Acta Mathematica Universitatis Ostraviensis

Chawalit Boonpok

∂-closed sets in biclosure spaces

Acta Mathematica Universitatis Ostraviensis, Vol. 17 (2009), No. 1, 51--66

Persistent URL: `http://dml.cz/dmlcz/137527`

∂-Closed Sets in Biclosure Spaces

Chawalit Boonpok

Abstract. In the present paper, we introduce and study the concept of ∂-closed sets in biclosure spaces and investigate its behavior. We also introduce and study the concept of ∂-continuous maps.

1 Introduction

A bitopological space $(X, \Im_1, \Im_2)$ is a set X together with two topologies $\Im_1$ and $\Im_2$ defined on X. The study of bitopological spaces was initiated by J. C. Kelly [6]. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. Closure spaces were studied in [1] (see also [2], [3], [9], [10]) as sets endowed with a grounded, extensive and monotone closure operator. In this paper, we introduce and study the concept of ∂-closed sets in biclosure spaces and characterize their properties. Moreover, we define the notions of ∂-continuity and ∂-irresoluteness by using ∂-closed sets and study some of their basic properties.

2 Preliminaries

A map $u\colon P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied:

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive* (respectively, *idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if

2000 Mathematics Subject Classification: 54A05

Key Words and Phrases: closure operator, closure space, biclosure space, ∂-closed set, ∂-continuous map.

its complement is closed. The empty set and the whole space are both open and closed.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too. A closure space (X, u) is said to be a T_0*-space* if, for any pair of points $x, y \in X$, from $x \in u\{y\}$ and $y \in u\{x\}$ it follows that $x = y$, and it is called a $T_{\frac{1}{2}}$*-space* if each singleton subset of X is closed or open.

Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f\colon (X, u) \to (Y, v)$ is continuous if and only if

$$uf^{-1}(B) \subseteq f^{-1}(vB)$$

for every subset $B \subseteq Y$. Clearly, if $f\colon (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$, is the closure space $\Big(\prod\limits_{\alpha\in I} X_\alpha, u\Big)$ where $\prod\limits_{\alpha\in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections

$$\pi_\alpha\colon \prod_{\alpha\in I}(X_\alpha, u) \to (X_\alpha, u)\,,$$

$\alpha \in I$, i.e., is defined by

$$uA = \prod_{\alpha\in I} u_\alpha \pi_\alpha(A)$$

for each $A \subseteq \prod\limits_{\alpha\in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta\colon \prod\limits_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition 1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Proof. Let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$. But

$$\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,,$$

hence $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$. Since π_β is closed,

$$\pi_\beta\Big(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = F$$

is a closed subset of (X_β, u_β). □

The following statement is evident:

Proposition 2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Definition 1. Let (X, u) be a closure space. A subset $A \subseteq X$ is called a *generalized closed* set, briefly a *g-closed* set, if $uA \subseteq G$ whenever G is an open subset of (X, u) with $A \subseteq G$. A subset $A \subseteq X$ is called a *generalized open* set, briefly a *g-open* set, if its complement is g-closed.

Proposition 3. *Let (X, u) be a closure space. A set $A \subseteq X$ is g-open if and only if $F \subseteq X - u(X - A)$ whenever F is a closed subset of (X, u) with $F \subseteq A$.*

Proof. Suppose that A is g-open and let $F \subseteq A$ be a closed subset of (X, u). Then $X - A \subseteq X - F$. But $X - A$ is g-closed and $X - F$ is open. It follows that $u(X - A) \subseteq X - F$ and hence $F \subseteq X - u(X - A)$.

