Antanas Laurinčikas

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Abstract. In the paper the necessary and sufficient conditions for the existence of joint limit distribution for real additive and complex-valued multiplicative function are presented.

1. Introduction.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, integers, real and complex numbers, respectively. We recall that a function $f : \mathbb{N} \to \mathbb{R}$ is called additive if $f(m \cdot n) = f(m) + f(n)$ for all $m, n \in \mathbb{N}$ such that $(m, n) = 1$, and a function $g : \mathbb{N} \to \mathbb{C}$ is said to be multiplicative if $g(m) \not\equiv 0$ and $g(m \cdot n) = g(m)g(n)$ for all $m, n \in \mathbb{N}$, $(m, n) = 1$. Hence we have that $f(1) = 0$, while $g(1) = 1$.

The classical probabilistic number theory investigates asymptotic probabilistic distribution laws for additive and multiplicative arithmetic functions. Let, for $n \in \mathbb{N}$,

$$\nu_n(...) = \frac{1}{n}\#\{1 \le m \le n : ...\},$$

where in place of dots a condition satisfied by m is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Value distribution of arithmetic functions usually is characterized by limit theorems in the sense of weak convergence of probability measures. We recall some results in the field.

Denote by p a prime number, and define

$$\|f(p)\| = \begin{cases} f(p) & \text{if } |f(p)| \le 1, \\ 1 & \text{if } |f(p)| > 1. \end{cases}$$

Theorem A. Let $f(m)$ be a real additive function. Then the probability measure

$$\nu_n\left(f(m) \in A\right), \quad A \in \mathcal{B}(\mathbb{R}),$$

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converges weakly to some probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as $n \to \infty$ if and only if the series

$$\sum_p \frac{\|f(p)\|}{p} \quad \text{and} \quad \sum_p \frac{\|f(p)\|^2}{p} \tag{1}$$

converge.

The sufficiency of Theorem A was proved by P. Erdös in [4], and a full proof was obtained in [6].

In the case of multiplicative functions, we define the m-weak convergence of probability measures. Let P_n and P be two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say that P_n converges m-weakly to P as $n \to \infty$ if P_n converges weakly to P and $P_n(\{0\}) \underset{n\to\infty}{\longrightarrow} P(\{0\})$. In the case $P(\{0\}) = 1$ the last condition is not needed.

The first attempt to prove the existence of limit distribution for multiplicative functions was made in [5].

Theorem B [5]. Let $g(m) \geq 0$ be a multiplicative function. Then the probability measure

$$\nu_n\left(g(m) \in A\right), \quad A \in \mathcal{B}(\mathbb{R}), \tag{2}$$

converges m-weakly to some probability measure P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $P(\{0\}) \neq 1$, as $n \to \infty$ if and only if the series

$$\sum_p \frac{\|g(p)-1\|}{p} \quad \text{and} \quad \sum_p \frac{\|g(p)-1\|^2}{p}$$

converge.

A. Bakštys obtained in [1] a limit theorem for multiplicative functions with positive and negative values. A probability measure P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is symmetric if $P(-\infty, a) = 1 - P(-\infty, a]$ for some $a \in \mathbb{R}$.

Theorem C [1]. Let $g(m)$ be a real multiplicative function. Then the probability measure (2) converges m-weakly to some non-symmetric probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as $n \to \infty$ if and only if the series

$$\sum_p \frac{\|g(p)-1\|}{p}, \quad \sum_p \frac{\|g(p)-1\|^2}{p} \quad \text{and} \sum_{g(p)<0} \frac{1}{p}$$

converge and there exists $\alpha \in \mathbb{N}$ such that $g(2^\alpha) \neq -1$.

