \quad \{f_i\}.$$

Since the right side is multiplicative, this already ensures multiplicativity, $(\sigma\sigma')f = \sigma(\sigma' f)$, of such a semigroup action. Thus the action is determined by that of a set of semigroup generators; when the Cayley representation is faithful, such a partially defined action has a (unique) extension to a full semigroup action.

Since the semigroup identity acts as the identity transformation, invertible elements act invertibly, so transpositions can be used to derive single substitution at every index and associativity at every argument from that at any one. If substitution is defined for the first argument (in every element), then that for the ith can be effected by preceding it with any invertible mutation which sends i to 1 (and following it with a mutation which rearranges the variables into a desired order). Then associativity at the first argument will entail it at every other.

This finally furnishes an (equational) axiomatization for the functional representability of a multi-argument function formalism with left identity which admits arbitrary composition and all argument mutations: viz., commutativity of mutation and composition (which it suffices to pose for semigroup generators of the mutations in the presence of axioms for those index selfmaps which move only finitely many integers — e.g., Jónsson's, in terms of the transpositions and replacements); and the superassociative law for the simultaneous substitution operation, or its equivalent in terms of individual substitution: commutativity of substitution at distinct arguments plus associativity at each (or any specified) argument.

$$\begin{aligned} e(f) &= f\,, \\ (\sigma f)(f_1,\dots,f_n) &= f(f_{\sigma 1},\dots,f_{\sigma n}) \qquad \text{for every sequence } \{f_i\}\,, \\ f(\dots,e,f_i,e,\dots)f_{i'} &= f(\dots,e,f_{i'},e,\dots)f_i\,, \\ f(f_1,\dots,f_i(g_1,\dots,g_n),f_{i+1},\dots) &= f(f_1,\dots,f_i,f_{i+1},\dots)(e,\dots,g_1,\dots,g_n,\dots)\,. \end{aligned}$$

Mal'cev's preiterative algebras

Besides substitution of function $g(x_1,\dots,x_m)$ at the first argument of $f(x_1,\dots,x_n)$, written $f\star g := f\big(g(x_1,\dots,x_m),x_{m+1},\dots,x_{m+n-1}\big)$, one has three mutational operations: ζ, which is to give cyclic permutation of arguments, τ which will exchange the first two while leaving any others fixed, and $\triangle$ leaving the first fixed and shifting all the others down one. Composing the index transformations these represent will yield the full transformation semigroup on n symbols: Indeed, repeatedly conjugating τ with ζ yields all transpositions while (2/1) can be derived from Δ by preceding it with the transposition of 2 with n and following it with a cyclic permutation of numeral 3 to n. Since (composed) mutations act faithfully on the indices of the displayed argument variables, they induce the full semigroup of finitely moving index transformations.[6] Therefore their composing to a representation of the full transformation semigroup acting on each f is required. With the full semigroup of mutations in place, single substitution of f_i at the ith argument may be obtained by conjugating $\star$ with the interchange of the first with the ith argument; as a consequence of this operation, the arguments in f beyond i have their indices each increased by $n_i - 1$; on substituting g for one of the arguments in f_i these indices are increased a further $m-1$: the total increase $n_i + m - 2$ is the same as the increase due to substituting the composite $f_i(\dots,g,\dots)$, which

[6]Formally, Mal'cev indicates only self-maps of the variables displayed in each function into themselves; however, the algebra can always be enlarged to accommodate arbitrary finite permutations, insofar as these are not understood to be present.

has $n_i + m - 1$ arguments, into f. Thus substitution is associative at each argument. (This does not hold for the form of multiple substitutions proposed by Mal'cev, next to last equation p. 396; it must be postulated if $\star$ is introduced as an abstract binary operation.) Then it suffices to postulate in addition commutativity of single substitutions at different arguments.

The "pure" combinatory formalism

It is a formalism for a groupoid (called "applicative structure") in which a finite number of special elements act on their immediately following symbols so as to effect the operations of composition and argument mutation on the latter's represented form as multi-argument functions.

