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Strict topologies and Banach-Steinhaus type theorems

MARIAN NOWAK

Abstract. Let X be a completely regular Hausdorff space, E a real Banach space, and let $C_b(X, E)$ be the space of all E -valued bounded continuous functions on X . We study linear operators from $C_b(X, E)$ endowed with the strict topologies β_z ($z = \sigma, \tau, \infty, g$) to a real Banach space $(Y, \|\cdot\|_Y)$. In particular, we derive Banach-Steinhaus type theorems for $(\beta_z, \|\cdot\|_Y)$ continuous linear operators from $C_b(X, E)$ to Y . Moreover, we study σ -additive and τ -additive operators from $C_b(X, E)$ to Y .

Keywords: vector-valued continuous functions, strict topologies, locally solid topologies, Dini-topologies, strong Mackey space, σ -additive operators, τ -additive operators

Classification: 47A70, 47B38, 46E10

1. Introduction

We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L, K \rangle$. Given a Hausdorff locally convex space (L, ξ) by L'_ξ we will denote its topological dual. Recall that (L, ξ) is said to be a *strong Mackey space* if every relatively countably $\sigma(L'_\xi, L)$ -compact subset of L'_ξ is ξ -equicontinuous (see [K, p.196], [KO₂, p.482]). Clearly, if (L, ξ) is a strong Mackey space, then $\xi = \tau(L, L'_\xi)$.

Let X be a completely regular Hausdorff space and $(E, \|\cdot\|_E)$ a real Banach space. Let $C_b(X, E)$ be the Banach space of all E -valued bounded continuous functions on X provided with supremum norm $\|\cdot\|_\infty$. We will write $C_b(X)$ instead of $C_b(X, \mathbb{R})$, where \mathbb{R} is the field of real numbers. For a function $f \in C_b(X, E)$ we will write $\|f\|(x) = \|f(x)\|_E$ for all $x \in X$. Then $\|f\| \in C_b(X)$.

A subset H of $C_b(X, E)$ is said to be *solid* whenever $\|f_1\| \leq \|f_2\|$ (i.e., $\|f_1(x)\|_E \leq \|f_2(x)\|_E$ for all $x \in X$) and $f_1 \in C_b(X, E)$, $f_2 \in H$ implies $f_1 \in H$. A linear topology ξ on $C_b(X, E)$ is said to be *locally solid* if it has a local base at 0 consisting of solid sets (see [K], [KO₂], [NR]).

In the topological measure theory a number of locally convex topologies β_z on $C_b(X, E)$, called strict topologies have been studied. Definitions of strict topologies β_z base in some natural way on the topology of X , or perhaps its Stone-Ćech compactification βX . Then $(C_b(X, E), \beta_z)'$ can be identified with some natural spaces $M_z(X, E')$ of E' -valued measures (see [F], [K], [KO₁], [KO₂], [KO₃], [KV], [NR]). In this paper we consider the strict topologies $\beta_\sigma, \beta_\tau, \beta_g$ and β_∞ on

$C_b(X, E)$. Note that in [F] and [K] the topologies β_σ and β_τ are denoted by β_1 and β respectively. It is well known that the strict topologies β_z ($z = \sigma, \tau, g$ and ∞) are locally solid (see [K, Theorem 8.1] for $z = \sigma, \tau, \infty$ and [KO₂, Theorem 6] for $z = g$). Moreover, $(C_b(X, E), \beta_z)$ is a strong Mackey space for $z = \sigma$ (see [KO₁, Corollary 6]); $z = \tau$ and X paracompact (see [K, Theorem 6.1]); $z = \infty$ (see [K, Theorem 3.7]) and $z = g$ (see [KO₂, Theorem 7]).

From now on $(Y, \|\cdot\|_Y)$ is a real Banach space, and let Y' stand for its Banach dual. Let $\mathcal{L}(C_b(X, E), Y)$ stand for the space of all bounded ($= (\|\cdot\|_\infty, \|\cdot\|_Y)$ -continuous) linear operators from $C_b(X, E)$ to Y . The *strong operator topology* (briefly SOT) is the locally convex topology on $\mathcal{L}(C_b(X, E), Y)$ defined by the family of seminorms $\{p_f : f \in C_b(X, E)\}$, where $p_f(T) = \|T(f)\|_Y$ for all $T \in \mathcal{L}(C_b(X, E), Y)$. The *weak operator topology* (briefly WOT) is the locally convex topology on $\mathcal{L}(C_b(X, E), Y)$ defined by the family of seminorms $\{p_{f,y'} : f \in C_b(X, E), y' \in Y'\}$, where $p_{f,y'}(T) = |\langle T(f), y' \rangle|$ for all $T \in \mathcal{L}(C_b(X, E), Y)$. In view of the Banach-Steinhaus theorem the space $\mathcal{L}(C_b(X, E), Y)$ provided with SOT is sequentially complete. By $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$ (for $z = \sigma, \tau, g$ and ∞) we will denote the subspace of $\mathcal{L}(C_b(X, E), Y)$ consisting of all those $T \in \mathcal{L}(C_b(X, E), Y)$ which are $(\beta_z, \|\cdot\|_Y)$ -continuous.