Conversely, let $X - A \subseteq G$ where G is open. Then $X - G \subseteq A$. Since $X - G$ is closed, $X - G \subseteq X - u(X - A)$. Therefore, $u(X - A) \subseteq G$. Hence, $X - A$ is g-closed and so A is g-open. □

Proposition 4. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is a g-open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Proof. Let F be a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$ such that $F \subseteq G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$. Then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is closed and G is g-open in (X_β, u_β),

$$\pi_\beta(F) \subseteq X_\beta - u_\beta(X_\beta - G)\,.$$

Therefore,

$$F \subseteq \pi_\beta^{-1}(X_\beta - u_\beta(X_\beta - G)) = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} u_\alpha \pi_\alpha\Big(\prod_{\alpha \in I} X_\alpha - G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big)\,.$$

By Proposition 4, $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

Conversely, let F be a closed subset of (X_β, u_β) such that $F \subseteq G$. Then

$$F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Since $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is closed and $G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-open in $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$,

$$F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$$

by Proposition 4. Therefore,

$$\prod_{\alpha\in I} u_\alpha\pi_\alpha\Big((X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) \subseteq \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Consequently, $u_\beta(X_\beta - G) \subseteq X_\beta - F$ implies $F \subseteq X_\beta - u_\beta(X_\beta - G)$. Hence, G is a g-open subset of (X_β, u_β). □

Proposition 5. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a g-closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Proof. Let F be a g-closed subset of (X_β, u_β). Then $X_\beta - F$ is a g-open subset of (X_β, u_β). By Proposition 5,

$$(X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$$

is a g-open subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$. Hence, $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let G be an open subset of (X_β, u_β) such that $F \subseteq G$. Then

$$F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Since $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-closed and $G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is open in $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$,

$$\prod_{\alpha\in I} u_\alpha\pi_\alpha\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\beta\Big) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Consequently, $u_\beta F \subseteq G$. Therefore, F is a g-closed subset of (X_β, u_β). □

Proposition 6. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. For each $\beta \in I$, let*

$$\pi_\beta \colon \prod_{\alpha \in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$$

be the projection map. If F is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$, then $\pi_\beta(F)$ is a g-closed subset of (X_β, u_β).

Proof. Let F be a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ and let G be an open subset of (X_β, u_β) such that $\pi_\beta(F) \subseteq G$. Then

$$F \subseteq \pi_\beta^{-1}(G) = G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha\,.$$

Since $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$,

$$\prod_{\alpha\in I} u_\alpha \pi_\alpha(F) \subseteq G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha\,.$$

Consequently, $u_\beta \pi_\beta(F) \subseteq G$. Hence, $\pi_\beta(F)$ is a g-closed subset of (X_β, u_β). □

Definition 2. A *biclosure space* is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X.

Definition 3. A subset A of a biclosure space (X, u_1, u_2) is called *closed* if $u_1u_2A = A$ and it is *open* if its complement is closed.

Clearly, A is a closed subset of a biclosure space (X, u_1, u_2) if and only if A is both a closed subset of (X, u_1) and (X, u_2).

Let A be a closed subset of a biclosure space (X, u_1, u_2). The following conditions are equivalent

(i) $u_2u_1A = A$,

(ii) $u_1A = A$, $u_2A = A$.

Proposition 7. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ if and only if F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$.*

Proof. Let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Then

$$F = \prod_{\alpha\in I} u^1_\alpha \pi_\alpha\Big(\prod_{\alpha\in I} u^2_\alpha \pi_\alpha(F)\Big)\,.$$

Since $F \subseteq \prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F)$,

$$\prod_{\alpha\in I} u^1_\alpha\pi_\alpha(F) \subseteq \prod_{\alpha\in I} u^1_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F)\Big) = F\,.$$

Hence, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$. Since $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F) \subseteq \prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F)$,

$$\prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F) \subseteq \prod_{\alpha\in I} u^1_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F)\Big) = F\,.$$

Therefore, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$.

Conversely, let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$. Then $F = \prod_{\alpha\in I} u^1_\alpha\pi_\alpha(F)$ and $F = \prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F)$. Consequently,

$$F = \prod_{\alpha\in I} u^1_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F)\Big)\,.$$

Hence, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. □

Proposition 8. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let F be a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then F is a closed subset of (X_β, u^1_β) and (X_β, u^2_β), respectively. Therefore, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$. Consequently, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$, respectively. Hence, F is a closed subset of (X_β, u^1_β) and (X_β, u^2_β). Consequently, F is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Definition 4. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. A map

$$f\colon (X, u_1, u_2) \to (Y, v_1, v_2)$$

is called *i-continuous* if the map $f\colon (X, u_i) \to (Y, v_i)$ is continuous. A map f is called *continuous* if f is i-continuous for each $i \in \{1, 2\}$.

