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# POINT SETS WITH LOW $L_p$-DISCREPANCY

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ABSTRACT. In this paper we study the $L_p$-discrepancy of digitally shifted Hammersley point sets. While it is known that the (unshifted) Hammersley point set (which is also known as Roth net) with $N$ points has $L_p$-discrepancy ($p$ an integer) of order $(\log N)/N$, we show that there always exists a shift such that the digitally shifted Hammersley point set has $L_p$-discrepancy ($p$ an even integer) of order $\sqrt{\log N}/N$ which is best possible by a result of W. Schmidt. Further we concentrate on the case $p = 2$. We give very tight lower and upper bounds for the $L_2$-discrepancy of digitally shifted Hammersley point sets which show that the value of the $L_2$-discrepancy of such a point set mostly depends on the number of zero coordinates of the shift and not so much on the position of these.

## 1. Introduction

For a point set $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$ of points in the 2-dimensional unit-interval $[0,1)^2$ the discrepancy function is defined as

$$\Delta(\alpha, \beta) := A_N\big([0,\alpha) \times [0,\beta)\big) - N\alpha\beta$$

for $0 < \alpha, \beta \leq 1$, where $A_N\big([0,\alpha) \times [0,\beta)\big)$ denotes the number of $n$ satisfying $0 \leq n \leq N-1$ and $\mathbf{x}_n \in [0,\alpha) \times [0,\beta)$. Now the $L_p$-discrepancy, for $p > 0$, of the point set is defined as the $L_p$-norm of the discrepancy function divided by the cardinality of the point set and is a measure for the irregularity of distribution of the point set over $[0,1)^2$ (see for example [1], [4], [9], [11]). I.e., for $0 < p < \infty$

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we set

$$L_{p,N}\big(\mathbf{x}_0,\dots,\mathbf{x}_{N-1}\big) := \frac{1}{N}\left(\int_0^1\int_0^1 |\Delta(\alpha,\beta)|^p \,\mathrm{d}\alpha\,\mathrm{d}\beta\right)^{\frac{1}{p}}.$$

For $p=\infty$, we get the usual star discrepancy

$$D_N^*\big(\mathbf{x}_0,\dots,\mathbf{x}_{N-1}\big) := \frac{1}{N}\sup_{0<\alpha,\beta\le 1}|\Delta(\alpha,\beta)|$$

of the point set.

From Schmidt [16] we know that for any $p>1$ there exists a constant $c_p>0$ such that for the $L_p$-discrepancy of any point set $\mathbf{x}_0,\dots,\mathbf{x}_{N-1}$ in $[0,1)^2$ we have

$$NL_{p,N} \ge c_p\sqrt{\log N}\,. \tag{1}$$

(For $p=2$ this result was proven by Roth [15].)

In this paper we consider the $L_p$-discrepancy of the digitally shifted Hammersley point set in base 2 with $N=2^m$ points. This is a generalization of the well known Hammersley point set in base 2 (which is also known as Roth net as it was first suggested by Roth [15]) and it is constructed as follows. Let $m\in\mathbb{N}$, $C_1$ the $m\times m$ identity matrix over $\mathbb{Z}_2$, and $C_2$ the $m\times m$ matrix given by

$$C_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1\\ 0 & 0 & \dots & 1 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 1 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Further choose vectors $\vec{\sigma}_1,\vec{\sigma}_2\in\mathbb{Z}_2^m$, $\vec{\sigma}_j=(\sigma_1^{(j)},\dots,\sigma_m^{(j)})^T$, $1\le j\le 2$.

For $n=0,\dots,2^m-1$ let $n=n_0+n_1 2+\dots+n_{m-1}2^{m-1}$ be the base 2 representation of $n$. Now for $j=1,2$ define

$$x_n^{(j)} := \frac{y_1^{(j)}(n)}{2}+\frac{y_2^{(j)}(n)}{2^2}+\dots+\frac{y_m^{(j)}(n)}{2^m},$$

where

$$\big(y_1^{(j)}(n),\dots,y_m^{(j)}(n)\big)^T = C_j(n_0,\dots,n_{m-1})^T+\vec{\sigma}_j\in\mathbb{Z}_2^m\,.$$

Then the point set $\mathbf{x}_0,\dots,\mathbf{x}_{2^m-1}$ with $\mathbf{x}_n=\big(x_n^{(1)},x_n^{(2)}\big)$ is the digitally shifted Hammersley point set in base 2 with $N=2^m$ points and shift vectors $\vec{\sigma}_1$ and $\vec{\sigma}_2$. If we choose $\vec{\sigma}_1=\vec{\sigma}_2=(0,\dots,0)^T$, then we obtain the classical Hammersley point set in base 2. Further we remark that any digitally shifted Hammersley point set in base 2 with $N=2^m$ points is a $(0,m,2)$-net in base 2 as defined in [12]; see also [13].

As pointed out by Kritzer [7], it is sufficient to consider shifts only in the second coordinate, i.e., $\vec{\sigma}_1 = (0,\dots,0)^T$, since for $\vec{\sigma}_1, \vec{\sigma}_2 \in \mathbb{Z}_2^m$ the shifts

$$\vec{\tau}_1 = (0,\dots,0)^T \qquad \text{and} \qquad \vec{\tau}_2 = C_2\vec{\sigma}_1 + \vec{\sigma}_2 \in \mathbb{Z}_2^m\,,$$

yield the same digitally shifted Hammersley point set.

From now on we consider the digitally shifted Hammersley point set in base 2 with $N = 2^m$ points and with shift $\vec{\sigma} = (\sigma_1,\dots,\sigma_m)^T \in \mathbb{Z}_2^m$ (in the second coordinate). We denote this point set by $H(\vec{\sigma})$. The classical Hammersley point set, i.e., $\vec{\sigma} = (0,\dots,0)^T$, will be simply denoted by $H$.

The star discrepancy of the Hammersley point set was studied in detail in [3], [5], [6], [10] and the star discrepancy of the shifted Hammersley point set was studied recently in [7], [8]. Here we deal with the $L_p$-discrepancy, $2 \le p < \infty$, $p$ even, of the digitally shifted Hammersley point set. In [14] it is shown that for any $p \in \mathbb{N}$ for the $L_p$-discrepancy of the classical Hammersley point set $H$ we have

$$\big(NL_{p,N}(H)\big)^p = \frac{m^p}{2^{3p}} + O\big((\log N)^{p-1}\big)\,,$$

where $N = 2^m$ and where the constant in the $O$-notation only depends on $p$. (See also Chen and Skriganov [2] for the special case $p = 2$.)