In Section 2 we study topological properties of the spaces $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$, where $z = \sigma, \tau, g$ and ∞ . In particular, we derive Banach-Steinhaus type theorems for $(\beta_z, \|\cdot\|_Y)$ -continuous linear operators from $C_b(X, E)$ to Y (see Theorem 2.5 and Corollary 2.6 below). In Section 3 we consider σ -additive and τ -additive operators from $C_b(X, E)$ to Y .

2. Linear operators on $C_b(X, E)$ with strict topologies

For a bounded linear operator $T : C_b(X, E) \rightarrow Y$ let $T' : Y' \rightarrow C_b(X, E)'$ denote its conjugate, i.e., $\langle f, T'(y') \rangle = \langle T(f), y' \rangle$ for $f \in C_b(X, E)$ and $y' \in Y'$.

Proposition 2.1. *Let $T : C_b(X, E) \rightarrow Y$ be a bounded linear operator and let $z = \sigma, g, \infty$ ($z = \tau$ and X is paracompact). Then the following statements are equivalent:*

- (i) $T'(Y') \subset C_b(X, E)'_{\beta_z}$, i.e., $y' \circ T \in C_b(X, E)'_{\beta_z}$ for each $y' \in Y'$;
- (ii) T is $(\sigma(C_b(X, E), C_b(X, E)'_{\beta_z}), \sigma(Y, Y'))$ -continuous;
- (iii) T is $(\beta_z, \|\cdot\|_Y)$ -continuous.

PROOF: (i) \iff (ii) General fact; see [AB, Proposition 9.26].

(ii) \iff (iii) It is known that T is $(\sigma(C_b(X, E), C_b(X, E)'_{\beta_z}), \sigma(Y, Y'))$ -continuous if and only if T is $(\tau(C_b(X, E), C_b(X, E)'_{\beta_z}), \tau(Y, Y'))$ -continuous (see [AB; Ex. 11, p. 149]). Since $\beta_z = \tau(C_b(X, E), C_b(X, E)'_{\beta_z})$ and $\tau(Y, Y')$ coincides with the $\|\cdot\|_Y$ -topology, the proof is complete.

(iii) \implies (i) It is obvious. □

Proposition 2.2. $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$ is a sequentially closed subspace of $\mathcal{L}(C_b(X, E), Y)$ for WOT, where $z = \sigma, g, \infty$ (resp. $z = \tau$ and X is paracompact).

PROOF: Let (T_n) be a sequence in $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$ such that $T_n \rightarrow T$ for WOT, where $T \in \mathcal{L}(C_b(X, E), Y)$. Given $y'_o \in Y'$, for each $f \in C_b(X, E)$ we get $(y'_o \circ T)(f) = \lim (y'_o \circ T_n)(f)$, where $y'_o \circ T_n \in C_b(X, E)'_{\beta_z}$ for $n \in \mathbb{N}$, and $y'_o \circ T \in C_b(X, E)'$. It follows that $(y'_o \circ T_n)$ is a $\sigma(C_b(X, E)'_{\beta_z}, C_b(X, E))$ -Cauchy sequence in $C_b(X, E)'_{\beta_z}$. Since the space $(C_b(X, E)'_{\beta_z}, \sigma(C_b(X, E)'_{\beta_z}, C_b(X, E)))$ is sequentially complete (see [KO₃, Theorem 3]), there exists $\Phi_o \in C_b(X, Y)'_{\beta_z}$ such that $\Phi_o(f) = \lim (y'_o \circ T_n)(f)$ for each $f \in C_b(X, E)$. Hence $y'_o \circ T = \Phi_o \in C_b(X, E)'_{\beta_z}$, and by Proposition 2.1 we get $T \in \mathcal{L}_{\beta_z}(C_b(X, E), Y)$. \square

Corollary 2.3. Let $z = \sigma, g, \infty$ ($z = \tau$ and X is paracompact). Then

- (i) $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$ is a sequentially closed subspace of $\mathcal{L}(C_b(X, E), Y)$ for SOT;
- (ii) the space $(\mathcal{L}_{\beta_z}(C_b(X, E), Y), \text{SOT})$ is sequentially complete.