Definition 5. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. A map

$$f : (X, u_1, u_2) \to (Y, v_1, v_2)$$

is called *i-closed* (resp. *i-open*) if the map $f : (X, u_i) \to (Y, v_i)$ is closed (resp. open). A map f is called *closed* (resp. *open*) if f is i-closed (resp. i-*open*) for each $i \in \{1, 2\}$.

3 ∂-Closed Sets

In this section, we introduce a new class of ∂-closed sets in biclosure spaces and study some of its properties.

Definition 6. A subset A of a biclosure space (X, u_1, u_2) is called a *∂-closed* if $u_2A \subseteq G$ whenever G is a g-open subset of (X, u_1) with $A \subseteq G$. The complement of a ∂-closed set is called *∂-open*.

Remark 1. *For a subset A of a biclosure space (X, u_1, u_2), the following implications hold:*

$$A \text{ is closed} \Rightarrow A \text{ is } \partial\text{-closed}$$

The implication is not reversible as shown by the following example.

Example 1. Let $X = \{a, b\}$ and define a closure operator u_1 on X by

$$u_1\emptyset = \emptyset, \quad u_1\{a\} = u_1\{b\} = u_1X = X .$$

Define a closure operator u_2 on X by

$$u_2\emptyset = \emptyset, \quad u_2\{a\} = \{a\}, \quad u_2\{b\} = u_2X = X .$$

Then $\{a\}$ is ∂-closed but it is not closed.

Proposition 9. *Let (X, u_1, u_2) be a biclosure space and let u_2 be additive. If A and B are ∂-closed subsets of (X, u_1, u_2), then $A \cup B$ is ∂-closed.*

Proof. Let U be a g-open subset of (X, u_1) such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are ∂-closed, $u_2A \subseteq U$ and $u_2B \subseteq U$. Since u_2 is additive,

$$u_2(A \cup B) = u_2A \cup u_2B \subseteq U .$$

Hence, $A \cup B$ is ∂-closed. □

Proposition 10. *Let (X, u_1, u_2) be a biclosure space and let u_2 be idempotent. If A is a ∂-closed subset and $A \subseteq B \subseteq u_2A$, then B is ∂-closed.*

Proof. Let G be a g-open subset of (X, u_1) such that $B \subseteq G$. Then $A \subseteq G$. Since A is ∂-closed, $u_2A \subseteq G$. Since u_2 is idempotent,

$$u_2B \subseteq u_2u_2A = u_2A \subseteq G .$$

Hence, B is ∂-closed. □

Proposition 11. *Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. If A is ∂-closed, then $u_2A - A$ has no nonempty g-closed subset of (X, u_1).*

Proof. Let F be a g-closed subset of (X, u_1) such that $F \subseteq u_2A - A$. Then $A \subseteq X - F$. Since A is ∂-closed and $X - F$ is a g-open subset of (X, u_1),

$$u_2A \subseteq X - F\,.$$

Hence, $F \subseteq X - u_2A$. Consequently,

$$F \subseteq (X - u_2A) \cap u_2A = \emptyset\,.$$

Therefore, $F = \emptyset$. □

Proposition 12. *Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then A is ∂-open if and only if*

$$F \subseteq X - u_2(X - A)$$

for every F is a g-closed subset of (X, u_1) with $F \subseteq A$.

Proof. Assume that A is ∂-open and let $F \subseteq A$ be a g-closed subset of (X, u_1). Then $X - A \subseteq X - F$. Since $X - A$ is ∂-closed and $X - F$ is g-open subset of (X, u_1),

$$u_2(X - A) \subseteq X - F\,.$$

Hence $F \subseteq X - u_2(X - A)$.