For $p = 2$ we have the following more exact result due to Vilenkin [17], Halton and Zaremba [6] and Pillichshammer [14],

$$\big(NL_{2,N}(H)\big)^2 = \frac{m^2}{64} + \frac{29m}{192} + \frac{3}{8} - \frac{m}{16\cdot 2^m} + \frac{1}{4\cdot 2^m} - \frac{1}{72\cdot 2^{2m}}\,,$$

where $N = 2^m$. Further Halton and Zaremba [6] gave a digital shift $\vec{\sigma}_{HZ}$ such that the $L_2$-discrepancy of the resulting point set $H\big(\vec{\sigma}_{HZ}\big)$ is given by

$$\big(NL_{2,N}\big(H(\vec{\sigma}_{HZ})\big)\big)^2 = \frac{5m}{192} + \frac{3}{8} - \frac{7\varepsilon_m}{64} + \frac{1}{4\cdot 2^m} + \frac{\varepsilon_m}{16\cdot 2^m} - \frac{1}{72\cdot 2^{2m}}\,, \tag{2}$$

where $N = 2^m$ and where $\varepsilon_m$ is zero if $m$ is even and one if $m$ is odd. The shift vector given by Halton and Zaremba is

$$\vec{\sigma}_{HZ} = \begin{cases} (1,0,1,0,\dots,1,0)^T & \text{if } m \text{ is even},\\ (1,0,1,0,\dots,1,0,1)^T & \text{if } m \text{ is odd}.\end{cases}$$

So the number of zero coordinates of $\vec{\sigma}_{HZ}$ is $m/2$ for even $m$ and $(m-1)/2$ for odd $m$.

It is the aim of this paper to show that for any even integer $p$ and $m \in \mathbb{N}$ there exists a shift $\vec{\sigma} \in \mathbb{Z}_2^m$ such that the $L_p$-discrepancy of the point set $H(\vec{\sigma})$ is of best possible order with respect to the result of Schmidt (Theorem 1 and Corollary 1). Further we prove very tight lower and upper bounds for the

$L_2$-discrepancy of a digitally shifted Hammersley point set (Theorem 2 and Theorem 3). We compare our results with the result from Halton and Zaremba and draw some interesting consequences. The results are presented in the subsequent Section 2. In Section 3 we collect some lemmas which will be needed in the proofs of our theorems. Finally the proofs of our theorems are given in Section 4 and Section 5.

## 2. The $L_p$-discrepancy of digitally shifted Hammersley point sets

First we have the following result which shows that on the average the $L_p$-discrepancy of a digitally shifted Hammersley point set is of best possible order with respect to Schmidt's lower bound.

**Theorem 1.** *Let $p$ be an even positive integer and let $m \in \mathbb{N}$. Then we have*

$$\frac{1}{2^m} \sum_{\vec{\sigma} \in \mathbb{Z}_2^m} \big(NL_{p,N}\big(H(\vec{\sigma})\big)\big)^p \leq \frac{2S(p,p/2)}{2^{2p}}\, m^{p/2} + O\big(m^{p/2-1}\big)\,,$$

*where $N = 2^m$, the constant in the $O$-notation only depends on $p$, and $S(p,p/2)$ is a Stirling number of the second kind.*

The proof of Theorem 1 is deferred to Section 4. From this theorem we obtain the following results.

**Corollary 1.** *Let $p$ be an even integer and let $m \in \mathbb{N}$. Then there exists a shift vector $\vec{\sigma}_* \in \mathbb{Z}_2^m$ such that the $L_p$-discrepancy of the digitally shifted Hammersley point set $H\big(\vec{\sigma}_*\big)$ is bounded by*

$$\big(NL_{p,N}\big(H(\vec{\sigma}_*)\big)\big)^p \leq \frac{2S(p,p/2)}{2^{2p}}\, m^{p/2} + O(m^{p/2-1})\,,$$

*where $N = 2^m$, the constant in the $O$-notation only depends on $p$, and $S(p,p/2)$ is a Stirling number of the second kind.*

**Remark 1.** The bound in Corollary 1 is best possible with respect to the lower bound from Schmidt (1).

**Corollary 2.** *Let $p$ be an even integer. For any $\varepsilon > 0$ and any $c > 0$ we have*

$$\lim_{m\to\infty} \frac{1}{2^m} \# \left\{\vec{\sigma} \in \mathbb{Z}_2^m :\ 2^m L_{p,2^m}\big(H(\vec{\sigma})\big) < c m^{\frac{1}{2}+\varepsilon}\right\} = 1\,.$$

Proof. With Theorem 1 it follows that

$$\begin{aligned}
&\frac{2S(p,p/2)}{2^{2p}}\, m^{p/2} + O\big(m^{p/2-1}\big)\\
\geq\ & \frac{1}{2^m}\sum_{\vec{\sigma}\in\mathbb{Z}_2^m} \big(2^m L_{p,2^m}\big(H(\vec{\sigma})\big)\big)^p\\
\geq\ & \frac{1}{2^m} c^p m^{(\frac{1}{2}+\varepsilon)p} \# \left\{\vec{\sigma}\in\mathbb{Z}_2^m:\ 2^m L_{p,2^m}\big(H(\vec{\sigma})\big) \geq c m^{\frac{1}{2}+\varepsilon}\right\}\\
=\ & \frac{1}{2^m} c^p m^{(\frac{1}{2}+\varepsilon)p} \Big(2^m - \# \left\{\vec{\sigma}\in\mathbb{Z}_2^m:\ 2^m L_{p,2^m}\big(H(\vec{\sigma})\big) < c m^{\frac{1}{2}+\varepsilon}\right\}\Big).
\end{aligned}$$

From this we obtain

$$\begin{aligned}
&\frac{1}{2^m}\#\left\{\vec{\sigma}\in\mathbb{Z}_2^m:\ 2^m L_{p,2^m}\big(H(\vec{\sigma})\big) < c^p m^{\frac{1}{2}+\varepsilon}\right\}\\
&\qquad \geq 1 - \frac{2S(p,p/2)}{2^{2p}c^p}\frac{1}{m^{\varepsilon p}} + O\left(\frac{1}{m^{1+\varepsilon p}}\right).
\end{aligned}$$

The result follows. □

If $p=2$, it is possible to obtain more precise results, which shall be outlined in the following. First we prove an upper bound for the $L_2$-discrepancy of a digitally shifted Hammersley point set.

**Theorem 2.** *Let $\vec{\sigma}\in\mathbb{Z}_2^m$ and let $l$ denote the number of zero coordinates of $\vec{\sigma}$. Then we have*

$$\begin{aligned}
&\big(NL_{2,N}\big(H(\vec{\sigma})\big)\big)^2\\
\leq\ & \frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{7}{16} + \frac{m}{8\cdot 2^m} - \frac{l}{4\cdot 2^m} + \frac{3}{16\cdot 2^m} - \frac{1}{72\cdot 4^m},
\end{aligned}$$

*where $N=2^m$.*

We also have the following lower bound.

