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Applications of Mathematics, Vol. 53 (2008), No. 6, 521–533

Persistent URL: http://dml.cz/dmlcz/140337

LARGE TIME BEHAVIOR OF SOLUTIONS TO A CLASS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS*

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(Received July 21, 2006, in revised version September 5, 2007)

Abstract. We study the large time asymptotic behavior of solutions of the doubly degenerate parabolic equation $u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) - u^q$ with an initial condition $u(x,0) = u_0(x)$. Here the exponents m, p and q satisfy $m+p \geqslant 3$, $p > 1$ and $q > m+p-2$.

Keywords: degenerate parabolic equation, large time asymptotic behavior

MSC 2010: 35K55, 35K65, 35B40

1. Introduction

The objective of this article is to study the large time asymptotic behavior of weak solutions of nonlinear parabolic equations of the type

$$u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) - u^q \quad \text{in } S = \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{on } \mathbb{R}^N. \tag{1.2}$$

Here $p > 1$, $m(p-1) > 1$, $q > 1$, $N \geqslant 1$ and $u_0(x) \in L^1(\mathbb{R}^N)$ is a nonnegative function. Equation (1.1) has been suggested as a mathematical model for a variety of problems in mechanics, physics and biology, one can see [3], [5], [1] etc. The existence of a nonnegative solution of (1.1)–(1.2), defined in some weak sense, is well established (see [12] and [8]). In this paper we are interested in the behavior of solutions as $t \to \infty$. The elliptic method was used in several papers (see e.g. [4], [9]) to study the asymptotic behavior of the solutions of the porous media and the p-Laplacian equations. Also by the elliptic method, J. Manfredi and V. Vespri studied the large

* The paper was supported by NSF of China (10571144), NSF for youth of Fujian province in China (2005J037) and NSF of Jimei University in China.

time behavior of the solution of the initial boundary problem without absorption $-u^q$ in [7]. In details the large time behavior of the solution of the problem

$$u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) \quad \text{in } \Omega\times(0,\infty), \tag{1.3}$$

$$u(x,t) = 0 \quad \text{in } \partial\Omega\times(0,\infty), \tag{1.4}$$

$$u(x,0) = u_0(x) \quad \text{on } \mathbb{R}^N \tag{1.5}$$

was considered in [7].

In our paper we will study problem (1.1)–(1.2) in a way different from the elliptic method which is used in [7], namely, we will compare the large time behavior of the general solution of (1.1)–(1.2) to the Barenblatt-type solution of (1.1)–(1.2).

We begin with some preliminaries.

It is not difficult to verify that

$$E_c = t^{-l/\mu}\Big\{\Big[b - \frac{m(p-1)-1}{mp}(N\mu)^{-1/(p-1)}(|x|t^{-l/\mu})^{p/(p-1)}\Big]_+\Big\}^{(p-1)/(m(p-1)-1)}$$

is the Barenblatt-type solution of the Cauchy problem

$$u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) \quad \text{in } S=\mathbb{R}^N\times(0,\infty), \tag{1.6}$$

$$u(x,0) = c\delta(x) \quad \text{on } \mathbb{R}^N \tag{1.7}$$

where $l = (1+(m-1)/(p-1))^{1-p}$, $\mu = m+p-3+p/N$, $c = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x$, b is a constant such that $b = \int_{\mathbb{R}^N} E_c(x,t)\,\mathrm{d}x$, and δ denotes the Dirac mass centered at the origin.

Let

$$B_R(x_0) = \{x\in\mathbb{R}^N\colon\ |x-x_0|<R\},\quad B_R = \{x\in\mathbb{R}^N\colon\ |x|<R\}.$$