Conversely, let U be a g-open subset of (X, u_1) such that $X - A \subseteq U$. Then $X - U \subseteq A$. Since $X - U$ is g-closed subset of (X, u_1),

$$X - U \subseteq X - u_2(X - A)\,.$$

Consequently, $u_2(X - A) \subseteq U$. Hence, $X - A$ is ∂-closed and so A is ∂-open. □

Proposition 13. *Let (X, u_1, u_2) be a biclosure space. If $A \subseteq X$ is ∂-closed, then $u_2A - A$ is ∂-open.*

Proof. Suppose that A is ∂-closed and let F be a g-closed subset of (X, u_1) such that $F \subseteq u_2A - A$. By Proposition 11, $F = \emptyset$ and hence

$$F \subseteq X - u_2(X - (u_2A - A))\,.$$

By Proposition 12, $u_2A - A$ is ∂-open. □

Proposition 14. *Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then G is a ∂-open subset of $(X_\beta, u_\beta^1, u_\beta^2)$ if and only if $G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-open subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$.*

Proof. Let F be a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ such that $F \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is g-closed in (X_β, u^1_β),

$$\pi_\beta(F) \subseteq X_\beta - u^2_\beta(X_\beta - G)\,.$$

Therefore,

$$F \subseteq \pi^{-1}_\beta(X_\beta - u^2_\beta(X_\beta - G)) = \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha \pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)\,.$$

By Proposition 12, $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let F be a g-closed subset of (X_β, u^1_β) such that $F \subseteq G$. Then

$$F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Since $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-closed and $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is ∂-open in $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$,

$$F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha \pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$$

by Proposition 12. Therefore,

$$\prod_{\alpha\in I} u^2_\alpha \pi_\alpha\Big((X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) \subseteq \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Consequently,

$$u^2_\beta(X_\beta - G) \subseteq X_\beta - F$$

implies

$$F \subseteq X_\beta - u^2_\beta(X_\beta - G)\,.$$

Hence, G is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 15. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let F be a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $X_\beta - F$ is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. By Proposition 14,

$$(X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$$

is a ∂-open subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Hence, $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let G be a g-open subset of (X_β, u^1_β) such that $F \subseteq G$. Then

$$F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Since $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed and $G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-open in $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha)$,

$$\prod_{\alpha\in I} u^2_\alpha \pi_\alpha\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\beta\Big) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Consequently, $u^2_\beta F \subseteq G$. Therefore, F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 16. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. For each $\beta \in I$, let*

$$\pi_\beta\colon \prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$$

be the projection map. Then

(i) If F is a ∂-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$, then $\pi_\beta(F)$ is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$.

(ii) If F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$, then $\pi^{-1}_\beta(F)$ is a ∂-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Proof. (i) Let F be a ∂-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ and let G be a g-open subset of (X_β, u^1_β) such that $\pi_\beta(F) \subseteq G$. Then

$$F \subseteq \pi^{-1}_\beta(G) = G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Since $G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha)$,

$$\prod_{\alpha\in I} u^2_\alpha \pi_\alpha(F) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Consequently, $u^2_\beta \pi_\beta(F) \subseteq G$. Hence, $\pi_\beta(F)$ is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$.

(ii) Let F be a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then

$$\pi^{-1}_\beta(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

By Proposition 15, $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$. Hence, $\pi_\beta^{-1}(F)$ is a ∂-closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$. □

4 ∂-Continuous Maps

In this section, we introduce the concept of ∂-continuous maps by using ∂-closed sets. These maps are investigated and studied.

Definition 7. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map

$$f\colon (X, u_1, u_2) \to (Y, v_1, v_2)$$

is called *∂-closed* (resp. *∂-open*) if $f(F)$ is a ∂-closed (resp. ∂-open) subset of (Y, v_1, v_2) for every closed (resp. open) subset of (X, u_1, u_2).

Proposition 17. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. If*

$$g \circ f\colon (X, u_1, u_2) \to (Z, w_1, w_2)$$

is ∂-closed and

$$f\colon (X, u_1, u_2) \to (Y, v_1, v_2)$$

is surjective and continuous, then

$$g\colon (Y, v_1, v_2) \to (Z, w_1, w_2)$$

is ∂-closed.