**Theorem 3.** *Let $\vec{\sigma}\in\mathbb{Z}_2^m$ and let $l$ denote the number of zero coordinates of $\vec{\sigma}$. Then we have*

$$\begin{aligned}
&\big(NL_{2,N}\big(H(\vec{\sigma})\big)\big)^2\\
\geq\ & \frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{5}{16} + \frac{m}{8\cdot 2^m} - \frac{l}{4\cdot 2^m} + \frac{5}{16\cdot 2^m} - \frac{1}{72\cdot 4^m},
\end{aligned}$$

*where $N=2^m$.*

The proofs of Theorem 2 and Theorem 3 will be given in Section 5. Observe that our upper and lower bound only differ by the almost negligible quantity $\frac{1}{8}(1-2^{-m})$.

**Remark 2.** With the help of Theorem 2 and Theorem 3 we can improve Theorem 1 in the case $p = 2$ to

$$\frac{m}{24} + \frac{5}{16} + \frac{5}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m} \leq \frac{1}{2^m} \sum_{\vec{\sigma} \in \mathbb{Z}_2^m} \big(NL_{2,N}\big(H(\vec{\sigma})\big)\big)^2$$
$$\leq \frac{m}{24} + \frac{7}{16} + \frac{3}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m},$$

where $N = 2^m$.

Now we obtain the following result.

**Corollary 3.** *Let $m \in \mathbb{N}$ and $N = 2^m$.*

(1) *If $m$ is even, let $\vec{\sigma} \in \mathbb{Z}_2^m$ with $l = m/2$ zero coordinates. Then we have*

$$\frac{5m}{192} + \frac{5}{16} + \frac{5}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m} \leq \big(NL_{2,N}\big(H(\vec{\sigma})\big)\big)^2$$
$$\leq \frac{5m}{192} + \frac{7}{16} + \frac{3}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m}.$$

(2) *If $m$ is odd, let $\vec{\sigma} \in \mathbb{Z}_2^m$ with $l = (m-1)/2$. We then get*

$$\frac{5m}{192} + \frac{13}{64} + \frac{7}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m} \leq \big(NL_{2,N}\big(H(\vec{\sigma})\big)\big)^2$$
$$\leq \frac{5m}{192} + \frac{21}{64} + \frac{5}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m}.$$

Of course the $L_2$-discrepancy (2) of the point set given by Halton and Zaremba lies between the bounds given in Corollary 3. Theorem 2 and Theorem 3 show that the value of the $L_2$-discrepancy of a digitally shifted Hammersley point set does mostly depend on the number of zero coordinates in the shift vector and not so much on the position of these. In fact, numerical results suggest that the value of the $L_2$-discrepancy of a digitally shifted Hammersley point set is exactly the same for all shift vectors with the same number of zeros. We remark that this is not the case for the star discrepancy of digitally shifted Hammersley point sets, see [8].

It is remarkable that it is possible to obtain better results than Halton and Zaremba by using the bounds outlined here.

**COROLLARY 4.** *Let $m \in \mathbb{N}$ and $N = 2^m$.*

(1) *If $m$ is even, let $\vec{\sigma} \in \mathbb{Z}_2^m$ with $l = (m-4)/2$ zero coordinates. Then we have*
$$\frac{5m}{192} + \frac{1}{16} + \frac{13}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m} \le \big(NL_{2,N}\big(H(\vec{\sigma})\big)\big)^2$$
$$\le \frac{5m}{192} + \frac{3}{16} + \frac{11}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m}\,.$$

(2) *If $m$ is odd, let $\vec{\sigma} \in \mathbb{Z}_2^m$ with $l = (m-3)/2$. We then get*
$$\frac{5m}{192} + \frac{5}{64} + \frac{11}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m} \le \big(NL_{2,N}\big(H(\vec{\sigma})\big)\big)^2$$
$$\le \frac{5m}{192} + \frac{13}{64} + \frac{9}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m}\,.$$

The upper bound given in Corollary 4 for even $m \ge 4$ is lower than the value (2) of Halton and Zaremba. The same is true for the upper bound for odd $m \ge 3$. The special choice of $l$ to obtain the latter bounds is motivated by the following observation. If we take, for fixed $m$,
$$g(l) := \frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{7}{16} + \frac{m}{8 \cdot 2^m} - \frac{l}{4 \cdot 2^m} + \frac{3}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m}\,,$$
as a function of $l$, we find that $g$ has a minimum at
$$l = \frac{m-4}{2} + \frac{1}{2^{m-1}}\,.$$

Finally we obtain the following interesting result.

**THEOREM 4.** *We have*
$$\lim_{m \to \infty} \left( \min_{\vec{\sigma} \in \mathbb{Z}_2^m} \frac{2^m L_{2,2^m}\big(H(\vec{\sigma})\big)}{\sqrt{m}} \right) = \sqrt{\frac{5}{192}}\,.$$

Proof. It is easy to show that
$$\frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{5}{16} \ge \frac{5m}{192}\,,$$
for any $l \in \{0, 1, \dots, m\}$. Therefore it follows from Theorem 3 that for any $m \in \mathbb{N}$ and any $\vec{\sigma} \in \mathbb{Z}_2^m$ it is
$$\big(2^m L_{2,2^m}\big(H(\vec{\sigma})\big)\big)^2 \ge \frac{5m}{192} - \frac{m}{8 \cdot 2^m} + \frac{5}{16 \cdot 2^m} - \frac{1}{72 \cdot 4^m}\,.$$
Together with Corollary 3 (or equality (2)) the result follows. □

## 3. Auxiliary results

In this section we collect some lemmas which will be used in the proofs of our results from the previous section. First we present a very useful formula for the discrepancy function of a digitally shifted Hammersley point set. Throughout the paper we use the following notation: for

$$\alpha = \frac{\alpha_1}{2} + \cdots + \frac{\alpha_m}{2^m} \qquad \text{and} \qquad \beta = \frac{\beta_1}{2} + \cdots + \frac{\beta_m}{2^m}$$

with $\alpha_i, \beta_i \in \{0,1\}$ we say in the following $\alpha$ (resp. $\beta$) is "$m$-bit".