Definition 1.1. A nonnegative function $u(x,t)$ is called a solution of (1.1)–(1.2) if u satisfies

$$u\in C(0,T;L^1(\mathbb{R}^N))\cap L^\infty(\mathbb{R}^N\times(\tau,T)),\ \ u^{(m-1)/(p-1)}Du\in L^p_{\mathrm{loc}}(\mathbb{R}^N\times(0,T)), \tag{1.8}$$
$$u_t\in L^1(\mathbb{R}^N\times(\tau,T)),\quad \forall\,\tau>0;$$

$$\int_S [u(x,t)\varphi_t(x,t) - u^{m-1}|Du|^{p-2}Du\cdot D\varphi - u^q\varphi]\,\mathrm{d}x\,\mathrm{d}t = 0,\quad \forall\,\varphi\in C_0^1(S); \tag{1.9}$$

$$\lim_{t\to0} |u(x,t)-u_0(x)|\,\mathrm{d}x = 0. \tag{1.10}$$

Definition 1.2. A nonnegative function $U\in C(\overline{S}\setminus(0))$, $U\neq0$ is called a very singular solution of (1.1), if U satisfies (1.1) in the sense of distributions in S and

$$\lim_{t\to0}\int_{B^R} U(x,t)\,\mathrm{d}x = 0,\quad \forall\,R>0.$$

Let $U(x,t)=t^{1/(q-1)}f(|x|t^{-1/\beta})$. Suppose f is the solution of the ordinary equation

$$(f^{m-1}|f'|^{p-2}f')'+\frac{1}{\eta}f^{m-1}|f'|^{p-2}f'+\frac{1}{\beta}\eta f'+\frac{1}{q}f-f^q=0,$$
$$f(\eta)\geqslant 0,\quad f'(0)=0,\quad \lim_{\eta\to\infty}\eta^{p/(q-(m+p-2))}f(\eta)=0.$$

Then we can prove that $U(x,t)$ is a very singular solution of (1.1); we will publish this result in another paper.

Theorem 1.3. *Let $m(p-1)>1$, $q>m+p-2$. If E_c is a unique solution of* (1.6)–(1.7), *then the solution u of* (1.1)–(1.2) *satisfies*

$$t^{l/\mu}|u(x,t)-E_c(x,t)|\to 0\quad \text{as } t\to\infty \tag{1.11}$$

uniformly on the sets $\{x\in\mathbb{R}^N\colon\ |x|<at^{-l/\mu N},\ a>0\}$, where

$$c=\int_{\mathbb{R}^N}u_0(x)\,\mathrm{d}x-\int_0^\infty\int_{\mathbb{R}^N}u^q(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Theorem 1.4. *Suppose $m(p-1)>1$, $q>m+p-2$ and*

$$|x|^\alpha u_0(x)\leqslant B,\qquad \lim_{|x|\to\infty}|x|^\alpha u_0(x)=C,$$

where α, B and C are constants with $\alpha\in(0,p/(q-(m+p-2)))$. Then the solution of (1.1)–(1.2) *satisfies*

$$t^{1/(q-1)}u(x,t)\to C^*\quad \text{as } t\to\infty$$

uniformly on the sets

$$\{x\in\mathbb{R}^N\colon\ |x|\leqslant at^{1/\beta},\ a>0\},$$

where $C^*=(1/(q-1))^{1/(q-1)}$ and $\beta=(q-1)/(q-(m+p-2))$.

Theorem 1.5. *Suppose $1<m(p-1)$, $m+p-2<q<m+p-2+p/N$ and*

$$|x|^\alpha u_0(x)\leqslant B,\quad a>\frac{p}{q-(m+p-2)},\quad \int_{\mathbb{R}^N}u_0(x)\,\mathrm{d}x>0.$$

Assume that (1.1) *has a unique very singular solution. Then the solution of* (1.1)–(1.2) *satisfies*

$$t^{1/(q-1)}|u(x,t)-U(x,t)|\to 0\quad \text{as } t\to\infty$$

uniformly on the sets

$$\{x \in \mathbb{R}^N\colon\ |x| \leqslant a t^{1/\beta}\},$$

where $\beta = (q-1)/(q-(m+p-2))$.

R e m a r k 1.6. For $m = 1$, the uniqueness of solutions of (1.6)–(1.7) is known (see [2]). For $m = 1, p = 2$, the uniqueness of the very singular solution of (1.1) is known, too (see [11]).