Finally, in [13] the problem of the existence of limit distribution for real multiplicative functions has been solved completely. Define, for $A \in \mathcal{B}(\mathbb{R})$,

$$P_a(A) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A. \end{cases}$$

Moreover, let

$$\|u\|_* = \begin{cases} u & \text{if } |u| \leq 1, \\ 1 & \text{if } u > 1, \\ -1 & \text{if } u < -1. \end{cases}$$

Theorem D [13]. *Let $g(m)$ be a real multiplicative function. The probability measure (2) converges m-weakly to some probability measure P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $P \neq P_a$ for every $a \in \mathbb{R}$, as $n \to \infty$ if and only if the series*

$$\sum_{\substack{p \\ g(p)\neq 0}} \frac{\| \log |g(p)| \|_*}{p}, \quad \sum_{\substack{p \\ g(p)\neq 0}} \frac{\| \log |g(p)| \|_*^2}{p} \quad \text{and} \quad \sum_{\substack{p \\ g(p)=0}} \frac{1}{p} \tag{3}$$

converge.

The case of complex-valued multiplicative functions is more complicated. Let $g(m$ be a complex-valued multiplicative function. Define

$$u_g(p) = \begin{cases} \frac{g(p)}{|g(p)|} & \text{if } g(p) \neq 0, \\ 0 & \text{if } g(p) = 0, \end{cases}$$

and

$$v_g(p) = \begin{cases} \log |g(p)| & \text{if } \frac{1}{\mathrm{e}} \leq |g(p)| \leq \mathrm{e}, \\ 1 & \text{if } |g(p)| < \frac{1}{\mathrm{e}} \text{ or } |g(p)| > \mathrm{e}. \end{cases}$$

Theorem E [3]. *Let $g(m)$ be a complex-valued multiplicative function. The probability measure*

$$\nu_n\left(g(m) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to a probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, $P(\{0\}) \neq 1$, as $n \to \infty$ if and only if the following hypotheses hold:

1^0 *The series*

$$\sum_p \frac{v_g(p)}{p} \quad \text{and} \quad \sum_p \frac{v_g^2(p)}{p}$$

converge;

2^0 *Either for all $m \in \mathbb{N}$ and all $t \in \mathbb{R}$*

$$\sum_p \frac{1 - \mathrm{Re}\ u_g^m(p) p^{-it}}{p} = +\infty,$$

or there exits at least one $m \in \mathbb{N}$ such that the series

$$\sum_p \frac{1 - u_g^m(p)}{p}$$

converges.

In [8] and [9] a joint limit theorem for real additive and real multiplicative functions has been obtained.

Let P_n and P be probability measures on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. We say that P_n converges $a, m-$weakly to P as $n \to \infty$ if P_n converges weakly to P and $P_n(\mathbb{R} \times \{0\}) \underset{n\to\infty}{\longrightarrow} P(\mathbb{R} \times \{0\})$.

Theorem F.[8], [9]. *Let $f(m)$ and $g(m)$ be a real additive and real multiplicative functions, respectively. The probability measure*

$$\nu_n\left((f(m), g(m)) \in A\right), \quad A \in \mathcal{B}(\mathbb{R}^2),$$

converges a, m-weakly to some probability measure P on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, $P(\mathbb{R} \times A) \neq P_a(A)$, $a \in \mathbb{R}$, as $n \to \infty$ if and only if the series (1) and (3) converge.

The aim of this paper is to obtain a joint limit theorem for a real additive and a complex-valued multiplicative function.

Let $\mathbb{X} = \mathbb{R} \times \mathbb{C}$, and let P_n and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We say that P_n converges a, m-weakly to in the sense of $\mathbb{X}$ P as $n \to \infty$ if P_n converges weakly to P and $P_n(\mathbb{R} \times \{0\}) \underset{n\to\infty}{\longrightarrow} P(\mathbb{R} \times \{0\})$.

Theorem 1. *Let $f(m)$ and $g(m)$ be a real additive and a complex-valued multiplicative function, respectively. The probability measure*

$$P_n(A) \stackrel{def}{=} \nu_n\left((f(m), g(m)) \in A\right), \quad A \in \mathcal{B}(\mathbb{X}),$$

converges a, m-weakly in the sense of $\mathbb{X}$ to some probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, $P(\mathbb{R} \times \{0\}) \neq 1$, as $n \to \infty$ if and only if the series (1) converge and the hypotheses of Theorem E are satisfied.

For the proof of Theorem 1 the method of characteristic transforms is applied.