**Lemma 1.** *Let $\alpha$ and $\beta$ be $m$-bit and let $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)^T \in \mathbb{Z}_2^m$. Then for the discrepancy function of $H(\vec{\sigma})$ we have*

$$\Delta(\alpha, \beta) = \sum_{u=0}^{m-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} \big(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\big) \,,$$

*where $\oplus$ denotes addition modulo 2, $\|\cdot\|$ is the distance to the nearest integer function, and where $j(u)$, $0 \le u \le m-1$, is defined by*

$$j(u) = \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } \alpha_{m+1-j} = \beta_j \oplus \sigma_j \\ & \text{for } j = 1, \ldots, u, \\ \max\{j \le u : \ \alpha_{m+1-j} \ne \beta_j \oplus \sigma_j\} & \text{else.} \end{cases}$$

*Further we set $\alpha_{m+1} := 0$.*

Proof. From [10, Theorem 1] this result follows for the (unshifted) Hammersley point set. It easily follows from the proof that the assertion is also true for the digitally shifted Hammersley point set. □

**Remark 3.** Let $\alpha, \beta \in [0,1]$ (not necessarily $m$-bit). Let again $H(\vec{\sigma})$ denote the Hammersley point set that is digitally shifted by an arbitrary vector $\vec{\sigma} \in \mathbb{Z}_2^m$. Since all points of $H(\vec{\sigma})$ have $m$-bit coordinates, it follows that

$$\Delta(\alpha, \beta) = \Delta\big(\alpha(m), \beta(m)\big) + 2^m \big(\alpha(m)\beta(m) - \alpha\beta\big) \,, \tag{3}$$

where $\alpha(m)$ and $\beta(m)$ are the smallest $m$-bit numbers larger than or equal to $\alpha$ and $\beta$, respectively. If $\alpha$ is greater than $1 - 2^{-m}$, choose $\alpha(m) = 1$. Similarly, choose $\beta(m) = 1$ if $\beta$ is greater than $1 - 2^{-m}$.

**Lemma 2.** *Let $\vec{\sigma}$ and $j(u)$ be defined as in* Lemma 1, *and choose an $m$-bit number $\beta$. Let $1 \le k \le m-1$ be an integer and $v_1, \ldots, v_k \in \{0, 1, \ldots, m-1\}$ with $v_i \neq v_j$ for $1 \le i \neq j \le m-1$. Then we have*

$$\sum_{2^m\alpha=1}^{2^m-1} \prod_{i=1}^{k} \bigl(\alpha_{m-v_i} \oplus \alpha_{m+1-j(v_i)}\bigr) = 2^{m-k}\,.$$

*(Here and in the following $\sum\limits_{2^m\alpha=1}^{2^m-1}$ means summation over all $\alpha > 0$ $m$-bit.)*

Proof. Observe that for given $u \in \{0, \ldots, m-1\}$, $j(u)$ depends only on $\alpha_{m+1-u}, \ldots, \alpha_m$ and not on $\alpha_1, \ldots, \alpha_{m-u}$. The rest of the proof follows exactly the lines of the proof of [14, Lemma 2]. □

**Lemma 3.** *Let $\vec{\sigma}$ and $j(u)$ be defined as in* Lemma 1. *Let $\alpha$ be $m$-bit and choose $u \in \{0, \ldots, m-1\}$ arbitrary. Then we have*

$$\begin{aligned} 2^{2m-2} - 2^{m-2} - 2^{m+u-2} &\le \sum_{2^m\alpha=1}^{2^m-1} 2^m\alpha\bigl(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\bigr) \\ &\le 2^{2m-2} - 2^{m-2} + 2^{m+u-2}\,. \end{aligned}$$

Proof. We have

$$\begin{aligned} &\sum_{2^m\alpha=1}^{2^m-1} 2^m\alpha\bigl(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\bigr) \\ &= \sum_{i=1}^{m} 2^{m-i} \sum_{\alpha_1,\ldots,\alpha_m=0}^{1} \alpha_i\bigl(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\bigr) \\ &= \sum_{i=1}^{m-u-1} 2^{m-i} \sum_{\substack{\alpha_1,\ldots,\alpha_m=0\\ \alpha_i\neq 0}}^{1} \bigl(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\bigr) + 2^u \sum_{\substack{\alpha_1,\ldots,\alpha_m=0\\ \alpha_{m-u}\neq 0}}^{1} \bigl(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\bigr) \\ &\qquad + \sum_{i=m+1-u}^{m} 2^{m-i} \sum_{\substack{\alpha_1,\ldots,\alpha_m=0\\ \alpha_i\neq 0}}^{1} \bigl(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\bigr) \end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^{m-u-1} 2^{m-i} \sum_{\alpha_{m+1-u},\dots,\alpha_m=0}^{1} \left( \sum_{\alpha_1,\dots,\alpha_{i-1}=0}^{1} \sum_{\alpha_{i+1},\dots,\alpha_{m-u}=0}^{1} \left(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\right)\right) \\ &\qquad\qquad + 2^u \sum_{\alpha_1,\dots,\alpha_{m-u-1}=0}^{1} \sum_{\alpha_{m+1-u},\dots,\alpha_m=0}^{1} \left(1 \oplus \alpha_{m+1-j(u)}\right) \\ &\quad + \sum_{i=m+1-u}^{m} 2^{m-i} \sum_{\alpha_{m+1-u},\dots,\alpha_{i-1}=0}^{1} \sum_{\alpha_{i+1},\dots,\alpha_m=0}^{1} \left( \sum_{\alpha_1,\dots,\alpha_{m-u}=0}^{1} \left(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\right)\right) \\ &=: \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

Since $j(u)$ depends only on $\alpha_{m+1-u},\dots,\alpha_m$ and not on $\alpha_1,\dots,\alpha_{m-u}$ and since $0 \le j(u) \le u$, it follows that

$$\Sigma_1 = \sum_{i=1}^{m-u-1} 2^{m-i} 2^{m-u-2} 2^u,$$

$$\Sigma_3 = \sum_{i=m+1-u}^{m} 2^{m-i} 2^{m-u-1} 2^{u-1}.$$

Further we have

$$0 \le \Sigma_2 \le 2^u 2^{m-1}.$$

This yields the result. □

**Lemma 4.** *Choose an $m$-bit number $\beta$. Let $1 \le k \le m-1$ be an integer and $v_1,\dots,v_k \in \{0,1,\dots,m-1\}$ with $v_i \ne v_j$ for $1 \le i \ne j \le m-1$. Further let $r_1,\dots,r_k$ be positive integers.*

(a) *We have*

$$\frac{2^m}{4^{r_1+\dots+r_k}} \le \sum_{2^m\beta=1}^{2^m-1} \|2^{v_1}\beta\|^{r_1} \cdots \|2^{v_k}\beta\|^{r_k} \le \frac{2^m}{2^{r_1+\dots+r_k} 2^k}.$$

*Moreover, we have equality for the lower and for the upper bound if $r_1 = \dots = r_k = 1$.*

(b) *For $0 \le u \le m-1$ we have*

$$\sum_{2^m\beta=1}^{2^m-1} \|2^u\beta\|^2 = \frac{2^{2m} + 2^{2u+1}}{3 \cdot 2^{m+2}}.$$

Proof. Part (b) and the upper bound in part (a) is [14, Lemma 3]. The proof of the lower bound in part (a) is similar to the proof of the upper bound. □

## 4. The proof of Theorem 1

The following lemma is the first step in the proof of Theorem 1.

