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EQUIVALENCE AND SYMMETRIES OF FIRST ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this article, the equivalence and symmetries of underdetermined differential equations and differential equations with deviations of the first order are considered with respect to the pseudogroup of transformations $\bar{x} = \varphi(x)$, $\overline{y} = \overline{y}(\bar{x}) = L(x)y(x)$. That means, the transformed unknown function $\overline{y}$ is obtained by means of the change of the independent variable and subsequent multiplication by a nonvanishing factor. Instead of the common direct calculations, we use some more advanced tools from differential geometry; however, the exposition is self-contained and only the most fundamental properties of differential forms are employed. We refer to analogous achievements in literature. In particular, the generalized higher symmetry problem involving a finite number of invariants of the kind $F^j = a_j y \prod |z_i|^{k_i^j} = a_j y|z_1|^{k_1^j} \ldots |z_m|^{k_m^j} = a_j(x)y|y(\xi_1)|^{k_1^j} \ldots |y(\xi_m)|^{k_m^j}$ is compared to similar results obtained by means of auxiliary functional equations.

Keywords: differential equations with deviations, equivalence of differential equations, symmetry of differential equation, differential invariants, moving frames

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1. Introduction

The most general pointwise transformations of the family of homogeneous linear differential equations with deviating arguments were investigated in [6], [10], [11], [12], [14], [16], [17], for example. They are given by the formula

$$\overline{y}(\varphi(x)) = L(x)y(x), \tag{1}$$

i.e., the transformations consist of a change of the independent variable and multiplication by a nonvanishing factor L. They coincide with the most general pointwise transformations of homogeneous linear differential equations of the n-th ($n \geqslant 2$) order without deviations, analyzed in more detail in the monograph [13]. Global transformations of the form (1) may serve for the investigation of oscillatory behaviour of solutions of certain classes of linear differential equations because each of global pointwise transformations preserves distribution of zeros of solutions of differential equations, see e.g., [6], [12], [13], [14].

However, transformations (1) can be applied to certain classes of nonlinear equations, as well. For instance, let us mention the family of all equations

$$y'(x) = \sum_{i=0}^{n} a_i(x) b_i(y(x)) \prod_{j=1}^{m} \delta_{i\,j}(y(\xi_j(x))), \quad \xi_0(x) = x,\ x \in \mathbf{j} \subseteq \mathbb{R}$$

($\prod \delta_j = \delta_1 \delta_2 \dots \delta_m$) derived in [18]. Here $b_i,\ \delta_{i\,j}$ are nontrivial solutions of Cauchy's functional equation $b(uv) = b(u)b(v)$ $(u, v \in \mathbb{R} - \{0\})$ with the general solutions continuous at a point $b(u) = 0$, $b(u) = |u|^c$, $b(u) = |u|^c$ sign u ($c \in \mathbb{R}$ being an arbitrary constant), see Aczél [1]. This result was obtained (without the regularity conditions) by the introduction of rather artificial functional equations assuming apriori the existence of differential equations of the form

$$\begin{aligned} L'(x) &= h(x, \varphi(x), L(x), L(\eta_1(x)), \dots, L(\eta_m(x))), \\ \varphi'(x) &= g(x, \varphi(x), L(x), L(\eta_1(x)), \dots, L(\eta_m(x))) \end{aligned}$$

for the functions L', φ' where the new deviations η_i satisfy $\xi_i(\varphi(x)) = \varphi(\eta_i(x)),\ x \in \mathbf{j} \subseteq \mathbb{R}$.

We shall see that certain results of this kind can be systematically obtained by employing a quite different and more natural method. In this paper we solve the symmetry and the equivalence problem (local approach) for the transformations (1) and formulate some results in terms of the global transformations by applying the moving frames (see also [19]) to the Monge equation $y' = f(x, y, z_1, \dots, z_m)$ with several unknowns $y, z_1, \dots, z_m$. The simple arrangement $z_i = y(\xi_i)$ then provides a large hierarchy of classes of differential equations of the first order with deviations ξ_i which admit the transformations (1).

For the most necessary information on technical tools used in this paper see e.g., [2], [3], [4], [5], [8], [9], [15].

For the convenience of the reader, let us recall some fundamental concepts with adaptations needed for the problem under consideration.

Let $\mathbf{M}$ be a topological space, Γ a family of homeomorphisms $\Phi\colon \mathscr{D}(\Phi) \to \mathscr{R}(\Phi)$ where $\mathscr{D}(\Phi), \mathscr{R}(\Phi) \subset \mathbf{M}$ are open subsets. We speak of a *pseudogroup* Γ of transformations if

(ι) the identity $\mathrm{id}\colon \mathscr{D}(\mathrm{id}) = \mathbf{M} \to \mathscr{R}(\mathrm{id}) = \mathbf{M}$ belongs to Γ;

($\iota\iota$) if $\Phi \in \Gamma$ and $\mathscr{D} \subset \mathbf{M}$ is an open subspace then the restriction of Φ to the subspace $\mathscr{D}(\Phi) \cap \mathscr{D}$ belongs to Γ;

($\iota\iota\iota$) if $\Phi \in \Gamma$ then $\Phi^{-1} \in \Gamma$;

($\iota\nu$) if $\Phi, \Psi \in \Gamma$ and $\mathscr{R}(\Phi) \cap \mathscr{D}(\Psi) \neq 0$ then the composition

$$\Psi \circ \Phi\colon\ \Phi^{-1}(\mathscr{R}(\Phi) \cap \mathscr{D}(\Psi)) \to \Psi(\mathscr{R}(\Phi) \cap \mathscr{D}(\Psi))$$

belongs to Γ;

(ν) if $\mathscr{D}, \mathscr{R}$ are open subsets and $\chi\colon \mathscr{D} \to \mathscr{R}$ a homeomorphism such that χ *locally coincides* with the mappings from Γ, then $\chi \in \Gamma$. (We suppose that for every $P \in \mathscr{D}$ there exists $\Phi \in \Gamma$ such that $\mathscr{D}(\Phi)$ is a neighbourhood of P and $\chi = \Phi$ on $\mathscr{D}(\Phi)$.)