2. Proof of Theorem 1.3

Let u be a solution of (1.1). We define the family of functions

$$u_k = k^N u(kx, k^{N\mu} t), \quad k > 0.$$

It is easy to see that they are solutions of the problems

$$u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) - k^{-v}u^q \quad \text{in } S = \mathbb{R}^N \times (0,\infty), \tag{2.1}$$

$$u(x,0) = u_{0k}(x) \quad \text{on } \mathbb{R}^N, \tag{2.2}$$

where $\mu = m+p-3+p/N$ as before and $v = q-m-p+2-p/N$, $u_{0k}(x) = k^N u_0(x)$.

Lemma 2.1. *For any* $s \in (0, m+p-2)$, u_k *satisfies*

$$\int_0^T\int_{B_R} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant c(s,R,|u_0|_{L^1}), \tag{2.3}$$

$$\int_0^T\int_{B_R} u_k^{m+p-2+p/N-s} \,\mathrm{d}x\,\mathrm{d}t \leqslant c(s,R,|u_0|_{L^1}). \tag{2.4}$$

P r o o f. From Definition 1.1, we are able to deduce (see [10]): $\forall\,\varphi \in C^1(\overline{S})$, $\varphi = 0$ when $|x|$ is large enough,

$$\begin{aligned}\int_{\mathbb{R}^N} u_k(x,t)\varphi \,\mathrm{d}x - \int_0^T\int_{\mathbb{R}^N} (u_k\varphi_t - u_k^{m-1}|Du_k|^{p-2}Du_k\cdot D\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \int_{\mathbb{R}^N} u_{0k}(x)\varphi(x,0)\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

Let

$$\psi_R \in C_0^\infty(B_{2R}), \quad 0 \leqslant \psi_R \leqslant 1, \quad \psi_R = 1 \text{ on } B_R, \quad |D\psi_R| \leqslant cR^{-1}. \tag{2.6}$$

By an approximate procedure we can choose $\varphi = (u_k^s/(1+u_k^s))\psi_R^p$ in (2.5); then

$$\begin{aligned}(2.7)\qquad &\int_{\mathbb{R}^N}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}\,\mathrm{d}z\psi_R^p(x)\,\mathrm{d}x \\ &\quad + s\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\psi_R^p(x)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant -p\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{s+m-1}}{1+u_k^s}|Du_k|^{p-2}\psi_R^{p-1}(x)Du_k\cdot D\psi_R\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + \int_{\mathbb{R}^N}\int_0^{u_k(x,h)} \frac{z^s}{1+z^s}\,\mathrm{d}z\psi_R^p(x)\,\mathrm{d}x,\end{aligned}$$

where $0 < h < t$. Notice that

$$\begin{aligned}(2.8)\qquad &\left|\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{s+m-1}}{1+u_k^s}|Du_k|^{p-2}\psi_R^{p-1}(x)Du_k\cdot D\psi_R\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\leqslant \int_h^t\int_{\mathbb{R}^N}\Bigg[\varepsilon\left(\frac{u_k^{(s+m-2)\cdot(p-1)/p}}{(1+u_k^s)^{2(p-1)/p}}|Du_k|^{p-1}\psi_R^{p-1}\right)^{p/(p-1)} \\ &\quad + c(\varepsilon)\left(\frac{u_k^{(s+m-1-(s+m-2))\cdot(p-1)/p}}{(1+u_k^s)^{1-2(p-1)/p}}|D\psi_R|\right)^p\Bigg]\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_h^t\int_{\mathbb{R}^N}\left[\varepsilon\left(\frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\psi_R^p + c(\varepsilon)\frac{u_k^{p+m-2}}{(1+u_k^s)^{2-p}}|D\psi_R|^p\right]\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

$$(2.9)\qquad \int_{\mathbb{R}^N}\int_0^{u_k(x,h)} \frac{z^s}{1+z^s}\,\mathrm{d}z\psi_R^p(x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} u(x,k^{N\mu}h)\,\mathrm{d}x,$$

hence by (2.7)–(2.9) we obtain

$$\begin{aligned}(2.10)\qquad &\sup_{0<t<T}\int_{\mathbb{R}^N}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}\,\mathrm{d}z\,\mathrm{d}x + \int_h^t\int_{\mathbb{R}^N} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\psi_R^p\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant c\int_{\mathbb{R}^N} u(x,k^{N\mu}h)\,\mathrm{d}x + c\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{p+s+m-2}}{(1+u_k^s)^{2-p}}|D\psi_R|^p\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Because $u_k \in L^\infty(\mathbb{R}^N\times(h,T))\cap L^1(S_T)$, $p+m-2>0$, we have

$$(2.11)\qquad \lim_{R\to\infty}\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{p+s+m-2}}{(1+u_k^s)^{2-p}}|D\psi_R|^p\,\mathrm{d}x\,\mathrm{d}\tau = 0.$$