**Lemma 5.** *Let $p$ be a positive integer and denote by $\Delta$ the discrepancy function of a digitally shifted Hammersley point set $H(\vec{\sigma})$ with $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m) \in \mathbb{Z}_2^m$. Then we have*

$$\frac{1}{2^m} \sum_{\sigma_1, \ldots, \sigma_m = 0}^{1} \frac{1}{2^{2m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right)^p = 0$$

*for odd $p$, and*

$$\begin{aligned} \frac{m^{p/2} S(p, p/2)}{2^{2p+p/2}} + O(m^{p/2-1}) &\leq \frac{1}{2^m} \sum_{\sigma_1, \ldots, \sigma_m = 0}^{1} \frac{1}{2^{2m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right)^p \\ &\leq \frac{m^{p/2} S(p, p/2)}{2^{2p}} + O\big(m^{p/2-1}\big) . \end{aligned}$$

*for even $p$, where the constant in the $O$-notation only depends on $p$ and $S(p, p/2)$ is a Stirling number of the second kind.*

Proof. We have

$$\begin{aligned} &\frac{1}{2^m} \sum_{\sigma_1, \ldots, \sigma_m = 0}^{1} \frac{1}{2^{2m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right)^p \\ &\qquad = \frac{1}{2^m} \sum_{\sigma_1, \ldots, \sigma_m = 0}^{1} \frac{1}{2^{2m}} \sum_{2^m\alpha=1}^{2^m-1} \sum_{2^m\beta=1}^{2^m-1} \Delta(\alpha, \beta)^p . \end{aligned}$$

Applying Lemma 1, the term above equals

$$\frac{1}{2^m} \sum_{\sigma_1, \ldots, \sigma_m = 0}^{1} \frac{1}{2^{2m}} \sum_{2^m\alpha=1}^{2^m-1} \sum_{2^m\beta=1}^{2^m-1} \prod_{i=1}^{p} \left( \sum_{u_i=0}^{m-1} \|2^{u_i}\beta\| \, (-1)^{\sigma_{u_i+1}} \big(\alpha_{m-u_i} \oplus \alpha_{m+1-j(u_i)}\big) \right)$$

which, by Lemma 2, is equal to

$$\frac{1}{2^{2m}} \sum_{u_1,\ldots,u_p=0}^{m-1} \frac{1}{2^m} \sum_{\sigma_1,\ldots,\sigma_m=0}^{1} \prod_{i=1}^{p} (-1)^{\sigma_{u_i+1}} \left( \sum_{2^m\beta=1}^{2^m-1} \prod_{i=1}^{p} \|2^{u_i}\beta\| \right) \times$$
$$\times \left( \sum_{2^m\alpha=1}^{2^m-1} \prod_{i=1}^{p} (\alpha_{m-u_i} \oplus \alpha_{m+1-j(u_i)}) \right)$$
$$= \frac{1}{2^{2m}} \sum_{u_1,\ldots,u_p=0}^{m-1} \frac{1}{2^m} \sum_{\sigma_1,\ldots,\sigma_m=0}^{1} \prod_{i=1}^{p} (-1)^{\sigma_{u_i+1}} \left( \sum_{2^m\beta=1}^{2^m-1} \prod_{i=1}^{p} \|2^{u_i}\beta\| \right) 2^{m-k(u_1,\ldots,u_p)}$$
$$= \frac{1}{2^m} \sum_{u_1,\ldots,u_p=0}^{m-1} \frac{1}{2^{k(u_1,\ldots,u_p)}} \sum_{2^m\beta=1}^{2^m-1} \prod_{i=1}^{p} \|2^{u_i}\beta\| \left( \frac{1}{2^m} \sum_{\sigma_1,\ldots,\sigma_m=0}^{1} \prod_{i=1}^{p} (-1)^{\sigma_{u_i+1}} \right),$$

where $k(u_1,\ldots,u_p)$ is the number of different $u_i$s. Let now $v_1,\ldots,v_k$ be the different $u_i$s ($k = k(u_1,\ldots,u_p)$) such that $v_1$ appears $r_1$ times, ..., $v_k$ appears $r_k$ times ($r_1 + \cdots + r_k = p$). Observe that

$$\frac{1}{2^m} \sum_{\sigma_1,\ldots,\sigma_m=0}^{1} \prod_{i=1}^{p} (-1)^{\sigma_{u_i+1}} = \frac{1}{2^m} 2^{m-k} \sum_{\sigma_1,\ldots,\sigma_k=0}^{1} \prod_{i=1}^{k} (-1)^{\sigma_k r_k}$$
$$= \frac{1}{2^k} \prod_{i=1}^{k} \sum_{\sigma=0}^{1} (-1)^{\sigma r_i}.$$

However, for each $1 \le i \le k$,

$$\sum_{\sigma=0}^{1} (-1)^{\sigma r_i} = \begin{cases} 2 & \text{if } r_i \equiv 0 \mod 2, \\ 0 & \text{else.} \end{cases}$$

Thus,

$$\frac{1}{2^m} \sum_{\sigma_1,\ldots,\sigma_m=0}^{1} \prod_{i=1}^{p} (-1)^{\sigma_{u_i+1}} = f(r_1,\ldots,r_k),$$

where $f(r_1,\ldots,r_k)$ is one if $r_i$ is even for all $i \in \{1,\ldots,k\}$ and zero otherwise. Therefore we find that

$$\frac{1}{2^m} \sum_{\sigma_1,\ldots,\sigma_m=0}^{1} \frac{1}{2^{2m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta \left( \frac{a}{2^m}, \frac{b}{2^m} \right)^p$$
$$= \frac{1}{2^m} \sum_{\substack{u_1,\ldots,u_p=0 \\ r_1,\ldots,r_k \text{ even}}}^{m-1} \frac{1}{2^{k(u_1,\ldots,u_p)}} \sum_{2^m\beta=1}^{2^m-1} \prod_{i=1}^{p} \|2^{u_i}\beta\|. \tag{4}$$

However, if $p$ is odd, it is impossible that all of the $r_i$, $1 \leq i \leq k$ are even, since $r_1 + \cdots + r_k = p$. Thus, the result is shown for odd $p$. We can therefore concentrate on even $p$. By Lemma 4 the latter term is bounded above by

$$\frac{1}{2^m} \sum_{\substack{u_1,\ldots,u_p=0 \\ r_1,\ldots,r_k \text{ even}}}^{m-1} \frac{1}{2^{k(u_1,\ldots,u_p)}} \frac{2^m}{2^p 2^{k(u_1,\ldots,u_p)}} = \frac{1}{2^p} \sum_{\substack{u_1,\ldots,u_p=0 \\ r_1,\ldots,r_k \text{ even}}}^{m-1} \frac{1}{2^{2k(u_1,\ldots,u_p)}}.$$