Proof. Let F be a closed subset of (Y, v_1, v_2). Then F is a closed subset of (Y, v_1) and (Y, v_2), respectively. Since f is continuous, $f^{-1}(F)$ is a closed subset of (X, u_1) and (X, u_2), respectively. Consequently, $f^{-1}(F)$ is a closed subset of (X, u_1, u_2). Since $g \circ f$ is ∂-closed and f is surjective, $g \circ f(f^{-1}(F)) = g(F)$ is a ∂-closed subset of (Z, w_1, w_2). Therefore, g is ∂-closed. □

Definition 8. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map

$$f\colon (X, u_1, u_2) \to (Y, v_1, v_2)$$

is called *∂-continuous* if $f^{-1}(F)$ is a ∂-closed subset of (X, u_1, u_2) for every closed subset F of (Y, v_1, v_2).

Clearly, it is easy to prove that

$$f\colon (X, u_1, u_2) \to (Y, v_1, v_2)$$

is ∂-continuous if and only if $f^{-1}(G)$ is a ∂-open subset of (X, u_1, u_2) for every open subset G of (Y, v_1, v_2).

Proposition 18. *Let* (X, u_1, u_2), (Y, v_1, v_2) *and* (Z, w_1, w_2) *be biclosure spaces. If*

$$g \circ f : (X, u_1, u_2) \to (Z, w_1, w_2)$$

is closed and

$$g : (Y, v_1, v_2) \to (Z, w_1, w_2)$$

is injective and ∂*-continuous, then*

$$f : (X, u_1, u_2) \to (Y, v_1, v_2)$$

is ∂*-closed.*

Proof. Let F be a closed subset of (X, u_1, u_2). Then F is a closed subset of (X, u_1) and (X, u_2), respectively. Since $g \circ f$ is closed, $g \circ f(F)$ is a closed subset of (Z, w_1) and (Z, w_2), respectively. Consequently, $g \circ f(F)$ is a closed subset of (Z, w_1, w_2). Since g is ∂-continuous and injective, $g^{-1}(g \circ f(F)) = f(F)$ is a ∂-closed subset of (Y, v_1, v_2). Therefore, f is ∂-closed. □

Definition 9. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map

$$f : (X, u_1, u_2) \to (Y, v_1, v_2)$$

is called ∂*-irresolute* if $f^{-1}(F)$ is a ∂-closed subset of (X, u_1, u_2) for every ∂-closed subset F of (Y, v_1, v_2).

Clearly, it is easy to prove that

$$f : (X, u_1, u_2) \to (Y, v_1, v_2)$$

is ∂-irresolute if and only if $f^{-1}(G)$ is a ∂-open subset of (X, u_1, u_2) for every ∂-open subset G of (Y, v_1, v_2).

The following statement is obvious:

Proposition 19. *Let* (X, u_1, u_2) *and* (Y, v_1, v_2) *be biclosure spaces. If*

$$f : (X, u_1, u_2) \to (Y, v_1, v_2)$$

is ∂*-irresolute, then* f *is* ∂*-continuous.*

The converse need not be true as can be seen from the following example.

Example 2. Let $X = \{a, b\} = Y$ and define closure operators u_1 and u_2 on X by

$$u_1\emptyset = \emptyset\,, \quad u_1\{a\} = \{a\}\,, \quad u_1\{b\} = u_1X = X\,,$$
$$u_2\emptyset = \emptyset\,, \quad u_2\{a\} = u_2\{b\} = u_2X = X\,.$$

Define closure operators v_1 and v_2 on Y by

$$v_1\emptyset = \emptyset\,, \quad v_1\{b\} = \{b\}\,, \quad v_1\{a\} = v_1Y = Y\,,$$
$$v_2\emptyset = \emptyset\,, \quad v_2\{a\} = v_2\{b\} = v_2Y = Y\,.$$

Let

$$f\colon (X,u_1,u_2)\to (Y,v_1,v_2)$$

be the identity map. Then f is ∂-continuous but it is not ∂-irresolute because $\{b\}$ is a ∂-closed subset of (Y,v_1,v_2) but $f^{-1}(\{b\})=\{b\}$ is not ∂-closed subset of (X,u_1,u_2).

Definition 10. A biclosure space (X,u_1,u_2) is called a $T^*_{\frac{1}{2}}$*-biclosure space* if every ∂-closed subset of (X,u_1,u_2) is a closed subset of (X,u_2).