Assume moreover that $\mathbf{M}$ is a manifold and Γ includes all smooth transformations $\Phi\colon \mathscr{D}(\Phi) \to \mathscr{R}(\Phi)$ that preserve a given family of functions f (*invariants* of Γ) and differential 1-forms ω (*Maurer-Cartan forms* of Γ), that is, $\Phi^* f = f$ and $\Phi^*\omega = \omega$ for all f and ω under consideration. Then we speak of a *Lie-Cartan pseudogroup* of transformations.

We shall deal with rather particular transformations $\Phi\colon \mathscr{D}(\Phi) \to \mathscr{R}(\Phi)$ defined by

$$\Phi(x, y, z_1, \ldots, z_m) = (\varphi(x), L(x)y, L_1(x)z_1, \ldots, L_m(x)z_m) \quad (m \geqslant 1) \tag{2}$$

where $\varphi\colon \mathscr{D}(\varphi) \to \mathscr{R}(\varphi)$ are invertible diffeomorphisms between the open subsets $\mathscr{D}(\varphi), \mathscr{R}(\varphi) \subset \mathbb{R}$, the functions $L(x), L_1(x), \ldots, L_m(x)$ are smooth and nonvanishing on $\mathscr{D}(\varphi)$, therefore $\mathscr{D}(\Phi) = \mathscr{D}(\varphi) \times \mathbb{R}^{m+1}$, $\mathscr{R}(\Phi) = \mathscr{R}(\varphi) \times \mathbb{R}^{m+1} \subset \mathbb{R}^{m+2}$. Later in Section 4 we will prove that these transformations Φ constitute a certain Lie-Cartan pseudogroup Γ.

Transformations Φ will be applied to curves $y = y(x), z_i = z_i(x)$ $(i = 1, \ldots, m)$ in such a manner that each curve is transformed into another one of the form

$$\overline{y} = \overline{y}(\bar{x}) = L(x)y(x),\ \overline{z}_i = \overline{z}_i(\bar{x}) = L_i(x)z_i(x) \quad (\bar{x} = \varphi(x),\ i = 1, \ldots, m).$$

Our first task is the (local) equivalence problem: to determine when a given *under-determined differential equation*

$$y'(x) = f(x, y(x), z_1(x), \dots, z_m(x)) \quad ({}' = \mathrm{d}/\mathrm{d}x) \tag{3}$$

is transformed into another equation

$$\overline{y}'(\bar{x}) = f(\bar{x}, \overline{y}(\bar{x}), \overline{z}_1(\bar{x}), \dots, \overline{z}_m(\bar{x})), \quad ({}' = \mathrm{d}/\mathrm{d}\bar{x}). \tag{4}$$

Then the results will be applied to obtain *global results* for the equivalence problem of the *differential equation with deviations*

$$y'(x) = f(x, y(x), y(\xi_1(x)), \dots, y(\xi_m(x))) \tag{5}$$

by inserting $y(\xi_i(x)) = z_i(x)$ where $\xi_i\colon \mathscr{D}_i \to \mathscr{R}_i$ are given diffeomorphisms of open subsets $\mathscr{D}_i, \mathscr{R}_i \subset \mathbb{R}$. Then the corresponding equation

$$\overline{y}'(\bar{x}) = f(\bar{x}, \overline{y}(\bar{x}), \overline{z}_1(\bar{x}), \dots, \overline{z}_m(\bar{x}))$$

can be again interpreted as a differential equation with deviations if $\overline{z}_i(\bar{x}) = \overline{y}(\bar{\xi}_i(\bar{x}))$ for certain new diffeomorphisms $\bar{\xi}_i\colon \overline{\mathscr{D}}_i \to \overline{\mathscr{R}}_i$. However, we restrict ourselves only to such functions φ that satisfy the commutativity requirements

$$\bar{\xi}_i(\varphi(x)) = \varphi(\xi_i(x)) \quad (i = 1, \dots, m) \tag{6}$$

whenever the right-hand sides are defined. (In more detail, if $x \in \mathscr{D}(\varphi) \cap \mathscr{D}$ for certain i, then $\bar{\xi}_i(\bar{x})$ is determined at the point $\bar{x} = \varphi(x) \in \mathscr{R}(\varphi)$.)

3. Preliminary results

To make the paper shorter, we give auxiliary results without (easy direct) proofs.

Observation 1. *Let $c = c(x, z_1, \dots, z_m), z_i \neq 0$, be a nonzero real function with continuous partial derivatives $c_{z_i} = \partial c/\partial z_i$ satisfying the conditions $z_i c_{z_i}/c = r_i \in \mathbb{R}$, $i = 1, 2, \dots, m$. Then $c(x, z_1, \dots, z_m) = q(x)|z_1|^{r_1} \dots |z_m|^{r_m} = q(x) \prod |z_i|^{r_i}$ where $q(x)$ is an arbitrary nonzero function.*