Note that $r_1, \ldots, r_k$ can only be even if $k \leq \frac{p}{2}$. Let us denote the number of tuples $(u_1, \ldots, u_p) \in \{0, \ldots, m-1\}^p$ such that $k$ different $u_i$s occur by $\#(p,k,m)$. This is the number of mappings from $\{1, \ldots, p\}$ to $\{0, \ldots, m-1\}$ such that the range has cardinality $k$. It is well known from combinatorics that the $\#(p,k,m)$ are closely related to Stirling numbers of the second kind, $S(p,k)$, via

$$\#(p,k,m) = k! \binom{m}{k} S(p,k).$$

This follows easily from the fact that the number of surjective mappings from $\{1, \ldots, p\}$ to $\{1, \ldots, k\}$ is given by $k!S(p,k)$. Since $k \leq \frac{p}{2}$, we have

$$\begin{aligned}
\frac{1}{2^m} \sum_{\sigma_1,\ldots,\sigma_m=0}^{1} \frac{1}{2^{2m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right)^p &\leq \frac{1}{2^p} \sum_{k=0}^{p/2} \frac{1}{2^{2k}} \#(p,k,m) \\
&= \frac{1}{2^p} \sum_{k=0}^{p/2} \frac{1}{2^{2k}} k! \binom{m}{k} S(p,k) \\
&= \frac{1}{2^{2p}} (p/2)! \binom{m}{p/2} S(p,p/2) + \frac{1}{2^p} \sum_{k=0}^{p/2-1} \frac{1}{2^{2k}} k! \binom{m}{k} S(p,k) \\
&\leq \frac{1}{2^{2p}} m(m-1)\cdots(m-p/2+1) S(p,p/2) + \\
&\quad + \frac{1}{2^p} m(m-1)\cdots(m-p/2+2) c(p) \\
&= \frac{m^{p/2} S(p,p/2)}{2^{2p}} + O(m^{p/2-1}),
\end{aligned}$$

where $c(p) := \sum_{k=0}^{p/2-1} \frac{1}{2^{2k}} S(p,k)$.

On the other hand, for even $p$, by Equation (4) and Lemma 4 we have

$$\begin{aligned}
&\frac{1}{2^m}\sum_{\sigma_1,\ldots,\sigma_m=0}^{1}\frac{1}{2^{2m}}\sum_{a=1}^{2^m-1}\sum_{b=1}^{2^m-1}\Delta\left(\frac{a}{2^m},\frac{b}{2^m}\right)^p \\
&= \frac{1}{2^m}\sum_{\substack{u_1,\ldots,u_p=0\\ r_1,\ldots,r_k \text{ even}}}^{m-1}\frac{1}{2^{k(u_1,\ldots,u_p)}}\sum_{2^m\beta=1}^{2^m-1}\prod_{i=1}^{p}\|2^{u_i}\beta\| \\
&\geq \frac{1}{2^m}\sum_{\substack{u_1,\ldots,u_p=0\\ r_1,\ldots,r_k \text{ even}}}^{m-1}\frac{1}{2^{k(u_1,\ldots,u_p)}}\frac{2^m}{4^p} \\
&\geq \frac{1}{2^{2p}}\sum_{\substack{u_1,\ldots,u_p=0\\ r_1=\cdots=r_k=2}}^{m-1}\frac{1}{2^{p/2}} \\
&= \frac{1}{2^{2p+p/2}}\#(p,p/2,m) \\
&= \frac{1}{2^{2p+p/2}}(p/2)!\binom{m}{p/2}S(p,p/2) \\
&= \frac{m^{p/2}S(p,p/2)}{2^{2p+p/2}} + O\big(m^{p/2-1}\big)\,.
\end{aligned}$$

Now we can give the proof of Theorem 1.

P r o o f. From equality (3) it follows that

$$\Delta\big(\alpha(m),\beta(m)\big) \leq \Delta(\alpha,\beta) \leq \Delta\big(\alpha(m),\beta(m)\big) + 2\,.$$

For even $p$, the function $x \mapsto x^p$ is convex and hence it follows that

$$\begin{aligned}
\Delta(\alpha,\beta)^p &\leq \max\Big(\Delta\big(\alpha(m),\beta(m)\big)^p, \big(\Delta\big(\alpha(m),\beta(m)\big)+2\big)^p\Big) \\
&\leq \Delta\big(\alpha(m),\beta(m)\big)^p + \big(\Delta\big(\alpha(m),\beta(m)\big)+2\big)^p\,.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\int_0^1\int_0^1 \Delta(\alpha,\beta)^p \,\mathrm{d}\alpha\,\mathrm{d}\beta \\
&\le \int_0^1\int_0^1 \left[\Delta\big(\alpha(m),\beta(m)\big)^p + \big(\Delta\big(\alpha(m),\beta(m)\big)+2\big)^p\right] \mathrm{d}\alpha\,\mathrm{d}\beta \\
&= \int_0^1\int_0^1 \left[2\Delta\big(\alpha(m),\beta(m)\big)^p + \sum_{l=0}^{p-1}\binom{p}{l}\Delta\big(\alpha(m),\beta(m)\big)^l 2^{p-l}\right] \mathrm{d}\alpha\,\mathrm{d}\beta \\
&= \frac{2}{2^{2m}}\sum_{a,b=1}^{2^m}\Delta\left(\frac{a}{2^m},\frac{b}{2^m}\right)^p + \sum_{l=0}^{p-1}\binom{p}{l}2^{p-l}\frac{1}{2^{2m}}\sum_{a,b=1}^{2^m}\Delta\left(\frac{a}{2^m},\frac{b}{2^m}\right)^l .
\end{aligned}$$

Since for any shift $\vec{\sigma}\in\mathbb{Z}_2^m$ the point set $H(\vec{\sigma})$ is a $(0,m,2)$-net in base 2 it follows that $\Delta\left(\frac{a}{2^m},\frac{b}{2^m}\right)=0$ whenever $a=2^m$ or $b=2^m$. Now the result follows from Lemma 5. □

## 5. The proofs of Theorem 2 and Theorem 3

The following lemma will be the basic tool in the proofs of Theorem 2 and Theorem 3.

**Lemma 6.** *Let $\vec{\sigma}\in\mathbb{Z}_2^m$ and let $\Delta$ be the discrepancy function of the digitally shifted Hammersley point set $H(\vec{\sigma})$. Further let $l$ denote the number of zero coordinates of $\vec{\sigma}$. Then we have*

$$\frac{1}{2^{2m}}\sum_{a=1}^{2^m-1}\sum_{b=1}^{2^m-1}\Delta\left(\frac{a}{2^m},\frac{b}{2^m}\right)^2 = \frac{1}{576}(9m^2+15m-36lm+36l^2+16-4^{2-m}).$$

Proof. We have

$$\frac{1}{2^{2m}}\sum_{a=1}^{2^m-1}\sum_{b=1}^{2^m-1}\Delta\left(\frac{a}{2^m},\frac{b}{2^m}\right)^2 = \frac{1}{2^{2m}}\sum_{2^m\alpha=1}^{2^m-1}\sum_{2^m\beta=1}^{2^m-1}\Delta(\alpha,\beta)^2 .$$