2. Characteristic transforms

First we recall some results on probability measures and their convergence on $\mathbb{C}$. Denote points of $\mathbb{C}$ by $z = r\mathrm{e}^{i\varphi}$. Let P be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. The function $w(\tau, k)$ defined by the equality

$$w(\tau, k) = \int\limits_{\mathbb{C}\setminus\{0\}} r^{i\tau}\mathrm{e}^{ik\varphi}\mathrm{d}P, \quad \tau \in \mathbb{R},\ k \in \mathbb{Z},$$

is called the characteristic transform of P.

The measure P is uniquely determined by its characteristic transform $w(\tau, k)$.

Let P and P_n be probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. We say that P_n converges weakly in sense of $\mathbb{C}$ to P as $n \to \infty$ if P_n converges weakly to P and $\lim\limits_{n\to\infty} P_n(\{0\}) = P(\{0\})$.

Lemma 2. *Let $\{P_n\}$ be a sequence of probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ and let $\{w_n(\tau, k)\}$ be the sequence of corresponding characteristic transforms. Suppose that $\lim\limits_{n\to\infty} w_n(\tau, k) = w(\tau, k)$ for all $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$, and that the function $w(\tau, 0)$ is continuous at the point $\tau = 0$. Then there exists a probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that P_n converges weakly in sense of $\mathbb{C}$ to P as $n \to \infty$. In this case, $w(\tau, k)$ is the characteristic transform of the measure P.*

Lemma 2 and other elements of the theory of probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ can be found in [10].

For points of the space $\mathbb{X}$ we will use the notation $\left(x, r\mathrm{e}^{i\varphi}\right)$. Let P be a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and

$$P_{\mathbb{R}}(A) = P(A \times \mathbb{C}), \quad A \in \mathcal{B}(\mathbb{R}).$$

The functions

$$w(\tau) = \int\limits_{\mathbb{R}} \mathrm{e}^{i\tau x}\mathrm{d}P_{\mathbb{R}}, \quad \tau \in \mathbb{R},$$

and

$$w(\tau_1,\tau_2,k)=\int\limits_{\mathbb{X}} \mathrm{e}^{i(\tau_1 x+k\varphi)} r^{i\tau_2}\mathrm{d}P, \quad \tau_1,\tau_2\in\mathbb{R},\ k\in\mathbb{Z},$$

where the last integrand is zero if $r=0$, are called the characteristic transforms of the measure P.

For the proof of Theorem 1 we need the continuity theorems for probability measures on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$.

In [11] it was proved that a probability measure P on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$ is uniquely determined by its characteristic transforms $(w(\tau), w(\tau_1,\tau_2,k))$. Moreover, two following statements in [11] were obtained.

Lemma 3. *Let $\{P_n\}$ be a sequence of probability measures on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$, and let $\{(w_n(\tau), w_n(\tau_1,\tau_2,k))\}$ be the corresponding sequence of characteristic transforms. Suppose that*

$$\lim_{n\to\infty} w_n(\tau)=w(\tau), \quad \tau\in\mathbb{R},$$

and

$$\lim_{n\to\infty} w_n(\tau_1,\tau_2,k)=w(\tau_1,\tau_2,k), \quad \tau_1,\tau_2\in\mathbb{R},\ k\in\mathbb{Z},$$

where the functions $w(\tau)$, $w(\tau_1,0,0)$ and $w(0,\tau_2,0)$ are continuous at the points $\tau=0$, $\tau_1=0$ and $\tau_2=0$, respectively. Then on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$ there exists a probability measure P such that P_n converges a,m-weakly in the sense of $\mathbb{X}$ to P as $n\to\infty$. In this case, $(w(\tau), w(\tau_1,\tau_2,k))$ are the characteristic transforms of the measure P.

Lemma 4. *Let $\{P_n\}$ and $\{(w_n(\tau), w_n(\tau_1,\tau_2,k))\}$ be the same as in Lemma 2. Suppose that P_n converges a,m-weakly in the sense of $\mathbb{X}$ to some probability measure P on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$ as $n\to\infty$. Then*

$$\lim_{n\to\infty} w_n(\tau)=w(\tau), \quad \tau\in\mathbb{R},$$

and

$$\lim_{n\to\infty} w_n(\tau_1,\tau_2,k)=w(\tau_1,\tau_2,k), \quad \tau_1,\tau_2\in\mathbb{R},\ k\in\mathbb{Z},$$

where $(w(\tau), w(\tau_1,\tau_2,k))$ are the characteristic transforms of the measure P.