Let $R\to\infty$, $h\to 0$ in (2.10). Then

$$\begin{aligned}(2.12)\qquad &\sup_{0<t<T}\int_{\mathbb{R}^N}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}\,\mathrm{d}z\,\mathrm{d}x + \iint_{S_t} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant c\int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x.\end{aligned}$$

Thus

$$\sup_{0<t<T}\int_{B_{2R}} u_k(x,t)\,\mathrm{d}x+\int_0^T\!\!\int_{B_{2R}}\frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}\tau\leqslant c(R). \tag{2.13}$$

Let

$$u_1=\max\{u_k(x,t),1\},\quad w=u_1^{(m+p-2-s)/p}.$$

By Sobolev's imbedding inequality (see [6]), for $\xi\in C_0^1(B_{2R})$, $\xi\geqslant 0$, we have

$$\begin{aligned}&\Big(\int_{\mathbb{R}^N}\xi^p w^r\,\mathrm{d}x\Big)^{1/r}\\&\quad\leqslant c\Big(\int_{\mathbb{R}^N}|D(\xi w)|^p\Big)^{s/p}\Big(\int_{B_{2R}} w^{p/(m+p-2-s)}\,\mathrm{d}x\Big)^{((1-\theta)(m+p-2)-s)/p},\end{aligned}$$

where

$$\begin{aligned}\theta&=\Big(\frac{m+p-2-s}{p}-\frac{1}{r}\Big)\Big(\frac{1}{N}-\frac{1}{p}+\frac{m+p-2-s}{p}\Big)^{-1},\\ r&=\frac{p(m+p-2+p/N-s)}{m+p-2-s}.\end{aligned}$$

It follows that

$$\begin{aligned}&\iint_{S_T}\xi^p w^r\,\mathrm{d}x\,\mathrm{d}t\\&\quad\leqslant c\iint_{S_T}|D(\xi w)|^p\,\mathrm{d}x\,\mathrm{d}t\\&\qquad\times\sup_{t\in(0,T)}\Big(\int_{B_{2R}} w^{p/(m+p-2-s)}\,\mathrm{d}x\Big)^{(r-p)(m+p-2-s)/p}.\end{aligned}\tag{2.14}$$

Since

$$|Dw|^p\leqslant c\,\frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\ \text{ a.e. on }\ \{u_k\geqslant 1\}\quad\text{and}\quad |Dw|=0\ \text{ on }\ \{u_k\leqslant 1\},$$

we have

$$\begin{aligned}\iint_{S_T}|D(\xi w)|^p\,\mathrm{d}x\,\mathrm{d}t&\leqslant c\iint_{S_T}(\xi^p|Dw|^p+w^p|D\xi|^p)\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant c\Big[\iint_{S_T}|D\xi|^p u_1^{p+m-2-s}\,\mathrm{d}x\,\mathrm{d}t\\&\qquad+\int_0^T\!\!\int_{B_{2R}}\frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}t\Big].\end{aligned}\tag{2.15}$$

Hence, by (2.14), (2.15) and (2.13), we get

$$\iint_{S_T} \xi^p u_1^{m+p-2+p/N-s}\,\mathrm{d}x\,\mathrm{d}t \leqslant c(s,R,|u_0|_{L^1})\Big(1+\iint_{S_T} |D\xi|^p u_1^{p+m-2-s}\,\mathrm{d}x\,\mathrm{d}t\Big).$$

Let $\xi=\psi_R^b$, where ψ_R is the function satisfying (2.6) and $b=N(m+p-2-s)/p$. Then

$$\iint_{S_T} \psi_R^{pb} u_1^{m+p-2+p/N-s}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant c(s,R,|u_0|_{L^1})\Big(1+\iint_{S_T} \psi_R^{pb} u_1^{p+m-2+p/N-s}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(m+p-2-s)/(m+p-2+p/N-s)},$$

which implies (2.4) is true. □

Let $Q_\varrho=B_\varrho(x_0)\times(t_0-\varrho^p,t_0)$ with $t_0>(2\varrho)^p$ and $u_{k1}=\max\{u_k,1\}$.