The latter term equals

$$\frac{1}{2^{2m}}\sum_{2^m\alpha=1}^{2^m-1}\sum_{2^m\beta=1}^{2^m-1}\left(\sum_{u=0}^{m-1}\|2^u\beta\|\,(-1)^{\sigma_{u+1}}\big(\alpha_{m-u}\oplus\alpha_{m+1-j(u)}\big)\right)^2 =: A+2B+C,$$

where

$$A = \frac{1}{2^{2m}} \sum_{2^m\alpha=1}^{2^m-1} \sum_{2^m\beta=1}^{2^m-1} \Bigg( \sum_{\substack{u=0 \\ \sigma_{u+1}=0}}^{m-1} \|2^u\beta\| \left(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\right) \Bigg)^2 ,$$

$$B = -\frac{1}{2^{2m}} \sum_{2^m\alpha=1}^{2^m-1} \sum_{2^m\beta=1}^{2^m-1} \Bigg( \sum_{\substack{u_1=0 \\ \sigma_{u_1+1}=0}}^{m-1} \|2^{u_1}\beta\| \left(\alpha_{m-u_1} \oplus \alpha_{m+1-j(u_1)}\right) \Bigg) \times$$
$$\times \Bigg( \sum_{\substack{u_2=0 \\ \sigma_{u_2+1}=1}}^{m-1} \|2^{u_2}\beta\| \left(\alpha_{m-u_2} \oplus \alpha_{m+1-j(u_2)}\right) \Bigg) ,$$

and

$$C = \frac{1}{2^{2m}} \sum_{2^m\alpha=1}^{2^m-1} \sum_{2^m\beta=1}^{2^m-1} \Bigg( \sum_{\substack{u=0 \\ \sigma_{u+1}=1}}^{m-1} \|2^u\beta\| \left(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}\right) \Bigg)^2 .$$

Making use of Lemma 2 and Lemma 4, we find that

$$A = \frac{1}{2^{2m}} \underbrace{\sum_{\substack{u_1=0 \\ \sigma_{u_1+1}=0}}^{m-1} \sum_{\substack{u_2=0 \\ \sigma_{u_2+1}=0}}^{m-1}}_{u_1 \neq u_2} \frac{2^m}{2^4} 2^{m-2} + \frac{1}{2^{2m}} \sum_{\substack{u=0 \\ \sigma_{u+1}=0}}^{m-1} \frac{2^{2m} + 2^{2u+1}}{3 \cdot 2^{m+2}} 2^{m-1} ,$$

$$B = -\frac{1}{2^{2m}} \sum_{\substack{u_1=0 \\ \sigma_{u_1+1}=0}}^{m-1} \sum_{\substack{u_2=0 \\ \sigma_{u_2+1}=1}}^{m-1} \frac{2^m}{2^4} 2^{m-2} ,$$

and

$$C = \frac{1}{2^{2m}} \underbrace{\sum_{\substack{u_1=0 \\ \sigma_{u_1+1}=1}}^{m-1} \sum_{\substack{u_2=0 \\ \sigma_{u_2+1}=1}}^{m-1}}_{u_1 \neq u_2} \frac{2^m}{2^4} 2^{m-2} + \frac{1}{2^{2m}} \sum_{\substack{u=0 \\ \sigma_{u+1}=1}}^{m-1} \frac{2^{2m} + 2^{2u+1}}{3 \cdot 2^{m+2}} 2^{m-1} .$$

Hence it follows that

$$\begin{aligned} A + 2B + C &= \frac{1}{2^{2m}} \sum_{u=0}^{m-1} \frac{2^{2m} + 2^{2u+1}}{3 \cdot 2^{m+2}} 2^{m-1} \\ &\quad + \frac{1}{2^{2m}} \frac{2^m}{2^4} 2^{m-2} \left( l^2 - l - 2(m-l)l + (m-l)^2 - (m-l) \right) . \end{aligned}$$

Straightforward computation yields the result. $\square$

**Remark 4.** Setting $l = m$ in Lemma 6 yields the third part of [14, Lemma 4].

Now we can prove Theorem 2.

P r o o f. Observe that

$$\begin{aligned}
&\big(NL_2(H(\vec{\sigma}))\big)^2 \\
&= \int_0^1 \int_0^1 \big(\Delta(\alpha,\beta)\big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\
&= \int_0^{1-2^{-m}} \int_0^{1-2^{-m}} \big(\Delta(\alpha,\beta)\big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta + \int_0^{1-2^{-m}} \int_{1-2^{-m}}^{1} \big(\Delta(\alpha,\beta)\big)^2 \,\mathrm{d}\beta\,\mathrm{d}\alpha \\
&\quad + \int_{1-2^{-m}}^{1} \int_0^{1-2^{-m}} \big(\Delta(\alpha,\beta)\big)^2 \,\mathrm{d}\beta\,\mathrm{d}\alpha + \int_{1-2^{-m}}^{1} \int_{1-2^{-m}}^{1} \big(\Delta(\alpha,\beta)\big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\
&=: I_1 + I_2 + I_3 + I_4\,.
\end{aligned}$$

Using $\Delta(\alpha, 1) = 0$ for $m$-bit $\alpha$ and Remark 3, we find for $I_2$ that

$$\begin{aligned}
I_2 &= \int_0^{1-2^{-m}} \int_{1-2^{-m}}^{1} \big(\Delta(\alpha,\beta)\big)^2 \,\mathrm{d}\beta\,\mathrm{d}\alpha \\
&= \int_0^{1-2^{-m}} \int_{1-2^{-m}}^{1} \Big(\Delta\big(\alpha(m),1\big) + 2^m\big(\alpha(m) - \alpha\beta\big)\Big)^2 \,\mathrm{d}\beta\,\mathrm{d}\alpha \\
&= 2^{2m} \sum_{a=1}^{2^m-1} \int_{\frac{a-1}{2^m}}^{\frac{a}{2^m}} \int_{1-2^{-m}}^{1} \Big(\frac{a}{2^m} - \alpha\beta\Big)^2 \,\mathrm{d}\beta\,\mathrm{d}\alpha \\
&= \frac{25}{36\cdot 2^m} - \frac{5}{9\cdot 4^m} - \frac{25}{36\cdot 4^m} + \frac{2}{3\cdot 8^m} - \frac{1}{9\cdot 16^m}\,.
\end{aligned}$$

$I_3$ has the same value as $I_2$ and is calculated similarly. Calculating $I_4$ is no problem by making use of Remark 3 again and employing the fact that $\Delta(1,1)=0$.

$$\begin{aligned} I_4 &= \int_{1-2^{-m}}^{1}\int_{1-2^{-m}}^{1} \big(\Delta(\alpha,\beta)\big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= \int_{1-2^{-m}}^{1}\int_{1-2^{-m}}^{1} \big(\Delta(1,1)+2^m(1-\alpha\beta)\big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= 2^{2m}\int_{1-2^{-m}}^{1}\int_{1-2^{-m}}^{1} (1-\alpha\beta)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= \frac{7}{6\cdot 4^m}+\frac{1}{9\cdot 16^m}-\frac{2}{3\cdot 8^m}. \end{aligned}$$