PROOF: (i) It follows from Proposition 2.2 because $\text{WOT} \subset \text{SOT}$.

(ii) It follows from (i) because the space $(\mathcal{L}(C_b(X, E), Y), \text{SOT})$ is sequentially complete. \square

The following general result will be of importance (see [SZ, Theorem 2]).

Proposition 2.4. Let \mathcal{K} be a SOT-compact subset of $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$, where $z = \sigma, g, \infty$ (resp. $z = \tau$ and X is paracompact). If H is a $\sigma(Y', Y)$ -closed and $\|\cdot\|_Y$ -equicontinuous subset of Y' , then the set $\bigcup\{T'(H) : T \in \mathcal{K}\}$ is a $\sigma(C_b(X, E)'_{\beta_z}, C_b(X, E))$ -compact subset of $C_b(X, E)'_{\beta_z}$.

Now we are ready to prove Banach-Steinhaus type theorems for $(\beta_z, \|\cdot\|_Y)$ -continuous linear operators from $C_b(X, E)$ to Y .

Theorem 2.5. Let \mathcal{K} be a SOT-compact subset of $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$, where $z = \sigma, g, \infty$ (resp. $z = \tau$ and X is paracompact). Then \mathcal{K} is a $(\beta_z, \|\cdot\|_Y)$ -equicontinuous.

PROOF: Since the closed unit ball $B_{Y'}$ in Y' is $\sigma(Y', Y)$ -closed and $(\|\cdot\|_Y)$ -equicontinuous (see [AB, Theorem 9.21]), by Proposition 2.4 the set $Z = \bigcup\{T'(B_{Y'}) : T \in \mathcal{K}\}$ is $\sigma(C_b(X, E)'_{\beta_z}, C_b(X, E))$ -compact subset of $C_b(X, E)'_{\beta_z}$. Since $(C_b(X, E), \beta_z)$ is a strong Mackey space, the set $Z = \{y' \circ T : T \in \mathcal{K}, y' \in B_{Y'}\}$ is $(\beta_z, \|\cdot\|_Y)$ -equicontinuous. Then for given $\varepsilon > 0$ there exists a neighbourhood V_ε of 0 for β_z such that for all $f \in V_\varepsilon$,

$$\sup_{T \in \mathcal{K}} \|T(f)\|_Y = \sup\{|(y' \circ T)(f)| : T \in \mathcal{K}, y' \in Y'\} \leq \varepsilon.$$

This means that \mathcal{K} is $(\beta_z, \|\cdot\|_Y)$ -equicontinuous, as desired. \square

Corollary 2.6. *Let $T_k : C_b(X, E) \rightarrow Y$ be $(\beta_z, \|\cdot\|_Y)$ -continuous linear operators for $z = \sigma, g, \infty$ (resp. $z = \tau$ and X is paracompact) and $k \in \mathbb{N}$. Assume that $T(f) := \lim T_k(f)$ exists in $(Y, \|\cdot\|_Y)$ for each $f \in C_b(X, E)$. Then T is $(\beta_z, \|\cdot\|_Y)$ -continuous and the set $\{T_k : k \in \mathbb{N}\}$ is $(\beta_z, \|\cdot\|_Y)$ -equicontinuous.*

PROOF: In view of the Banach-Steinhaus theorem T is bounded. Hence by Corollary 2.3 T is $(\beta_z, \|\cdot\|_Y)$ -continuous. Then $T_k \rightarrow T$ in $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$ for SOT, so $\{T_k : k \in \mathbb{N}\} \cup \{T\}$ is a SOT-compact subset of $\mathcal{L}_{\beta_z}(C_b(X, E), Y)$. Hence by Theorem 2.5 the set $\{T_k : k \in \mathbb{N}\}$ is $(\beta_z, \|\cdot\|_Y)$ -equicontinuous. \square

3. σ -additive and τ -additive operators on $C_b(X, E)$

We start by recalling definitions of σ -Dini and Dini topologies on $C_b(X, E)$ (see [NR, Definition 5.1]). For a net (u_α) in $C_b(X)$ we will write $u_\alpha \downarrow 0$ whenever $u_\alpha(x) \downarrow 0$ for all $x \in X$.