Observation 2. *Let $c = c(x, z_1, \dots, z_m) = q(x) \prod |z_i|^{r_i}$, $z_i \neq 0$, $r_i \in \mathbb{R}$, $\sum r_i^2 \neq 0$ $(i = 1, \dots, m)$ and let $b = b(x, z_1, \dots, z_m)$ be a nonzero real function with continuous partial derivatives b_{z_i} satisfying the conditions $z_i b_{z_i} = s_i c$, $s_i \in \mathbb{R} - \{0\}$, $i = 1, 2, \dots, m$. Then there exists $k \in \mathbb{R} - \{0\}$ such that $b(x, z_1, \dots, z_m) = k \cdot c(x, z_1, \dots, z_m) + p(x)$ and $s_i = k \cdot r_i$ where $p(x)$ is an arbitrary nonzero function.*

Observation 3. *Let* $b = b(x, z_1, \dots, z_m)$*,* $z_i \neq 0$*, be a nonzero real function with continuous partial derivatives* b_{z_i} *satisfying the conditions* $z_i b_{z_i} = s_i q(x)$*,* $s_i \in \mathbb{R}$*,* $i = 1, 2, \dots, m$*. Then* $b(x, z_1, \dots, z_m) = q(x) \cdot \sum \ln |z_i|^{s_i} + p(x)$ *where* $p(x)$ *is an arbitrary nonzero function,* $i = 1, \dots, m$*.*

Observation 4. *Let* $c = c(x, z_1, \dots, z_m)$*,* $z_i \neq 0$*, be a nonzero real function with continuous partial derivatives* c_{z_i} *satisfying the conditions* $z_i c_{z_i} = q(x) \cdot I^i(F)$*,* $F = a(x)|z_i|^{k_1} \dots |z_m|^{k_m} = a(x) \prod |z_i|^{k_i} \neq 0$*,* $k_i \in \mathbb{R}$*, for appropriate nonzero functions* $a(x)$*,* $q(x)$ *and certain differentiable functions* $I^i(F)$*,* $i = 1, \dots, m$*. Then there exists a function* $I(F)$ *such that* $I^i(F) = k_i I(F)$ *for* i *with* $k_i \neq 0$ *and* $c = q(x)(\hat{I}(F) + \sum_{j \in \mathscr{J}} I^j(F) \ln |z_j|) + p(x)$*,* $\hat{I}(F) = \int (I(F)/F)\, \mathrm{d}F$ *where* $p(x)$ *is an arbitrary function and* $j \in \mathscr{J}$ *if* $k_j = 0$*.*

Observation 5. *Let* $c = c(x, z_1, \dots, z_m)$*,* $z_i \neq 0$*, be a nonzero real function with continuous partial derivatives* c_{z_i} *satisfying the conditions* $z_i c_{z_i}/c = I^i(F)$*,* $F = a(x)|z_i|^{k_1} \dots |z_m|^{k_m} = a(x) \prod |z_i|^{k_i}$*,* $k_i \in \mathbb{R}$*, for an appropriate nonzero function* $a(x)$ *and certain differentiable functions* $I^i(F)$*,* $i = 1, \dots, m$*. Then there exists* $I(F)$ *such that* $I^i(F) = k_i I(F)$ *is satisfied for* i *with* $k_i \neq 0$ *and* $c = p(x) \mathrm{e}^{\hat{I}(F)} \prod_{j \in \mathscr{J}} |z_j|^{I^j(F)}$*,* $\hat{I}(F) = \int (I(F)/F)\, \mathrm{d}F$ *where* $p(x)$ *is an arbitrary nonzero function and* $j \in \mathscr{J}$ *if* $k_j = 0$*.*

4. Main results—an underdetermined equation

Let us follow the idea of Section 2 (see also [19]) and analyze the equivalence problem for underdetermined differential equations

$$y' = f(x, y, z_1, \dots, z_m), \quad \overline{y}' = \overline{f}(\bar{x}, \overline{y}, \overline{z}_1, \dots, \overline{z}_m) \tag{7}$$

with respect to the pseudogroup of (invertible, local) transformations

$$\bar{x} = \varphi(x), \quad \overline{y} = L(x)y, \quad \overline{z}_i = L_i(x) z_i$$
$$(\varphi'(x) L(x) L_1(x) \dots L_m(x) \neq 0, \ i = 1, \dots, m).$$

In accordance with the applications to (5), we suppose $(f/y)_y \neq 0$ from now on.

Lemma 1. *The forms* $\omega = A\,\mathrm{d}x$ $(A \neq 0)$, $\omega_0 = \mathrm{d}y/y - B\,\mathrm{d}x$, $\omega_i = \mathrm{d}z_i/z_i - B^i\,\mathrm{d}x$ $(i = 1, \dots, m)$ *with new variables* A, B, $B^1, \dots, B^m$ *are the invariant forms for the transformations* (2).

P r o o f. The equality $\omega = \overline{\omega}(A\,\mathrm{d}x = \bar{A}\,\mathrm{d}\bar{x})$ means that $\bar{x} = \varphi(x)$ is a certain function of x and, moreover, $A = \bar{A}\varphi' \neq 0$ which ensures the invertibility of φ. Analogously $\omega_0 = \overline{\omega}_0$ reads $\mathrm{d}\ln\overline{y}/y = (\overline{B} - B)\,\mathrm{d}x$, i.e., $\ln\overline{y}/y$ is a function of x which means that $\overline{y} = L(x)y$. The remaining relations $\overline{z}_i = L_i(x)z_i$ follow similarly from $\omega_i = \overline{\omega}_i$, $i = 1, \dots, m$. □