Proposition 20. *Let (X,u_1,u_2) be a biclosure space. Then (X,u_1,u_2) is a $T^*_{\frac{1}{2}}$-biclosure space if and only if every singleton subset of X is either a g-closed subset of (X,u_1) or an open subset of (X,u_2).*

Proof. Let $x\in X$ and suppose that $\{x\}$ is not a g-closed subset of (X,u_1). Then $X-\{x\}$ is not a g-open subset of (X,u_1). The only g-open subset of (X,u_1) containing $X-\{x\}$ is X, hence $X-\{x\}$ is a ∂-closed subset of (X,u_1,u_2). Since (X,u_1,u_2) is a $T^*_{\frac{1}{2}}$-biclosure space, $X-\{x\}$ is a closed subset of (X,u_2). Consequently, $\{x\}$ is an open subset of (X,u_2).

Conversely, let A be a ∂-closed subset of (X,u_1,u_2). Suppose that $x\notin A$. Then $\{x\}\subseteq X-A$ and we have $A\subseteq X-\{x\}$. If $\{x\}$ is an open subset of (X,u_2), then $X-\{x\}$ is a closed subset of (X,u_2). Consequently,

$$u_2A\subseteq u_2(X-\{x\})=X-\{x\}\,,$$

thus $x\notin u_2A$. If $\{x\}$ is a g-closed subset of (X,u_1), then $X-\{x\}$ is a g-open subset of (X,u_1). Since A is a ∂-closed, $u_2A\subseteq X-\{x\}$. Therefore, $x\notin u_2A$. So, we always have $u_2A\subseteq A$. Thus $u_2A=A$ or, equivalently, A is a closed subset of (X,u_2). Therefore, (X,u_1,u_2) is a $T^*_{\frac{1}{2}}$-biclosure space. □

Proposition 21. *Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. Let*

$$f\colon (X,u_1,u_2)\to (Y,v_1,v_2)$$

be surjective, 2-closed and ∂-irresolute. If (X,u_1,u_2) is a $T^*_{\frac{1}{2}}$-biclosure space, then (Y,v_1,v_2) is a $T^*_{\frac{1}{2}}$-biclosure space.

Proof. Let F be a ∂-closed subset of (Y,v_1,v_2). Since f is ∂-irresolute, $f^{-1}(F)$ is a ∂-closed subset of (X,u_1,u_2). Since (X,u_1,u_2) is a $T^*_{\frac{1}{2}}$-biclosure space, $f^{-1}(F)$ is a closed subset of (X,u_2). Since f is 2-closed and surjective, F is a closed subset of (Y,v_2). Hence, (Y,v_1,v_2) is a $T^*_{\frac{1}{2}}$-biclosure space. □

Proposition 22. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. Let*

$$f\colon (X,u_1,u_2)\to (Y,v_1,v_2)$$

and

$$g\colon (Y,v_1,v_2)\to (Z,w_1,w_2)$$

be maps. Then

(i) $g \circ f$ *is* ∂*-continuous if* g *is continuous and* f *is* ∂*-continuous.*

(ii) $g \circ f$ *is* ∂*-irresolute if* f *and* g *are* ∂*-irresolute.*

(iii) $g \circ f$ *is* ∂*-continuous if* g *is* ∂*-continuous and* f *is* ∂*-irresolute.*

Proof. (i) Let F be a closed subset of (Z, w_1, w_2). Then F is a closed subset of (Z, w_1) and (Z, w_2), respectively. Since g is continuous, $g^{-1}(F)$ is a closed subset of (Y, v_1) and (Y, v_2), respectively. Consequently, $g^{-1}(F)$ is closed subset of (Y, v_1, v_2). Since f is ∂-continuous, $f^{-1}(g^{-1}(F))$ is a ∂-closed subset of (X, u_1, u_2). Therefore, $(g \circ f)^{-1}(F)$ is a ∂-closed subset of (X, u_1, u_2). Hence, $g \circ f$ is ∂-continuous.

The proofs of (ii)–(iii) are similar. □

Proposition 23. *Let* $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ *be a family of biclosure spaces. Then for each* $\beta \in I$*, the projection map*

$$\pi_\beta \colon \prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$$

is continuous.