Definition 3.1. Let ξ be a locally solid topology on $C_b(X, E)$.

(i) ξ is said to be a σ -Dini topology if, whenever (f_n) is a sequence in $C_b(X, E)$ such that $\|f_n\| \downarrow 0$, then $f_n \rightarrow 0$ for ξ .

(ii) ξ is said to be a Dini topology if, whenever (f_α) is a net in $C_b(X, E)$ such that $\|f_\alpha\| \downarrow 0$, then $f_\alpha \rightarrow 0$ for ξ .

We have (see [NR, Theorem 5.2]):

Proposition 3.1. β_σ (resp. β_τ) is the finest locally convex σ -Dini (resp. Dini) topology on $C_b(X, E)$.

Following [F, Definition 2.1] we can extend the definition of σ -additive and τ -additive linear functionals on $C_b(X, E)$ to linear operators from $C_b(X, E)$ to Y .

Definition 3.2. Let $T : C_b(X, E) \rightarrow Y$ be a bounded linear operator, and let $B_\infty = \{g \in C_b(X, E) : \|g\|_\infty \leq 1\}$.

(i) T is said to be σ -additive if $\sup_{g \in B_\infty} \|T(u_n g)\|_Y \xrightarrow{n} 0$ for each sequence (u_n) in $C_b(X)$ such that $u_n \downarrow 0$.

(ii) T is said to be τ -additive if $\sup_{g \in B_\infty} \|T(u_\alpha g)\|_Y \xrightarrow{\alpha} 0$ for each net (u_α) in $C_b(X)$ such that $u_\alpha \downarrow 0$.

By $L_\sigma(C_b(X, E), Y)$ (resp. $L_\tau(C_b(X, E), Y)$) we will denote the set of all σ -additive (resp. τ -additive) operators from $C_b(X, E)$ to Y .

Proposition 3.2. For a bounded linear operator $T : C_b(X, E) \rightarrow Y$ the following statements are equivalent:

- (i) $y' \circ T \in C_b(X, E)'_{\beta_\sigma}$ for each $y' \in Y'$;
- (ii) T is $(\beta_\sigma, \|\cdot\|_Y)$ -continuous;
- (iii) T is $(\beta_\sigma, \|\cdot\|_Y)$ -sequentially continuous;
- (iv) T is σ -additive.

PROOF: (i) \iff (ii) It follows from Proposition 2.1.

(ii) \implies (iii) It is obvious.

(iii) \implies (iv) Assume that T is $(\beta_z, \|\cdot\|_Y)$ -sequentially continuous, and let $u_n \downarrow 0$ in $C_b(X)$. Note that $\sup_{g \in B_\infty} \|T(u_n g)\|_Y \leq \|T\| \cdot \|u_1\|_\infty < \infty$. Let $\varepsilon > 0$ be given. Then for each $n \in \mathbb{N}$ there exists $g_n \in B_\infty$ such that

$$\sup_{g \in B_\infty} \|T(u_n g)\|_Y \leq \|T(u_n g_n)\|_Y + \frac{\varepsilon}{2}.$$

For a fixed $e_o \in S_E$ (= the closed unit sphere in E), we have $\|(u_n g)(x)\|_E \leq \|(u_n \otimes e_o)(x)\|_E = \|u_n(x)e_o\|_E = u_n(x) \downarrow 0$ for all $x \in X$. Since β_σ is a σ -Dini topology, we obtain that $u_n \otimes e_o \longrightarrow 0$ for β_σ . Hence $u_n g \longrightarrow 0$ for β_σ because β_σ is locally solid. It follows that $\|T(u_n g_n)\|_Y \longrightarrow 0$. Choose $n_\varepsilon \in \mathbb{N}$ such that $\|T(u_n g_n)\|_Y \leq \frac{\varepsilon}{2}$ for $n \geq n_\varepsilon$. It follows that $\sup_{g \in B_\infty} \|T(u_n g)\|_Y \leq \varepsilon$ for $n \geq n_\varepsilon$, i.e., T is σ -additive.

(iv) \implies (i) It follows from [F, Theorem 2.3]. □

Hence we have

$$\mathcal{L}_{\beta_\sigma}(C_b(X, E), Y) = L_\sigma(C_b(X, E), Y).$$

Now we are in position to state a Banach-Steinhaus type theorem for σ -additive operators from $C_b(X, E)$ to Y .