3. Mean values of multiplicative functions

We say that a multiplicative function $g(m)$ has the mean value $M(g)$ if the limit

$$\lim_{x\to\infty}\frac{1}{x}\sum_{m\le x} g(m)=M(g)$$

exists.

Lemma 5. *In order that the mean value of the multiplicative function $g(m)$, $|g(m)|\le 1$, exist and be zero, it is necessary and sufficient that one of the following conditions should be satisfied:*

1^0 *For every* $u\in\mathbb{R}$, $\displaystyle\sum_p \frac{1-\operatorname{Re} g(p)p^{-iu}}{p}=\infty;$

2^0 *There exists a number* $u_0 \in \mathbb{R}$ *such that the series*

$$\sum_p \frac{1 - \text{Re } g(p)p^{-iu_0}}{p}$$

converges, and $2^{-riu_0}g(2^r) = -1$ *for all* $r \in \mathbb{N}$.

The lemma is a corollary of results from [7].

Lemma 6. *Let* $g(m) = g(m; t_1, ..., t_r)$, $|g(m)| \leq 1$, *be a multiplicative function, and the series*

$$\sum_p \frac{1 - \text{Re } g(p; t_1, ..., t_r)p^{-ia(t_1,...,t_2)}}{p}$$

converges uniformly in t_j, $|t_j| \leq T$, $j = 1, ..., r$. *Then, as* $x \to \infty$,

$$\frac{1}{x}\sum_{m \leq x} g(m; t_1, ..., t_r) = \frac{x^{ia(t_1,...,t_r)}}{1 + a(t_1, ..., t_r)} \times$$

$$\times \prod_{p \leq x}\left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha; t_1, ..., t_r)}{p^{\alpha(1+ia(t_1,...,t_r))}}\right) + o(1)$$

uniformly in t_j, $|t_j| \leq T$, $j = 1, ..., n$.

The lemma is a special case of a result from [12].

4. Sufficiency

We suppose that $0_z = 0$, for $z \in \mathbb{C}$.

Let $(w_n(\tau), w_n(\tau_1, \tau_2, k))$ be the characteristic transforms of the measure P_n. Then we have that

$$w_n(\tau) = \frac{1}{n}\sum_{m=1}^{n} \mathrm{e}^{i\tau f(m)}$$

and

$$w_n(\tau_1, \tau_2, k) = \frac{1}{n}\sum_{m=1}^{n} \mathrm{e}^{i\tau_1 f(m) + ik \arg g(m)}|g(m)|^{i\tau_2}.$$

It is easily seen that the series

$$\sum_p \frac{1 - \text{Re } \mathrm{e}^{-i\tau f(p)}}{p} \ll \sum_{\substack{p \\ |f(p)|>1}} \frac{1}{p} +$$

$$+ \sum_{\substack{p \\ |f(p)|\leq 1}} \frac{\sin^2 \frac{\tau f(p)}{2}}{p} \ll (\tau^2 + 1)\sum_p \frac{\|f(p)\|^2}{p} \tag{4}$$

in view of the convergence of series (1) converges uniformly in $|\tau| \leq T$. Therefore, by Lemma 6, as $n \to \infty$,

$$w_n(\tau) = \prod_{p \leq n}\left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{\mathrm{e}^{i\tau f(p^\alpha)}}{p^\alpha}\right) + o(1)$$

uniformly in $|\tau| \le T$. Hence, taking into account the convergence of series (1) again, we find that

$$\lim_{n\to\infty} w_n(\tau) = w(\tau), \tag{5}$$

where

$$w(\tau) = \prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{e^{i\tau f(p^\alpha)}}{p^\alpha}\right)$$

is continuous at $\tau = 0$.