Remark 1. The result can be interpreted in two ways. Either ω, $\omega_1, \dots, \omega_m$ can be regarded for differential forms *depending on parameters* A, B, $B^1, \dots, B^m$ in the space $\mathbf{M} = \mathbb{R}^{m+2}$ with coordinates x, y, $z_1, \dots, z_m$ and we deal with the local transformations $\Phi\colon \mathscr{D}(\Phi) \to \mathscr{R}(\Phi)$ in the space $\mathbf{M}$. The invariants (absent here) and the Maurer-Cartan forms may depend on additional parameters (which can be changed by applying the transformation Φ). This is, however, a little unorthodox conception. A better approach is as follows. Together with the previous coordinates x, y, $z_1, \dots, z_m$, we introduce *additional coordinates* A, B, $B^1, \dots, B^m$ (the parameters appearing in forms $\omega, \omega_1, \dots, \omega_m$) and introduce the extended space $\overline{\mathbf{M}} = \mathbf{M} \times \mathbb{R}^{m+2} = \mathbb{R}^{2m+4}$ with the coordinates

$$x, y, z_1, \dots, z_m, A, B, B^1, \dots, B^m.$$

Then the original transformation Φ on $\mathbf{M}$ induces a certain extension $\overline{\Phi}\colon \mathscr{D}(\Phi) \times \mathbb{R}^{m+2} \to \mathscr{R}(\Phi) \times \mathbb{R}^{m+2}$ on the open subsets of $\overline{\mathbf{M}}$ and we may introduce the pseudogroup $\overline{\Gamma}$ of these mappings $\overline{\Phi}$. The extendend pseudogroup has the *true* Maurer-Cartan forms $\omega, \omega_1, \dots, \omega_m$ (depending only on *coordinates* of the underlying space $\overline{\mathbf{M}}$). Both the conceptions are (in principle) quite reasonable, however, they provide a slightly different interpretation of the subsequent calculations.

Lemma 2. *The equivalence problem is alternatively expressed by the invariance requirements*

$$\begin{aligned} \omega &= \overline{\omega} \quad (\omega = A\,\mathrm{d}x,\ A \neq 0), \\ \omega_0 &= \overline{\omega}_0 \quad (\omega_0 = \mathrm{d}y/y - B\,\mathrm{d}x,\ B = f/y), \\ \omega_i &= \overline{\omega}_i \quad (\omega_i = \mathrm{d}z_i/z_i - B^i\,\mathrm{d}x,\ i = 1, \dots, m) \end{aligned}$$

for the transformation (2).

P r o o f. The equation (3) can be represented by the Pfaffian equation $\eta_1 = \mathrm{d}y - f\,\mathrm{d}x = 0$. Equivalence of the equations (3), (4) takes place if and only if this form $\eta_1 = \mathrm{d}y - f\,\mathrm{d}x$ is transformed into a multiple of $\overline{\eta}_1 = \mathrm{d}\overline{y} - \overline{f}\,\mathrm{d}\bar{x}$. In more symmetric terms: the form $C\eta_1$ (where $C \neq 0$ is a new variable) should be transformed into the form $\overline{C}\overline{\eta}_1$ (with an appropriate $\overline{C}$ depending on C) or, briefly saying, *the form $C\eta_1$ is preserved.* However,

$$C\eta_1 = Cy\omega_0 + \frac{C}{A}(By - f)\omega = \overline{C}\overline{\eta}_1 = \overline{C}\overline{y}\overline{\omega}_0 + \frac{\overline{C}}{\overline{A}}(\overline{B}\overline{y} - \overline{f})\overline{\omega}$$

in terms of the Maurer-Cartan forms and it follows that necessarily

$$Cy = \overline{C}\overline{y}, \quad \frac{C}{A}(By - f) = \frac{\overline{C}}{\overline{A}}(\overline{B}\overline{y} - \overline{f}).$$

In particular, if the uncertain parameters C, B are specified to satisfy $Cy = 1$ and $By - f = 0$, we obtain the *invariant form*

$$\frac{1}{y}(\mathrm{d}y - f\,\mathrm{d}x) = \omega_0 = \frac{1}{\overline{y}}(\mathrm{d}\overline{y} - \overline{f}\,\mathrm{d}\bar{x}) = \overline{\omega}_0$$

where $C = 1/y$, $\overline{C} = 1/\overline{y}$, $B = f/y$, $\overline{B} = \overline{f}/\overline{y}$ is substituted into ω_0, $\overline{\omega}_0$. □

Remark 2. Our next strategy is as follows. The exterior derivatives of the invariant forms ω, ω_0, ω_i are also invariant forms: $\mathrm{d}\omega = \mathrm{d}\overline{\omega}$, $\mathrm{d}\omega_0 = \mathrm{d}\overline{\omega}_0$, $\mathrm{d}\omega_i = \mathrm{d}\overline{\omega}_i$. They can be expressed in terms of the original forms ω, ω_0, ω_i and then the coefficients (clearly) are invariant functions, see 4.1 below. Moreover, with these invariants available, the other invariant functions may be derived by using the covariant derivatives. The differential of any function $F = F(x, y, z_1, \ldots, z_m)$ admits the unique development

$$\mathrm{d}F = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + \sum F_{z_i}\,\mathrm{d}z_i = \frac{\partial F}{\partial \omega}\omega + \frac{\partial F}{\partial \omega_0}\omega_0 + \sum \frac{\partial F}{\partial \omega_i}\omega_i \tag{8}$$

where we have introduced the *covariant derivatives*

$$\frac{\partial F}{\partial \omega} = \frac{1}{A}\Big(F_x + yF_yB + \sum z_iF_{z_i}B^i\Big), \ \frac{\partial F}{\partial \omega_0} = yF_y, \ \frac{\partial F}{\partial \omega_i} = z_iF_{z_i} \ (i = 1\ldots, m). \tag{9}$$

If F is an invariant (i.e., $F = \overline{F}$ is preserved in the equivalences) then all the covariant derivatives of F are invariants, too.