The elements of a groupoid can be construed as selfmaps, say by left translation: f sends each g to fg. The form fgh could be interpreted either as the value of f at gh or the value of fg at h (different unless the groupoid is associative); it could also be interpreted as the value of f as a function of two variables, conventionally derived from the latter of the two unary interpretations, which is written fgh without parentheses: i.e. association is understood to be from the left unless parenthesized to indicate the contrary. (By holding the intermediate variable fixed one recovers this unary composite from its binary interpretation.) More generally, for every $n > 0$ each f gives rise to the n-place selfmap which sends $g_1, \ldots, g_n$ to $fg_1 \ldots g_n$, understood as association to the left. (Hence these are composites of the successive unary such functions: the n-place function is synthesized from these by successively applying the value qua unary function to the next argument.) Thus every groupoid term describes a unique (multi-place) selfmap in which substitutions (as indicated by parentheses, may) have been made.

Observe that the set of selfmaps obtained is closed under composition and argument mutation, the latter commuting with substitution; superassociativity will turn out to be the only requirement for functional representability. Applied to the unary selfmaps, "super" becomes ordinary associativity, so the groupoid would have to be a semigroup; conversely, every semigroup is seen to yield a superassociative system when the terms $fg_1 \ldots g_n$ are interpreted as the values of the n-place selfmaps f: Every bracketing of such a term would yield the same element as value also for the construed smaller placed f in which functions g have been substituted for some arguments.

Besides the internal binary operation of application, resulting in strings of functional symbols (plus parentheses), the combinatory formalism features operators of transformation, called combinators, whose symbols are however embedded in the strings on a par with the function symbols and operate by transforming the immediately following symbols in a specified manner. We select those corresponding to Mal'cev's $\star, \tau, \Delta$. They are the binary elementary compositor $B(f,g) = f(g)$, i.e. $Bfgh = f(gh)$ and the unary elementary permutator C of interchange, and duplicator W of identification, of the first two arguments in the next following function symbol.

These suffice to obtain all the argument mutations: e.g. $BCfghk = C(fg)hk = fgkh$, $BWfghk = W(fg)hk = fghh$ for the second and third arguments; and by prefixing further B's one can interchange or identify any pair of successive arguments; these transformations can be combined to yield any finite argument mutation. (The mutations are generated by the transpositions $(n+1, n+2)$ of arguments, obtained from $B^n C$ and the replacement $(2/1)$ of the second argument by the first, obtained from W.) Under the interpretation $f(g, h, \ldots)$ of $fgh\ldots$, these operations effect the argument mutations they promise.

Finally, $BBfghk = B(fg)hk = fg(hk)$ gives substitution into the second argument of f, $B(BB)f$ into the third, and so on (the inductive step is to write the next prefix of B's as B followed by the preceding prefix in parentheses): thus one can realize the simultaneous substitution $f(f_1, \ldots, f_n)$. Similarly, by starting with f and successively appending B to the preceding prefix parenthesized with f, one obtains combinators which yield $f(f_1 g_1 \cdots g_n)$ from $f f_1 g_1 \cdots g_n$ for every n. Combining with substitution at any argument as just described enables one to formulate the associative axiom schema as an equality in combinators.

Thus a groupoid containing elements which satisfy the defining identities of B, C and W may be construed as an abstract function system equipped with internal operations of simultaneous substitution and mutation.[7] Then (super) associativity is the condition that the functional representation preserves these abstract operations.

Remarks

Rosenberg has developed a set of 19 axioms for preiterative algebras (universal algebraic version) without however characterizing the functional representable ones. His proposed axiomatization construes ζ and Δ as simple unary operators: thus, they are not actual mutations since their action does not depend on the list of variables displayed in the function element to which they are applied, and so they could not be synthesized mutationally. He defines the "arity" abstractly as the smallest number of variable identifications which move

[7]The result furnishes a version of "combinatorial completeness": Every form in n (not necessarily distinct) symbols drawn from $x_1, \ldots, x_n$ can be realized by some combinator in B, C, W applied to this string.

the function; but since this could be smaller than the number of the displayed variables, it fails to pin down the desired action.

The Basic Representation (for a slightly different system) may be found in Dicker, Whitlock, Lausch–Nöbauer and the writings of Schweizer-Sklar.

Apart from the difference in the way substitution at arguments other than the first is handled, Mal'cev's formalism is seen to have been anticipated by Schönfinkel. More precisely, this is seen to hold for his "iterative algebras" which have the additional ingredient of "selectors" or "projections".

It might be noted that an iterative algebra is the same thing as a "clone" since the selectors or projections can be used to synthesize all argument mutations and to effect substitution with identification of arguments (see Menger); conversely, a preiterative algebra can be embedded in an iterative simply by adjoining the projections and "dummy" variables to each of the elements.

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