Now we consider the series

$$S(\tau_1,\tau_2,k) \overset{def}{=} \sum_p \frac{1-\text{Re } e^{i\tau_1 f(p)+ik\arg g(p)}|g(p)|^{i\tau_2}}{p}.$$

Using the identity

$$1 - z_1z_2z_3 = z_2z_3(1-z_1) + z_3(1-z_2) + (1-z_3), \tag{6}$$

we find that uniformly in $|\tau_j| \le T$, $j = 1,2$,

$$S(\tau_1,\tau_2,k) \ll \sum_{g(p)=0}\frac{1}{p} + \sum_p \frac{1-\text{Re } e^{i\tau_1 f(p)}}{p} + \sum_{\substack{p\\ g(p)\ne 0}} \frac{1-\text{Re } e^{ik\arg g(p)}}{p} +$$
$$+ \sum_{\substack{p\\ g(p)\ne 0}} \frac{1-e^{i\tau_2\log|g(p)|}}{p} + \left(\sum_p \frac{1-\text{Re } e^{i\tau_1 f(p)}}{p}\right)^{\frac12} \times$$
$$\times \left(\sum_{\substack{p\\ g(p)\ne 0}} \frac{1-\text{Re } e^{ik\arg g(p)}}{p}\right)^{\frac12} + \left(\sum_p \frac{1-\text{Re } e^{i\tau_1 f(p)}}{p}\right)^{\frac12} \times$$
$$\left(\sum_{\substack{p\\ g(p)\ne 0}} \frac{1-\text{Re } e^{i\tau_2\log|g(p)|}}{p}\right)^{\frac12} + \left(\sum_{\substack{p\\ g(p)\ne 0}} \frac{1-\text{Re } e^{ik\arg p}}{p}\right)^{\frac12} \times$$
$$\times \left(\sum_{\substack{p\\ g(p)\ne 0}} \frac{1-\text{Re } e^{i\tau_2\log|g(p)|}}{p}\right)^{\frac12}. \tag{7}$$

Suppose that there exists $k_0 \in \mathbb{N}$ such that the series

$$\sum_p \frac{1-u_g^{k_0}(p)}{p}$$

converges. Then it can be proved, see [2], p. 224 -227, that there exists $q \in \mathbb{N}$ such that the series

$$\sum_p \frac{1-u_g^{k}(p)}{p}$$

converges if and only if $q|k$. For these k, we have that the series

$$\sum_{\substack{p\\ g(p)\ne 0}} \frac{1-\text{Re } e^{ik\arg g(p)}}{p}$$

converges.

We already have seen that in view of the convergence of series (1) the series

$$\sum_p \frac{1 - \text{Re } e^{i\tau_1 f(p)}}{p}$$

converges uniformly in $|\tau_1| \le T$. Moreover, condition 1^0 of Theorem E shows that the series

$$\sum_{\substack{p \\ g(p)\neq 0}} \frac{1 - \text{Re } e^{i\tau_2 \log|g(p)|}}{p} \ll \sum_{\substack{p \\ v_g(p)=1}} \frac{1}{p} + \frac{\tau_2^2}{2} \sum_p \frac{v_g^2(p)}{p} \tag{8}$$

also converges uniformly in $|\tau_2| \le T$. These three remarks and (7) yield the uniform convergence in $|\tau_j| \le T$, $j = 1, 2$, for the series $S(\tau_1, \tau_2, k)$ if $q|k$. Therefore, if $q|k$, then by Lemma 6, as $n \to \infty$,

$$w_n(\tau_1, \tau_2, k) = \prod_{p \le n} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{e^{i\tau_1 f(p^\alpha) + ik \arg g(p^\alpha)} |g(p^\alpha)|^{i\tau_2}}{p^\alpha}\right) + o(1) \tag{9}$$

uniformly in $|\tau_j| \le T$, $j = 1, 2$. We have that

$$\frac{e^{i\tau_1 f(p) + ik \arg g(p)} |g(p)|^{i\tau_2} - 1}{p} =$$

$$= \begin{cases} O\left(\frac{1}{p}\right) & \text{if } \ g(p) = 0, \\ O\left(\frac{1}{p}\right) & \text{if } \ |f(p)| > 1 \\ & \text{or } \ |\log|g(p)|| > 1, \\ \frac{u_g^k(p) - 1}{p} + \frac{i\tau_1 f(p)}{p} + \frac{i\tau_1 (u_g^k(p) - 1) f(p)}{p} + & \\ \quad + \frac{i\tau_2 v_g(p)}{p} + \frac{i\tau_2 (u_g^k(p) - 1) v_g(p)}{p} - & \text{if } \ |f(p)| \le 1 \\ \quad - \frac{\tau_1 \tau_2 f(p) v_g(p) u_g^k(p)}{p} + & \text{and } \ |\log|g(p)|| \le 1. \\ \quad + O\left(\frac{|\tau_1| f^2(p)}{p}\right) + O\left(\frac{|\tau_2| v_g^2(p)}{p}\right) & \end{cases} \tag{10}$$