4.1. The exterior derivatives.

Let us start with the formula

$$d\omega_0 = \omega \wedge \Big(\frac{yB_y}{A}\omega_0 + \sum \frac{z_iB_{z_i}}{A}\omega_i\Big)$$

where $yB_y = y\,(f/y)_y \neq 0$. Then

$$\mathrm{d}\omega_0 = \omega \wedge \Big(\omega_0 + \sum I^i\omega_i\Big),\quad I^i = \frac{z_i B_{z_i}}{yB_y},\quad i = 1,\ldots,m \tag{10}$$

by introducing the interrelation $yB_y/A = 1$ between the uncertain parameters A, B (and analogously for their conterparts $\bar{A}$, $\overline{B}$), hence by choosing

$$A = yB_y. \tag{11}$$

The equation $\mathrm{d}\omega_0 = \mathrm{d}\overline{\omega}_0$ implies the invariance of the corresponding coefficients $I^i = \overline{I}^i$ $(i = 1,\ldots,m)$ and the *functions* I^i *can be realized as invariants* (of the differential equation (6)). In a similar way,

$$\mathrm{d}\omega = \Big(J\omega_0 + \sum J^i\omega_i\Big) \wedge \omega,\quad J = \frac{(yB_y)_y}{B_y},\quad J^i = z_i\frac{B_{yz_i}}{B_y},\quad i = 1,\ldots,m \tag{12}$$

and J, J^i *are invariants*, too. Unlike $B = f/y$, the parameters B^i are not specified and remain independent variables. Hence

$$\mathrm{d}\omega_i = \mathrm{d}x \wedge \mathrm{d}B^i,\quad i \in \{1,\ldots,m\}. \tag{13}$$

The identity

$$\mathrm{d}^2\omega_0 = \Big(J\omega_0 \wedge \omega + \sum J^i\omega_i \wedge \omega\Big) \wedge \Big(\omega_0 + \sum I^j\omega_j\Big) + \Big(\sum \mathrm{d}I^i \wedge \omega_i\Big) \wedge \omega = 0$$

yields the important *Bianchi identities*

$$\frac{\partial I^i}{\partial \omega_0} = J^i - JI^i,\quad \frac{\partial I^i}{\partial \omega_j} - \frac{\partial I^j}{\partial \omega_i} = J^iI^j - I^iJ^j, \tag{14}$$

$i, j \in \{1,\ldots,m\}, i \neq j$, if the developments (8) of the differentials $\mathrm{d}I^i$ are inserted. Analogously,

$$\mathrm{d}^2\omega = \Big(\mathrm{d}J \wedge \omega_0 + \sum \mathrm{d}J^i \wedge \omega_i\Big) \wedge \omega = 0$$

is equivalent to the identities

$$\frac{\partial J}{\partial \omega_i} = \frac{\partial J^i}{\partial \omega_0},\quad \frac{\partial J^i}{\partial \omega_j} = \frac{\partial J^j}{\partial \omega_i},\quad i, j \in \{1,\ldots,m\}. \tag{15}$$

Remark 3. In accordance with Lemma 2 and (11), the equivalence problem for differential equations (3), (4) is expressed by the invariance requirements

$$\omega = (f_y - f/y)\,\mathrm{d}x = (\overline{f}_{\overline{y}} - \overline{f}/\overline{y})\,\mathrm{d}\bar{x} = \overline{\omega}$$

and also

$$\omega_0 = \frac{1}{y}(\mathrm{d}y - f\,\mathrm{d}x) = \frac{1}{\overline{y}}(\mathrm{d}\overline{y} - \overline{f}\,\mathrm{d}\bar{x}) = \overline{\omega}_0$$

and

$$\omega_i = \mathrm{d}z_i/z_i - B^i\,\mathrm{d}x = \mathrm{d}\overline{z}_i/\overline{z}_i - \overline{B}^i\,\mathrm{d}\bar{x} = \overline{\omega}_i, \quad i = 1, \ldots, m,$$

respectively. Inserting the transformation relations (2) we obtain equivalent conditions

$$f_y - \frac{1}{y}f = \overline{f}_{\overline{y}}\varphi' - \frac{1}{Ly}\overline{f}\varphi' \quad (yB_y = \overline{y}\overline{B}_{\overline{y}}\varphi') \tag{16}$$

and also

$$L'y = \overline{f}\varphi' - fL \quad (L' = (\overline{B}\varphi' - B)L) \tag{17}$$

and

$$L'_i = (\overline{B}^i\varphi' - B^i)L_i, \quad i = 1, \ldots, m, \tag{18}$$

respectively. Moreover, we have the invariants (10)–(12) satisfying $F = \overline{F}$ for $F \in \{I^i, J, J^i\}$ and many other invariants arising by the repeated covariant derivatives (9). One can observe that invariance of the forms ω_i with the variables B^i provides the prolongation transformation for the derivatives $\mathrm{d}z_i/\,\mathrm{d}x$ and that equations (7) are independent of these derivatives.

4.2. Constant invariants.

Theorem 1. *Let all invariants be constant. Then:*
(ι) *For* $J \neq 0$,

$$B = \frac{1}{J}q(x)|y|^J \prod |z_i|^{JI^i} + b(x) \; (= f/y)$$

($b(x)$ *is an arbitrary function*) *and the symmetry equivalence problem* (*relations* (16)–(18)) *is expressed by* $f = yB$ *and by the invariance requirements*

$$q = \bar{q}(\varphi)\varphi'|L|^J \prod |L_j|^{JI^j}, \quad L'/L = \bar{b}(\varphi)\varphi' - b, \quad L'_i/L_i = \overline{B}^i\varphi' - B^i.$$

($\iota\iota$) *For* $J = 0$,

$$B = q(x)\left(\ln|y| + \sum I^j \ln|z_j|\right) + p(x)$$

($p(x)$ *is an arbitrary function*) *and the symmetry equivalence problem* (*relations* (16)–(18)) *is expressed by* $f = yB$ *and by the invariance requirements*

$$q = \bar{q}(\varphi)\varphi', \quad \frac{L'}{L} - q(x)\Bigl(\ln|L| + \sum I^j \ln|L_j|\Bigr) = \bar{p}(\varphi)\varphi' - p,$$
$$L_i'/L_i = \overline{B}^i\varphi' - B^i.$$

(Parameters B^i are not specified here.)