It remains to analyze $I_1$.

$$\begin{aligned} I_1 &= \int_{0}^{1-2^{-m}}\int_{0}^{1-2^{-m}} \big(\Delta(\alpha,\beta)\big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= \int_{0}^{1-2^{-m}}\int_{0}^{1-2^{-m}} \Big(\Delta\big(\alpha(m),\beta(m)\big)+2^m\big(\alpha(m)\beta(m)-\alpha\beta\big)\Big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= \int_{0}^{1-2^{-m}}\int_{0}^{1-2^{-m}} \Big(\Delta\big(\alpha(m),\beta(m)\big)\Big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &\quad + 2^{2m}\int_{0}^{1-2^{-m}}\int_{0}^{1-2^{-m}} \Big(\big(\alpha(m)\beta(m)-\alpha\beta\big)\Big)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &\quad + 2^{m+1}\int_{0}^{1-2^{-m}}\int_{0}^{1-2^{-m}} \Delta\big(\alpha(m),\beta(m)\big)\big(\alpha(m)\beta(m)-\alpha\beta\big) \,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= \frac{1}{2^{2m}}\sum_{a=1}^{2^m-1}\sum_{b=1}^{2^m-1}\Delta\left(\frac{a}{2^m},\frac{b}{2^m}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&+ 2^{2m} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \int_{\frac{a-1}{2^m}}^{\frac{a}{2^m}} \int_{\frac{b-1}{2^m}}^{\frac{b}{2^m}} \left(\frac{ab}{2^{2m}} - \alpha\beta\right)^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \\
&+ 2^{m+1} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \int_{\frac{a-1}{2^m}}^{\frac{a}{2^m}} \int_{\frac{b-1}{2^m}}^{\frac{b}{2^m}} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \left(\frac{ab}{2^{2m}} - \alpha\beta\right) \,\mathrm{d}\alpha\,\mathrm{d}\beta \\
=: \;& \Sigma_1 + \Sigma_2 + \Sigma_3 \,.
\end{aligned}
$$

From Lemma 6 we find that

$$\Sigma_1 = \frac{1}{576}(9m^2 + 15m - 36lm + 36l^2 + 16 - 4^{2-m}) \,.$$

Analyzing $\Sigma_2$ is a matter of straightforward computation and yields

$$\Sigma_2 = -\frac{1}{72 \cdot 16^m}(2^m - 1)^2(32 \cdot 2^m - 25 \cdot 4^m - 8) \,.$$

So it remains to deal with $\Sigma_3$. Here, we find that

$$
\begin{aligned}
\frac{1}{2^m}\Sigma_3 \;&=\; 2 \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \frac{ab}{2^{2m}} \frac{1}{2^{2m}} \\
&\quad - 2 \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \frac{1}{4} \frac{1}{2^{4m}} (4ab - 2(a+b) + 1) \\
&=\; \frac{1}{2^{4m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} (a+b)\Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \\
&\quad - \frac{1}{2^{4m+1}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \\
&=: \; \Sigma_4 - \Sigma_5 \,.
\end{aligned}
$$

We start with $\Sigma_4$.

$$
\begin{aligned}
\Sigma_4 \;&=\; \frac{1}{2^{4m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} a\Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) + \frac{1}{2^{4m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} b\Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \\
&=: \; \frac{1}{2^{4m}} \left(\Sigma_{4,1} + \Sigma_{4,2}\right) .
\end{aligned}
$$

Using Lemma 1, we find that

$$\begin{aligned}\Sigma_{4,1} &= \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1} \sum_{2^m\beta=1}^{2^m-1} \|2^u\beta\| \sum_{2^m\alpha=1}^{2^m-1} 2^m\alpha\big(\alpha_{m-u}\oplus\alpha_{m+1-j(u)}\big)\\ &\quad - \sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1} \sum_{2^m\beta=1}^{2^m-1} \|2^u\beta\| \sum_{2^m\alpha=1}^{2^m-1} 2^m\alpha\big(\alpha_{m-u}\oplus\alpha_{m+1-j(u)}\big)\,.\end{aligned}$$

By Lemma 3 and Lemma 4, the latter term is bounded above by

$$\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1} \frac{2^m}{4}\big(2^{2m-2}-2^{m-2}+2^{m+u-2}\big) - \sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1} \frac{2^m}{4}\big(2^{2m-2}-2^{m-2}-2^{m+u-2}\big)$$

$$=\left(\big(l-(m-l)\big)2^{2m-2}+\big((m-l)-l\big)2^{m-2}+\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1} 2^{m+u-2}+\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1} 2^{m+u-2}\right)\frac{2^m}{4}\,.$$

It follows that

$$\frac{1}{2^{4m}}\Sigma_{4,1} \le \frac{1}{16\cdot 2^m}-\frac{1}{16\cdot 4^m}+\frac{l}{8\cdot 2^m}-\frac{l}{8\cdot 4^m}+\frac{m}{16\cdot 4^m}-\frac{m}{16\cdot 2^m}\,.$$

On the other hand, using Lemma 1 and Lemma 2, we have

$$\Sigma_{4,2} = \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1} 2^{2m-1} \sum_{2^m\beta=1}^{2^m-1} \beta\,\|2^u\beta\| - \sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1} 2^{2m-1} \sum_{2^m\beta=1}^{2^m-1} \beta\,\|2^u\beta\|\,.$$

It was shown in the proof of [14, Theorem 2] that

$$\sum_{2^m\beta=1}^{2^m-1} \beta\,\|2^u\beta\| = \frac{2^m}{8}\,.$$

It is now a matter of straightforward computation to show

$$\frac{1}{2^{4m}}\Sigma_{4,2} = \frac{1}{2^m}\left(\frac{l}{8}-\frac{m}{16}\right).$$

Finally we analyze $\Sigma_5$ by using Lemma 2 and Lemma 4. Here, we find that

$$\Sigma_5 = \frac{1}{2^{2m}}\left(\frac{l}{8}-\frac{m}{16}\right).$$

Putting these results together, it is no problem to obtain

$$\begin{aligned} I_1+I_2+I_3+I_4 &\le \frac{m^2}{64}-\frac{19m}{192}-\frac{lm}{16}+\frac{l^2}{16}+\frac{l}{4}+\frac{7}{16} \\ &\quad +\frac{m}{8\cdot 2^m}-\frac{l}{4\cdot 2^m}+\frac{3}{16\cdot 2^m}-\frac{1}{72\cdot 4^m}. \end{aligned}$$

This is the desired result. □

We prove Theorem 3.

Proof. The proof is similar to that of Theorem 2, with the only difference that one has to establish a lower bound on $\Sigma_{4,1}$, which is no problem due to Lemma 3. □

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