Lemma 2.2. *Each u_k satisfies*

$$\sup_{Q_\varrho} u_k \leqslant c(\varrho,s_1)\Big(\iint_{Q_{2\varrho}} u_{k1}^{p+m-3+s_1}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/s_1}, \tag{2.16}$$

where $c(\varrho,s_1)$ depends on ϱ and s_1, and s_1 can be any number satisfying $0<s_1<1+p/N$.

Lemma 2.3. *Each u_k satisfies*

$$\int_\tau^T\!\!\int_{B_R} u_k^{m-1}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}t\leqslant c(\tau,R),\qquad \int_\tau^T\!\!\int_{B_R}|u_{kt}|^p\,\mathrm{d}x\,\mathrm{d}t\leqslant c(\tau,R). \tag{2.17}$$

P r o o f. By Lemma 2.1 and 2.2, u_k are uniformly bounded on every compact set $K\subset S_T$. Let ψ_R be a function satisfying (2.6) and let $\xi\in C_0^1(0,T+1)$ with $0\leqslant\xi\leqslant1$, $\xi=1$ if $t\in(\tau,T)$. We choose $\eta=\psi_R^p\xi u_k$ in (2.5) to obtain

$$\begin{aligned}\frac12\int_{\mathbb{R}^N} u_k^2(x,T)\psi_R^p\,\mathrm{d}x+\iint_{S_T} u_k^{m-1}|Du_k|^p\psi_R^p\xi\,\mathrm{d}x\,\mathrm{d}t\\ \leqslant\frac12\iint_{S_T}u_k^2\xi'\psi_R^p\,\mathrm{d}x\,\mathrm{d}t-p\iint_{S_T}u_k^m|Du_k|^{p-2}Du_k\cdot D\psi_R\psi_R^{p-1}\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.18}$$

Notice that

$$\begin{aligned}&\iint_{S_T} u_k^m|Du_k|^{p-1}|D\psi_R|\psi_R^{p-1}\xi\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leqslant\varepsilon\iint_{S_T}u_k^{m-1}|Du_k|^p\psi_R^p\xi\,\mathrm{d}x\,\mathrm{d}t+c(\varepsilon)\iint_{S_T}u_k^{p+m-1}|D\psi_R|^p\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.19}$$

By (2.18), (2.19), one knows that the first inequality of (2.17) is true.

Now we will prove the second inequality of (2.17). Let

$$v(x,t) = u_{kr}(x,t) = ru_k(x, r^{m+p-3}t), \quad r \in (0,1).$$

Then

$$v_t(x,t) = \operatorname{div}(v^{m-1}|Dv|^{p-2}Dv) - r^{m+p-2-q}k^{-v}v^q, \tag{2.20}$$

$$v(x,0) = ru_k(x,0). \tag{2.21}$$

Notice that $r^{m+p-2-q}k^{-v} > k^{-v}$ using the argument similar to that in the proof of Theorem 1 of [12], one can prove

$$u_k \geqslant u_{kr}.$$

It follows that

$$\frac{u_k(x, r^{m+p-3}t) - u_k(x,t)}{(r^{m+p-3} - 1)t} \geqslant \frac{r-1}{(1 - r^{m+p-3})t} u_k(x, r^{m+p-3}t).$$