P r o o f. Let all the invariants

$$J = \frac{(yB_y)_y}{B_y}, \ J^i = z_i\frac{B_{yz_i}}{B_y}, \ I^i = \frac{z_iB_{z_i}}{yB_y}, \ i = 1, \dots, m$$

be constant. In such a case

$$J^i = JI^i, \quad i = 1, \dots, m \tag{19}$$

follows from (14) and we do not have any other invariants. We get

$$yB_y = C(x, z_1, \dots, z_m)|y|^J \tag{20}$$

through integration of $(yB_y)_y/B_y = J \in \mathbb{R}$. Substituting (20) into $z_i(yB_y)_{z_i}/(yB_y) = J^i \in \mathbb{R}$ we obtain the conditions $z_iC_{z_i} = J^iC$, hence $C = q(x)\prod |z_i|^{J^i}$ follows from Observation 1. Altogether,

$$B_y = q(x)\frac{1}{y}|y|^J \prod |z_i|^{JI^i} \tag{21}$$

and there are two different subcases for $J \neq 0$ and $J = 0$.

The equivalence of equations $y' = f$, $\overline{y}' = \overline{f}$ is possible if and only if $J = \overline{J}$, $I^i = \overline{I}^i$ are the same constants ($i \in \{1, \dots, m\}$). It is determined by the completely integrable system

$$\omega = \overline{\omega}, \quad \omega_0 = \overline{\omega}_0, \quad \omega_i = \overline{\omega}_i \quad (i = 1, \dots, m) \tag{22}$$

where $\omega = yB_y\,\mathrm{d}x$, $\omega_0 = \mathrm{d}y/y - B\,\mathrm{d}x$, $\omega_i = \mathrm{d}z_i/z_i - B^i\,\mathrm{d}x$ and we have the corresponding structural formulae

$$\begin{aligned} \mathrm{d}\omega &= \Bigl(J\omega_0 + \sum J^j\omega_j\Bigr) \wedge \omega = J\Bigl(\omega_0 + \sum I^j\omega_j\Bigr) \wedge \omega, \\ \mathrm{d}\omega_0 &= \omega \wedge \Bigl(\omega_0 + \sum I^j\omega_j\Bigr), \\ \mathrm{d}\omega_i &= \mathrm{d}x \wedge \mathrm{d}B^i. \end{aligned} \tag{23}$$

in accordance with (10), (12), (13), (19).

(ι) *The subcase* $J \neq 0$. The following identity

$$B = \frac{1}{J} q(x) |y|^J \prod |z_i|^{JI^i} + b(x, z_1, \dots, z_m)$$

holds true by virtue of (21). Conditions $I^i = z_i B_{z_i}/(yB_y) = J^i/J + (z_i b_{z_i}/q(x)) \times \left(|y|^J \prod |z_j|^{I^j}\right)^{-1}$ are equivalent to $z_i b_{z_i} = (I^i - J^i/J) q(x) |y|^J \prod |z_j|^{I^j} = 0$ due to (19). Thus $b = b(x)$ and

$$B = \frac{1}{J} q(x) |y|^J \prod |z_i|^{JI^i} + b(x) \ (= f/y). \tag{24}$$

The forms ω, ω_0 are determined through (24) and we are passing to the structural formulae (23). Let us introduce the form $\eta_0 = J\omega_0 + \omega$. Then $\mathrm{d}\eta_0 = 0$, therefore η_0 is a total differential and the system (22) can be replaced by the simpler system

$$\begin{aligned} \omega &= \overline{\omega}, & \omega &= q(x) |y|^J \prod |z_j|^{JI^j} \, \mathrm{d}x, \\ \eta_0 &= \overline{\eta}_0, & \eta_0 &= J(\mathrm{d}y/y - b(x)\,\mathrm{d}x), \\ \omega_i &= \overline{\omega}_i, & \omega_i &= \mathrm{d}z_i/z_i - B^i\,\mathrm{d}x. \end{aligned} \tag{25}$$

We obtain the transformation relations

$$\begin{aligned} \omega = \overline{\omega} &\iff q = \bar{q}(\varphi)\varphi' |L|^J \prod |L_j|^{JI^j}, \\ \eta_0 = \overline{\eta}_0 &\iff L'/L = \bar{b}(\varphi)\varphi' - b, \\ \omega_i = \overline{\omega}_i &\iff L_i'/L_i = \overline{B}^i \varphi' - B^i \end{aligned} \tag{26}$$

for the differential equations $y' = f$, $\overline{y}' = \overline{f}$ by inserting the transformation relations $\bar{x} = \varphi(x), \overline{y} = L(x)y, \overline{z}_i = L_i(x) z_i$ into (25), $i = 1, \dots, m$.