From the hypotheses of Theorem 1 it follows that

$$\sum_{\substack{p \\ |f(p)|\le 1}} \frac{\text{Re } (1 - u_g^k(p)) f(p)}{p} \le \sum_{\substack{p \\ |f(p)|\le 1}} \frac{1 - \text{Re } u_g^k(p)}{p} < \infty,$$

$$\sum_{\substack{p \\ |f(p)|\le 1}} \frac{\text{Im } (1 - u_g^k(p)) f(p)}{p} = \sum_{\substack{p \\ |f(p)|\le 1}} \frac{-\text{Im } u_g^k(p) f(p)}{p} \le$$

$$\le \left(\sum_p \frac{|\text{Im } u_g^k(p)|^2}{p}\right)^{\frac{1}{2}} \left(\sum_{\substack{p \\ |f(p)|\le 1}} \frac{f^2(p)}{p}\right)^{\frac{1}{2}} \le$$

$$\leq 2\left(\sum_{p}\frac{1-\text{Re }u_g^k(p)}{p}\right)^{\frac{1}{2}}\left(\sum_{\substack{p\\|f(p)|\leq 1}}\frac{f^2(p)}{p}\right)^{\frac{1}{2}}$$

converges if $q|k$. Hence the series

$$\sum_{\substack{p\\|f(p)|\leq 1}}\frac{(u_g^k(p)-1)f(p)}{p}$$

converges. Similarly, we find that the series

$$\sum_{\substack{p\\|\log|g(p)||\leq 1}}\frac{(u_g^k(p)-1)v_g(p)}{p}$$

and

$$\sum_{|f(p)|\leq 1,\ \log|g(p)|\leq 1}\frac{f(p)v_g(p)u_g^k(p)}{p}$$

also converges. Therefore, (9) and (10) show that uniformly in $|\tau_j|\leq T$, $j=1,2$,

$$\lim_{n\to\infty} w_n(\tau_1,\tau_2,k)=w(\tau_1,\tau_2,k),$$

where

$$w(\tau_1,\tau_2,k)=\prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{\mathrm{e}^{i\tau_1 f(p^\alpha)+ik\arg g(p^\alpha)}|g(p^\alpha)|^{i\tau_2}}{p^\alpha}\right).$$

Clearly, $w(\tau_1,0,0)$ and $w(0,\tau_2,0)$ are continuous at τ_1 and $\tau_2=0$, respectively.

Now suppose that $q\nmid k$. Then, by repeating the arguments of [2], it can be proved that

$$\sum_p\frac{1-\text{Re }u_g^k(p)p^{-iu}}{p}=\infty \tag{11}$$

for all $u\in\mathbb{R}$. Then, using (6), we have

$$\sum_{p\leq n}\frac{1-\text{Re }\mathrm{e}^{i\tau_1 f(p)}u_g^k(p)|g(p)|^{i\tau_2}p^{-iu}}{p}\geq\sum_{\substack{p\leq n\\g(p)=0}}\frac{1}{p}+$$

$$+\sum_{\substack{p\leq n\\g(p)\neq 0}}\frac{1-\text{Re }u_g^k(p)p^{-iu}}{p}-c_1\sum_{\substack{p\leq n\\g(p)\neq 0}}\frac{1-\text{Re }\mathrm{e}^{i\tau_1 f(p)}|g(p)|^{i\tau_2}}{p}-$$