Theorem 3.3. *Let \mathcal{K} be a SOT-compact subset of $L_\sigma(C_b(X, E), Y)$. Then \mathcal{K} is uniformly σ -additive, i.e.,*

$$\sup_{T \in \mathcal{K}} (\sup_{g \in B_\infty} \|T(u_n g)\|_Y) \xrightarrow{n} 0 \text{ whenever } u_n \downarrow 0 \text{ in } C_b(X).$$

PROOF: By Theorem 2.5 the set \mathcal{K} is $(\beta_\sigma, \|\cdot\|_Y)$ -equicontinuous. Let $\varepsilon > 0$ be given. Then there exists a solid neighbourhood V_ε of 0 for β_σ such that $\|T(f)\|_Y \leq \varepsilon$ for all $f \in V_\varepsilon$ and each $T \in \mathcal{K}$. Now let $u_n \downarrow 0$ in $C_b(X)$, and let $e_o \in S_E$ be fixed. Then for each $g \in B_\infty$ we have $\|(u_n g)(x)\|_E \leq \|(u_n \otimes e_o)(x)\|_E = u_n(x) \downarrow 0$ for all $x \in X$. Since β_σ is a σ -Dini topology, there exists $n_\varepsilon \in \mathbb{N}$ such that $u_n \otimes e_o \in V_\varepsilon$ for $n \geq n_\varepsilon$. Hence for each $g \in B_\infty$, we get $u_n g \in V_\varepsilon$ for $n \geq n_\varepsilon$, because V_ε is a solid subset of $C_b(X, E)$. It follows that for $n \geq n_\varepsilon$, $\sup_{T \in \mathcal{K}} (\sup_{g \in B_\infty} \|T(u_n g)\|_Y) \leq \varepsilon$. □

As an application of Corollary 2.6 and Theorem 3.3 we have

Corollary 3.4. *Let $T_k : C_b(X, E) \longrightarrow Y$ be σ -additive operators for $k \in \mathbb{N}$, and assume that $T(f) := \lim_k T_k(f)$ exists in $(Y, \|\cdot\|_Y)$ for each $f \in C_b(X, E)$. Then T is a σ -additive operator and the set $\{T_k : k \in \mathbb{N}\}$ is uniformly σ -additive, i.e., $\sup_k (\sup_{g \in B_\infty} \|T_k(u_n g)\|_Y) \xrightarrow{n} 0$ whenever $u_n \downarrow 0$ in $C_b(X)$.*

Following the proofs of Proposition 3.2, Theorem 3.3 and Corollary 3.4 we can derive analogous results for τ -additive operators from $C_b(X, E)$ to Y .

Proposition 3.5. Assume that X is paracompact. Then for a bounded linear operator $T : C_b(X, E) \rightarrow Y$ the following statements are equivalent:

- (i) $y' \circ T \in C_b(X, E)'_{\beta_\tau}$ for each $y' \in Y'$;
- (ii) T is $(\beta_\tau, \|\cdot\|_Y)$ -continuous;
- (iii) T is τ -additive.

Hence, if X is paracompact, then

$$\mathcal{L}_{\beta_\tau}(C_b(X, E), Y) = L_\tau(C_b(X, E), Y).$$

Theorem 3.6. Assume that X is paracompact. Let \mathcal{K} be a SOT-compact subset of $L_\tau(C_b(X, E), Y)$. Then \mathcal{K} is uniformly τ -additive, i.e.,

$$\sup_{T \in \mathcal{K}} (\sup_{g \in B_\infty} \|T(u_\alpha g)\|_Y) \xrightarrow{\alpha} 0 \text{ whenever } u_\alpha \downarrow 0 \text{ in } C_b(X).$$

Corollary 3.7. Assume that X is paracompact. Let $T_k : C_b(X, E) \rightarrow Y$ be τ -additive operators for $k \in \mathbb{N}$, and assume that $T(f) := \lim T_k(f)$ exists in $(Y, \|\cdot\|_Y)$ for each $f \in C_b(X, E)$. Then $T : C_b(X, E) \rightarrow Y$ is a τ -additive operator and the set $\{T_k : k \in \mathbb{N}\}$ is uniformly τ -additive, i.e.,

$$\sup_k (\sup_{g \in B_\infty} \|T_k(u_\alpha g)\|_Y) \xrightarrow{\alpha} 0 \text{ whenever } u_\alpha \downarrow 0 \text{ in } C_b(X).$$

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