($\iota\iota$) *The subcase* $J = 0$. In this subcase $J^i = 0$ $(i = 1, \dots, m)$ and $yB_y = q(x)$ (see (19), (21)). Thus

$$B = q(x) \ln |y| + b(x, z_1, \dots, z_m)$$

and we can write $I^i = z_i B_{z_i}/(yB_y) = z_i b_{z_i}/q(x)$ and $b = q(x) \ln \prod |z_j|^{I^j} + p(x) = q(x) \sum I^j \ln |z_j| + p(x)$ in accordance with Observation 3. The resulting function

$$B = q(x) \left(\ln |y| + \sum I^j \ln |z_j| \right) + p(x) \ (= f/y) \tag{27}$$

follows easily. The structural formulae (23) become

$$\begin{aligned} \mathrm{d}\omega &= 0, \\ \mathrm{d}\omega_0 &= \omega \wedge \left(\omega_0 + \sum I^j \omega_j \right), \\ \mathrm{d}\omega_i &= \mathrm{d}x \wedge \mathrm{d}B^i \end{aligned} \tag{28}$$

in this particular subcase and the system (22) is of the form

$$
\begin{aligned}
(29)\qquad \omega &= \overline{\omega}, & \omega &= q(x)\,\mathrm{d}x,\\
\omega_0 &= \overline{\omega}_0, & \omega_0 &= \mathrm{d}y/y - \Big(q(x)\Big(\ln|y| + \sum I^j \ln|z_j|\Big) + p(x)\Big)\,\mathrm{d}x,\\
\omega_i &= \overline{\omega}_i, & \omega_i &= \mathrm{d}z_i/z_i - B^i\,\mathrm{d}x.
\end{aligned}
$$

We obtain the transformation relations

$$
\begin{aligned}
(30)\qquad \omega = \overline{\omega} &\Longleftrightarrow q = \bar{q}(\varphi)\varphi',\\
\omega_0 = \overline{\omega}_0 &\Longleftrightarrow L'/L - q(x)\Big(\ln|L| + \sum I^j \ln|L_j|\Big) = \bar{p}(\varphi)\varphi' - p,\\
\omega_i = \overline{\omega}_i &\Longleftrightarrow L_i'/L_i = \overline{B}^i\varphi' - B^i
\end{aligned}
$$

for the differential equations $y' = f$, $\overline{y}' = \overline{f}$ by inserting the relations $\bar{x} = \varphi(x)$, $\overline{y} = L(x)y$, $\overline{z}_i = L_i(x)z_i$ into (29), $i = 1, \dots, m$. $\square$

4.3. Nonconstant invariants.

Let $F = F(x, y, z_1, \dots, z_m)$ be an invariant ($F = \overline{F}$) and let all invariants be functions of F only. Then the covariant derivatives satisfy

$$
(31)\qquad \Big(\frac{\partial F}{\partial \omega_0} =\Big)\ yF_y = G_0(F),\ \Big(\frac{\partial F}{\partial \omega_i} =\Big)\ z_iF_{z_i} = G_i(F)\quad (i = 1, \dots, m).
$$

Theorem 2. *Let $F = F(x, y, z_1, \dots, z_m)$ be a nonconstant invariant ($F = \overline{F}$) and let all invariants be functions of F only, $F_y = 0$. Then we have the following possibilities.*

(ι) $J(F) = 0$,

$$
\frac{f}{y} = B = q(x)\bigg(\ln|y| + \hat{I}(F) + \sum_{j\in\mathscr{J}} I^j(F)\ln|z_j|\bigg) + p(x),\ F = a(x)\prod |z_i|^{k_i},
$$

$k_i \in \mathbb{R}$, $j \in \mathscr{J}$ if $k_j = 0$. The invariance requirements become

$$
F = \overline{F},\ q = \bar{q}(\varphi)\varphi',\ \frac{L'}{L} = q\bigg(\ln|L| + \sum_{j\in\mathscr{J}} I^j(F)\ln|L_j|\bigg) + \bar{p}(\varphi)\varphi' - p,\ \omega_i = \overline{\omega}_i
$$

($i = 1, \dots, m$; the parameters B^i are not specified).

($\iota\iota$)$_1$ $J(F)J'(F) \neq 0$,

$$
\frac{f}{y} = B = \frac{p(x)}{J(F)}|y|^{J(F)}\prod |z_i|^{J^i(F)} + h(x),\quad F = a(x) \neq 0
$$

with the invariance requirements

$$a=\bar{a}(\varphi),\ p=\bar{p}(\varphi)\varphi'|L|^{J(F)}\prod|L_i|^{J^i(F)},\ \frac{L'}{L}=\bar{h}(\varphi)\varphi'-h,\ \omega_i=\overline{\omega}_i,\ i=1,\dots,m$$

(the parameters B^i are not specified).

$(\iota\iota)_2$ $J(F)\equiv J=$ *const.* $\neq 0$,

$$\frac{f}{y}=B=\frac{1}{J}p(x)|y|^J\mathrm{e}^{\hat{I}(F)}\prod_{j\in\mathscr{J}}|z_j|^{J^j(F)}+h(x),\hat{I}(F)=\int\frac{I(F)}{F}\,\mathrm{d}F,$$

$F=a(x)\prod|z_i|^{k_i},\ k_i\in\mathbb{R}\ (i=1,\dots,m),\ \sum k_i^2\neq 0,\ J^i(F)=k_iI(F)$ *for* $k_i\neq 0$, $j\in\mathscr{J}$ *if* $k_j=0$. *The invariance requirements this time become*

$$F=\overline{F},\ p=\bar{p}(\varphi)\varphi'|L|^J\prod_{j\in\mathscr{J}}|L_j|^{J^j(F)},$$
$$L'/L=\bar{h}(\varphi)\varphi'-h,\ \omega_i=\overline{\omega}_i,\ i=1,\dots,m$$

(*the parameters B^i are not specified*).