$$-c_3\left(\sum_{\substack{p\leq n\\g(p)\neq 0}}\frac{1-\text{Re }\mathrm{e}^{i\tau_1 f(p)}|g(p)|^{i\tau_2}}{p}\right)^{\frac{1}{2}}\left(\sum_{\substack{p\leq n\\g(p)\neq 0}}\frac{1-\text{Re }u_g^k(p)p^{-iu}}{p}\right)^{\frac{1}{2}} \tag{12}$$

with some positive c_1, c_2 and c_3. However,

$$\sum_{\substack{p\leq n\\g(p)\neq 0}}\frac{1-\text{Re }\mathrm{e}^{i\tau_1 f(p)}|g(p|^{i\tau_2}}{p}\ll$$

$$\ll\sum_{p\leq n}\frac{1-\text{Re }\mathrm{e}^{i\tau_1 f(p)}}{p}+\sum_{\substack{p\leq n\\g(p)\neq 0}}\frac{1-\text{Re }\mathrm{e}^{i\tau_2\log|g(p)|}}{p}+$$

$$+\left(\sum_{p\le n}\frac{1-\mathrm{Re}\ \mathrm{e}^{i\tau_1 f(p)}}{p}\right)^{\frac{1}{2}}\left(\sum_{\substack{p\le n\\ g(p)\ne 0}}\frac{1-\mathrm{Re}\ \mathrm{e}^{i\tau_2\log|g(p|}}{p}\right)\ll 1$$

uniformly in $|\tau_j|\le T$, $j=1,2$, in view (4) and (8). From this, (11) and (12) we obtain that

$$\sum_p\frac{1-\mathrm{Re}\ \mathrm{e}^{i\tau_1 f(p)}u_g^k(p)|g(p)|^{i\tau_2}p^{-iu}}{p}=\infty$$

for all $\tau_1,\tau_2\in\mathbb{R}$ and all $u\in\mathbb{R}$. Consequently, by Lemma 5 we have in this case that

$$\lim_{n\to\infty}w_n(\tau_1,\tau_2,k)=0 \tag{13}$$

for all $\tau_1,\tau_2\in\mathbb{R}$.

If

$$\sum_p\frac{1-\mathrm{Re}\ u_g^k(p)p^{-iu}}{p}=+\infty$$

for all $k\in\mathbb{N}$ and $u\in\mathbb{R}$, then, reasoning similarly to the case $q\nmid k$, we obtain that

$$\lim_{n\to\infty}w_n(\tau_1,\tau_2,k)=0$$

for all $\tau_1,\tau_2\in\mathbb{R}$ and $k\in\mathbb{N}$.

Therefore, the sufficiency follows from Lemma 3.

5. Necessity

Suppose that the measure P_n converges a,m-weakly to some probability measure P on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$, $P(\mathbb{R}\times\{0\})\ne 1$, as $n\to\infty$. Let $(w_n(\tau),w_n(\tau_1,\tau_2,k))$ be the characteristic transforms of the measure P_n. Then by Lemma 4

$$\lim_{n\to\infty}w_n(\tau)=w(\tau),\quad \tau\in\mathbb{R},$$

and

$$\lim_{n\to\infty}w_n(\tau_1,\tau_2,k)=w(\tau_1,\tau_2,k),\quad \tau_1,\tau_2\in\mathbb{R},\ k\in\mathbb{Z}, \tag{14}$$

where $(w(\tau),w(\tau_1,\tau_2,k))$ are the characteristic transforms of P.

The function $w(\tau)$ is the characteristic function of the probability measure $P_{\mathbb{R}}(A\times\mathbb{C})$, $A\in\mathcal{B}(\mathbb{R})$. Therefore, $w(\tau)$ is continuous at $\tau=0$. Hence we obtain that the probability measure

$$\nu_n\left(f(m)\in A\right),\quad A\in\mathcal{B}(\mathbb{R}),$$

converges weakly to the probability measure $P_{\mathbb{R}}$ as $n\to\infty$. Therefore, by Theorem A, series (1) converges.