Letting $r \to 1$, we get

$$u_{kt} \geqslant -\frac{u_k}{(m+p-3)t}. \tag{2.22}$$

Denote $w = t^\beta u_k(x,t)$, $\beta = 1/(m+p-3)$. By (2.22), $w_t \geqslant 0$. By (2.1),

$$\begin{aligned} (2.23) \quad & \int_\tau^T \int_{B_{2R}} t^\beta w_t \psi_R \,\mathrm{d}x \,\mathrm{d}t \\ &= - \int_\tau^T \int_{B_{2R}} u_k^{m-1} |Du_k|^{p-2} Du_k \cdot D\psi_R \,\mathrm{d}x \,\mathrm{d}t \\ &\quad - \int_\tau^T \int_{B_{2R}} k^{-v} u_k^q \psi_R \,\mathrm{d}x \,\mathrm{d}t + \beta \int_\tau^T \int_{B_{2R}} t^{-1} u_k(x) \psi_R \,\mathrm{d}x \,\mathrm{d}t \\ &\leqslant \frac{\beta}{\tau} \int_\tau^T \int_{B_{2R}} u_k \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \left(\int_\tau^T \int_{B_{2R}} u_k^{m-1} |Du_k|^p \,\mathrm{d}x \,\mathrm{d}t \right)^{(p-1)/p} \left(\int_\tau^T \int_{B_{2R}} |D\psi_R|^p \,\mathrm{d}x \,\mathrm{d}t \right)^{1/p}. \end{aligned}$$

From (2.13), (2.16) and (2.23) we obtain (2.17). $\square$

P r o o f of Theorem 1.3.

By Lemmas 2.1–2.3 and [2], there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and a function v such that on every compact set $K \subset S$

$$u_{k_j} \to v \ \text{ in } \ C(K), \quad Du_{k_j}^m \rightharpoonup Dv^m \ \text{ in } \ L^p_{\rm loc}(S_T), \quad |u_{kt}|_{L^1_{\rm loc}(S_T)} \leqslant c.$$

Similar to what was done in the proof of Theorem 2 in [12], we can prove that v satisfies (1.1) in the sense of distributions.

We now prove $v(x,0) = c\delta(x)$. Let $\chi \in C_0^1(B_R)$. Then we have

$$\begin{aligned}(2.24)\qquad &\int_{\mathbb{R}^N} u_k(x,t)\chi \,\mathrm{d}x - \int_{\mathbb{R}^N} \varphi_k \chi \,\mathrm{d}x \\ &= -\int_0^t\int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k\cdot D\chi \,\mathrm{d}x\,\mathrm{d}s - k^{-v}\int_0^t\int_{\mathbb{R}^N} u_k^q\chi\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

To estimate $\int_0^t\int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k\cdot D\chi\,\mathrm{d}x\,\mathrm{d}s$, without losing generality, one can assume that $u_k > 0$. By Hölder inequality and Lemma 2.1,

$$\begin{aligned}(2.25)\qquad &\Big|\int_0^t\int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k\cdot D\chi\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &\leqslant c\Big(\int_0^T\int_{B_{2R}} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}t\Big)^{(p-1)/p} \\ &\quad\times\Big(\int_0^T\int_{B_{2R}} (1+u_k^s)^{2(p-1)}u_k^{(p-1)(2-s-m)}\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/p} \\ &\leqslant c\Big(\int_0^t\int_{B_{2R}} (u_{k1}^{(p-1)(2-s-m)} + u_{k1}^{(p-1)(2+s-m)}\,\mathrm{d}x\,\mathrm{d}\tau\big)^{1/p} \\ &\leqslant c\Big(\int_0^t\int_{B_{2R}} \big(u_{k1}^{(p-1)(2-s-m)}\big)^{\frac{m+p-2+p/N-s}{(p-1)(s+2-m)}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{(p-1)(s-2-m)}{m+p-2+p/N-s}\frac{1}{p}} t^d,\end{aligned}$$

where $s \in (0,1/N)$, $d = ((m-s-1)Np+(s-2)N+p-s+2)/((m+p-2)N+p-s) < 1$ because $p > (N+3)/(2N+1)$, $u_{k1} = \max(u_k,1)$.