Proof. Let $A \subseteq \prod\limits_{\alpha \in I} X_\alpha$. Then

$$\pi_\beta\Big(\prod_{\alpha \in I} u^1_\alpha \pi_\alpha(A)\Big) = u^1_\beta \pi_\beta(A)\,.$$

Hence,

$$\pi_\beta \colon \prod_{\alpha \in I} (X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)$$

is continuous. Similarly, since

$$\pi_\beta\Big(\prod_{\alpha \in I} u^2_\alpha \pi_\alpha(A)\Big) = u^2_\beta \pi_\beta(A)\,.$$

Therefore,

$$\pi_\beta \colon \prod_{\alpha \in I} (X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)$$

is continuous. Consequently,

$$\pi_\beta \colon \prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$$

is continuous. □

Proposition 24. *Let* (X, u_1, u_2) *be a biclosure space and let* $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ *be a family of biclosure spaces. Let* $f \colon X \to \prod\limits_{\alpha \in I} Y_\alpha$ *be a map. If*

$$f \colon (X, u_1, u_2) \to \prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha)$$

is ∂-continuous, then

$$\pi_\alpha \circ f\colon (X, u_1, u_2) \to (Y_\alpha, v_\alpha^1, v_\alpha^2)$$

is ∂-continuous for each $\alpha \in I$.

Proof. Let f be ∂-continuous. Since π_α is continuous for each $\alpha \in I$, also $\pi_\alpha \circ f$ is ∂-continuous for each $\alpha \in I$. □

Proposition 25. *Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha\colon X_\alpha \to Y_\alpha$ be a map and*

$$f\colon \prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha$$

be the map defined by $f((x_\alpha)_{\alpha\in I}) = (f_\alpha(x_\alpha))_{\alpha\in I}$. If

$$f\colon \prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2) \to \prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$$

is ∂-continuous, then

$$f_\alpha\colon (X_\alpha, u_\alpha^1, u_\alpha^2) \to (Y_\alpha, v_\alpha^1, v_\alpha^2)$$

is ∂-continuous for each $\alpha \in I$.

Proof. Let F be a closed subset of $(Y_\beta, v_\beta^1, v_\beta^2)$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$. Since f is ∂-continuous,

$$f^{-1}\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha\Big) = f_\beta^{-1}(F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$$

is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$. By Proposition 15, $f_\beta^{-1}(F)$ is a ∂-closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$. Hence, f_β is ∂-continuous. □

References

[1] E. Čech, Topological Spaces, *Topological Papers of Eduard Čech*, Academia, Prague (1968), 436–472.

[2] J. Chvalina, On homeomorphic topologies and equivalent set-systems, *Arch. Math. 2*, Scripta Fac. Sci. Nat. UJEP Brunensis, XII (1976), 107–116.

[3] J. Chvalina, Stackbases in power sets of neighbourhood spaces preserving the continuity of mappings, *Arch. Math. 2*, Scripta Fac. Sci. Nat. UJEP Brunensis, XVII (1981), 81–86.

[4] J. Dujunji, *Topology*, Allyn and Bacon, Boston, 1966.

[5] M. K. Gupta and Rupen Pratap Singh, On fuzzy pairwise s-continuous and fuzzy pairwise s-open mapping, *Int. J. Pure & Appl. Math. Sci.* 1 (2004), 101–109.

[6] K. C. Kelly, Bitopological spaces, *Pro. London Math. Soc.* 3(13) (1969), 71–79.

[7] T. Noiri and V. Popa, Some properties of weakly open functions in bitopological spaces, *Novi Sad J. Math.* 36 (1) (2006), 47–54.

[8] S. K. Sen and M. N. Mukherjee, On extension of pairwise θ-continuous maps, *Internat. J. Math. Sci.* 19 (1) (1966), 53–56.

[9] L. Skula, Systeme von stetigen abbildungen, *Czech. Math. J.*, 17 (92) (1967), 45–52.

[10] J. Šlapal, Closure operations for digital topology, *Theoret. Comput. Sci.*, 305 (2003), 457–471.

Author(s) Address(es):
Department of Mathematics, Faculty of Science, Mahasarakham University, Mahasarakham 44150, Thailand

E-mail Address: chawalit.b@msu.ac.th