We observe that the function $w(0,\tau_2,0)$ is continuous. Really,

$$w(0,\tau_2,0)=\int\limits_{\mathbb{X}}r^{i\tau_2}\mathrm{d}P=\int\limits_{\substack{\infty\\ r\ne 0}}^{\infty}r^{i\tau_2}\mathrm{d}\hat{P}, \tag{15}$$

where $\hat{P}$ is a probability measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. In place of the measure $\hat{P}$ we can use the distribution function

$$F(x)=\hat{P}(-\infty,x).$$

Define

$$\beta_0 = 1 - F(0), \quad \beta_1 = F(0).$$

Now let, for $\beta_j \neq 0$, $j = 1, 2$,

$$G_0(x) = \frac{F(\mathrm{e}^x) - F(0)}{\beta_0},$$

$$G_1(x) = \frac{F(0) - F(-\mathrm{e}^x)}{\beta_1}.$$

Then $G_0(x)$ and $G_1(x)$ are distribution functions, and we have in view of (15) that

$$w(0, \tau_2, 0) = \beta_0 f_0(\tau_2) + \beta_1 f_1(\tau_2), \tag{16}$$

where $f_j(\tau_2)$ is the characteristic function of the distribution function $G_j(x)$, $j = 0, 1$. (16) remains valid also in the case when $\beta_0 = 0$ or $\beta_1 = 0$, or $\beta_0 = 0$, $\beta_1 = 0$. In this case the corresponding terms on the right-hand side of (16) are zeros.

Since the characteristic functions $f_0(\tau_2)$ and $f_1(\tau_2)$ are continuous, equality (16) gives the continuity of $w(0, \tau_2, 0)$. By (14) the characteristic transform of the measure

$$\nu_n\left(g(m) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}), \tag{17}$$

converges to the function $w(0, \tau_2, k)$, and $w(0, \tau_2, k)$ is continuous at $\tau_2 = 0$. Hence, by Lemma 2, the measure (17) converges weakly in the sense of $\mathbb{C}$ to some probability measure P_1 on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Clearly, $P(\mathbb{R} \times A)$ coinsides with $P_1(A)$, $A \in \mathcal{B}(\mathbb{C})$. Since $P(\mathbb{R} \times \{0\}) \neq 0$, hence we have that $P_1(\{0\}) \neq 0$. Therefore, by Theorem E we obtain the conditions related to the function $g(m)$. The necessity is proved.

References

[1] A. Bakštys, On limit distribution laws for arithmetic multiplicative functions, *Liet. matem. rink.*, **8** (1968), 5–20 (in Russian).

[2] H. Delange, On the distribution modulo 1 of additive functions, *J. Indian Math. Soc.,* **39** (1970), 215–235.

[3] H. Delange, Sur la distribution des valeurs des fonctions multiplicative complexes, *C. R. Acad. Sc. Paris,* **276**, Série A (1973), 161–164.

[4] P. Erdös, On the density of some sequences of numbers, III, *J. London Math. Soc.* **13** (1938), 119–127.

5. P. Erdös, Some remarks about additive and multiplicative functions, *Bull. Amer. Math. Soc.*, **52** (1946), 527–537.

[6] P. Erdös, A. Wintner, Additive arithmetical functions and statistical independence, *Amer. J. Math.* **61** (1939), 713–721.

[7] G. Halász, Über die Mittelwerte multikativer zahlentheoreticher Funktionen, *Acta Math. Sci. Hung.*, **19**(1968), 365–403.

[8] A. Laurinčikas, On joint value distribution of additive and multiplicative functions, *Liet. matem. rink.*, **16**(2)(1976), 190–192 (in Russian).

[9] A. Laurinčikas, On joint distribution of values of arithmetical functions, *Liet. matem. rink.*, **31**(3)(1991), 433–454 (in Russian).

[10] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht, Boston, London, 1996.

[11] A. Laurinčikas, The joint distribution of the Riemann zeta-function, 2005, submitted.
[12] B. V. Levin, N. M. Timofeev, Analytic method in probabilistic number theory, *Uch. zap. Vladim. gos. ped. inst.,* **57**(2) (1971), 57–150 (in Russian).
[13] B. V. Levin, N. M. Timofeev, S. T. Tuliaganov, Distribution of values of multiplicative functions, *Liet. matem. rink.,* **13**(1) (1973), 87–100 (in Russian).

Author(s) Address(es):
Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, 03225 Vilnius, Lithuania

E-mail address: antanas.laurincikas@maf.vu.lt