Hence from (2.24) we get

$$\begin{aligned}(2.26)\qquad &\Big|\int_{\mathbb{R}^N} u_k(x,t)\chi\,\mathrm{d}x - \int_{\mathbb{R}^N}\varphi_k\chi\,\mathrm{d}x + k^{-v}\int_0^t\int_{\mathbb{R}^N} u_k^q\chi\,\mathrm{d}x\,\mathrm{d}s\Big| \\ &= \Big|\int_{\mathbb{R}^N} u_k(x,t)\chi\,\mathrm{d}x - \int_{\mathbb{R}^N}\varphi_k\chi(k^{-1}x)\,\mathrm{d}x + \int_0^{N\mu t}\int_{\mathbb{R}^N} u_k^q\chi(k^{-1}x)\,\mathrm{d}x\,\mathrm{d}\tau\Big| \leqslant ct^d.\end{aligned}$$

Letting now $k\to\infty$, $t\to 0$, we obtain

$$\lim_{t\to 0}\int_{\mathbb{R}^N} v(x,t)\chi\,\mathrm{d}x = \chi(0)\Big(\int_{\mathbb{R}^N}\varphi(x)\,\mathrm{d}x - \int_0^\infty\int_{\mathbb{R}^N} u^q\,\mathrm{d}x\,\mathrm{d}t\Big).$$

Thus

$$v(x,0)=c\delta(x), \quad c=\int_{\mathbb{R}^N}\varphi(x)\,\mathrm{d}x-\int_0^\infty\int_{\mathbb{R}^N}u^q\,\mathrm{d}x\,\mathrm{d}t,$$

$v(x,t)$ is a solution of (1.3)–(1.4). By the assumption on uniqueness of solution, we have $v(x,t)=E_c(x,t)$ and the whole sequence $\{u_k\}$ converges to E_c as $k\to\infty$. Set $t=1$. Then

$$u_k(x,1)=k^N u(kx,k^{N\mu})\to E_c(x,1)$$

uniformly on every compact subset of $\mathbb{R}^N$. Thus writing $kx=k'$, $k^{N\mu}=t'$, and dropping the prime again, we see that

$$t^{1/\mu}u(x,t)\to E_c(xt^{1/(N\mu)},1)=t^{1/\mu}E_c(x,t)$$

uniformly on the sets $\{x\in\mathbb{R}^N\colon\ |x|\leqslant at^{1/(N\mu)}\}$, $a>0$. Thus Theorem 1.3 is true.

□

3. Proofs of Theorem 1.4 and 1.5

Let u be a solution of (1.1)–(1.2) and let $u_k(x,t)=k^\delta u(kx,k^\beta t)$, $k>0$. If $\delta=1/(q-(m+p-2))$, $\beta=(q-1)/(q-(m+p-2))$, then

$$u_{kt}=\operatorname{div}(u_k^{m-1}|Du_k|^{p-2}Du_k)-u_k^q, \tag{3.1}$$

$$u_k(x,0)=\varphi_k(x)=k^\delta\varphi(kx). \tag{3.2}$$

Lemma 3.1. *The solution u_k of* (3.1)–(3.2) *satisfies*

$$u_k(x,t)\leqslant C^*t^{-1/(q-1)}, \quad C^*=\Big(\frac{1}{q-1}\Big)^{1/(q-1)}. \tag{3.3}$$

P r o o f. We consider the regularized problem of (3.1), namely,

$$u_{kt}=\operatorname{div}((u_k^{m-1}+\varepsilon)(|Du_k|^2+\varepsilon)^{(p-2)/2}Du_k)-u_k^q. \tag{3.4}$$

By the uniqueness of the solution of (3.1)–(3.2), we can prove that

$$u_{k\varepsilon}\to u_k \quad \text{as} \quad \varepsilon\to 0 \ \text{ in } \ C(K)$$

on every compact set $K\subset S$, where $u_{k\varepsilon}$ are the solutions of (3.4), (3.2). By computation, it is easy to show that $C^*(t-t_0)^{-1/(q-1)}$ is a solution of (3.4) in $\mathbb{R}^N\times(t_0,\infty)$, $t_0>0$. For any $\delta_1>0$, we choose $\delta_0\in(0,\delta_1)$ such that

$$|u_{k\varepsilon}(x,\delta_1)|_{L^\infty(\mathbb{R}^N)}\leqslant C^*(\delta_1-\delta_0)^{-1/(q-1)}.$$

Hence by the comparison principle, we have

$$u_{k\varepsilon}(x,t) \leqslant C^*(t-t_0)^{-1/(q-1)}, \quad t > \delta_1.$$

The proof of Lemma 3.1 is completed by letting $\delta_1 \to 0$ and $\varepsilon \to 0$. $\square$

Lemma 3.2. *Each u_k satisfies*

$$\int_\tau^T \int_{B_R} |Du_k|^p \leqslant c(\tau,R), \quad \int_\tau^T \int_{B_R} |u_{kt}| \, \mathrm{d}x \, \mathrm{d}t \leqslant c(\tau,R), \tag{3.5}$$

where $\tau \in (0,T)$.

Proof. The proof of Lemma 3.2 is similar to that of Lemma 2.3. $\square$

Proof of Theorem 1.4. By Lemma 3.1, $\{u_k\}$ are uniformly bounded on every compact set of S. Hence by [2], there exists a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that

$$u_{k_j} \to U \quad \text{in } C(K)$$

and

$$U(x,t) \leqslant C^* t^{-1/(q-1)}.$$

We now prove that $U(x,t) = C^* t^{-1/(q-1)}$. Let us introduce the function

$$\varphi_k^A = \min\{\varphi_k, A\} \tag{3.6}$$

and denote by $V_{K\varepsilon}^A$ the solution of (3.4) with initial value (3.6). By the comparison principle,

$$V_{K\varepsilon}^A \leqslant u_{k\varepsilon}, \tag{3.7}$$

where $u_{k\varepsilon}$ is the solution of (3.4), (3.2).

Define

$$V_A = C^*\Big(t + \frac{A^{1-q}}{q-1}\Big)^{-1/(q-1)},$$

which is the solution of (3.4) with initial value

$$V_A(x,0) = A. \tag{3.8}$$

Notice that

$$\lim_{k\to\infty} \varphi_k^A(x) = \lim_{k\to\infty} \min\Big\{A, \frac{\varphi(kx)|kx|^\alpha k^{\delta-\alpha}}{|x|^\alpha}\Big\} = A.$$

Using the uniqueness of solution of (3.4), (3.8), we can prove (see [6])

$$V_{k\varepsilon}^A \to V_A \quad \text{as } k \to \infty \text{ in } C(K),$$

where K is a compact set in S. Moreover, by [2] and [12]

$$V_{k\varepsilon}^A \to V_k^A u_{k\varepsilon} \to u_k \quad \text{as } k \to \infty \text{ in } C(K)$$

uniformly in K, where V_k^A is the solution of (1.1) with initial value (3.6). It follows that

$$V_k^A \to V_A \quad \text{as } k \to \infty \text{ in } C(K).$$

Letting $\varepsilon \to 0$ and $k \to \infty$ in turn in (3.7), we get

$$V_A(x,t) \leqslant V_\infty(x,t) = C^* t^{-1/(q-1)} \quad \text{in } S.$$

Since the lower bound holds for every $A > 0$, we conclude that

$$U(x,t) = V_\infty(x,t) = C^* t^{-1/(q-1)} \quad \text{in } S.$$

Thus

$$k^{p/(q-(m+p-2))} u(kx, k^\beta t) \to C^* t^{-1/(q-1)} \quad \text{as } k \to \infty.$$

Set $t = 1$. Then

$$k^{p/(q-(m+p-2))} u(kx, k^\beta) \to C^* \quad \text{as } k \to \infty$$

uniformly on every compact subset of $\mathbb{R}^N$. Therefore if we set $kx = x'$, $k^\beta = t'$, and omit the primes, we obtain

$$t^{1/(q-1)} u(x,t) \to C^* \quad \text{as } t \to \infty$$

uniformly on sets $\{x \in \mathbb{R}^N \colon |x| \leqslant \alpha t^{1/\beta}\}$ with $\alpha > 0$ for $t > 0$ and so Theorem 1.4 is proved. □

Proof of Theorem 1.5. By Lemma 3.1 and [2], there exist a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that

$$u_{k_j} \to U \quad \text{in } C(K). \tag{3.9}$$

By Lemma 3.2, we can prove that U satisfies (1.1) in the sense of distributions in a manner similar to Theorem 2 of [12]. □

Acknowledgement. The author most sincerely thanks his advisor, Professor Zhao Junning, for all his inspiring guidance, constructive discussions, and encouragement in preparation of this paper.

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