P r o o f. We get

$$F=a(x)|z_1|^{k_1}\dots|z_m|^{k_m}=a(x)\prod|z_i|^{k_i},\quad k_i\in\mathbb{R} \tag{32}$$

for $F_y=0$ by using (31_2). Then

$$\frac{(yB_y)}{yB_y}=\frac{1}{y}J=\frac{1}{y}J(F),\quad \text{i.e., } yB_y=c(x,z_1,\dots,z_m)|y|^{J(F)} \tag{33}$$

and we have two subcases $J(F)=0$ and $J(F)\neq 0$.

(ι) If $J=0$ then

$$yB_y=c(x,z_1,\dots,z_m)\neq 0$$

(in accordance with $(yB_y)_y/B_y=J(F)$) and

$$J^i=\frac{z_i(yB_y)_{z_i}}{yB_y}=\frac{z_ic_{z_i}}{c}=J^i(F)\quad(i=1,\dots,m).$$

Moreover,

$$B=c(x,z_1,\dots,z_m)\ln|y|+h(x,z_1,\dots,z_m),$$
$$\frac{z_iB_{z_i}}{yB_y}=\frac{z_ic_{z_i}\ln|y|+z_ih_{z_i}}{c}=\frac{z_ic_{z_i}}{c}\ln|y|+\frac{z_ih_{z_i}}{c}=J^i(F)\ln|y|+\frac{z_ih_{z_i}}{c}=I^i(F)$$

and

$$J^i(F) = 0 \quad (i = 1, \ldots, m)$$

since the functions b, c, F are independent of y. The conditions $J(F) = 0$, $J^i(F) = 0$ are equivalent to

$$(yB_y)_y = 0, \quad (yB_y)_{z_i} = 0 \quad (i = 1, \ldots, m), \text{ i.e.,}$$
$$B = q(x) \ln|y| + h(x, z_1, \ldots, z_m), \text{ where } \frac{z_i h_{z_i}}{q(x)} = I^i(F) \quad (i = 1, \ldots, m).$$

Thus we get

$$B = q(x)(\ln|y| + \hat{I}(F) + \sum_{j \in \mathscr{J}} I^j(F) \ln|z_j|) + p(x) \tag{34}$$

($F = a(x) \prod |z_i|^{k_i}$, $k_i \in \mathbb{R}$) in accordance with Observation 4. The invariance requirements become

$$\begin{aligned} & F = \overline{F}, \quad q = \bar{q}(\varphi)\varphi', \\ & \frac{L'}{L} = q(\ln|L| + \sum_{j \in \mathscr{J}} I^j(F) \ln|L_j|) + \bar{p}(\varphi)\varphi' - p, \\ & \omega_i = \overline{\omega}_i \quad (i = 1, \ldots, m). \end{aligned} \tag{35}$$

($\iota\iota$) The condition (33) holds for $J(F) \neq 0$ and

$$\begin{aligned} J^i(F) &= \frac{z_i (yB_y)_{z_i}}{yB_y} = \frac{z_i}{c|y|^{J(F)}} (c_{z_i} |y|^{J(F)} + c|y|^{J(F)} J'(F) F \ln|y|) \\ &= \frac{z_i c_{z_i}}{c} + k_i J'(F) F \ln|y| \end{aligned}$$

($i = 1, \ldots, m$) since $z_i F_{z_i} = k_i F$ in accordance with (32). We have two subcases $k_i = 0$ ($i = 1, \ldots, m$) or $J'(F) = 0$, by virtue of $F_y = c_y = 0$.

($\iota\iota$)$_1$ In the first subcase $c = p(x)|z_1|^{J^1(F)} \ldots |z_m|^{J^m(F)} = p(x) \prod |z_i|^{J^i(F)}$ for $F = a(x)$ (here $k_1 = \ldots = k_m = 0$). Hence

$$yB_y = |y|^{J(F)} p(x) \prod |z_i|^{J^i(F)}, \quad F = a(x), \quad J(F) \neq 0,$$

i.e.,

$$B = \frac{p(x)}{J(F)} |y|^{J(F)} \prod |z_i|^{J^i(F)} + h(x, z_1, \ldots, z_m), \quad F = a(x).$$

The conditions $h_{z_i} = 0$ ($i = 1, \ldots, m$) follow from

$$I^i(F) = \frac{z_i B_{z_i}}{yB_y} = \frac{J^i(F)}{J(F)} + \frac{z_i h_{z_i}}{yBy}, \quad F_y = 0.$$

We obtain

$$B = \frac{p(x)}{J(F)}|y|^{J(F)} \prod |z_i|^{J^i(F)} + h(x) = \frac{f}{y}, \quad F = a(x) \tag{36}$$

with the invariance requirements

$$a = \bar{a}(\varphi), \quad p = \bar{p}(\varphi)\varphi'|L|^{J(F)} \prod |L_i|^{J^i(F)}, \quad \frac{L'}{L} = \bar{h}(\varphi)\varphi' - h, \quad \omega_i = \overline{\omega}_i, \tag{37}$$

$i = 1, \ldots, m$.

$(\iota\iota)_2$ We consider $F = a(x) \prod |z_i|^{k_i}$, $k_i \in \mathbb{R}$ $(i = 1, \ldots, m)$, $\sum k_i^2 \neq 0$ and (33),

$$yB_y = c(x, z_1, \ldots, z_m)|y|^J = c|y|^J, \quad J = \text{const.} \neq 0.$$

Now we obtain

$$\frac{z_i B_{z_i}}{yB_y} = \frac{1}{J}\frac{z_i c_{z_i}}{c} + \frac{z_i h_{z_i}}{c|y|^J} = \frac{1}{J}J^i(F) + \frac{z_i h_{z_i}}{c|y|^J} = I^i(F)$$

by using

$$B = \frac{1}{J}c|y|^J + h(x, z_1, \ldots, z_m) \tag{38}$$

and

$$J^i(F) = \frac{z_i(zB_y)_{z_i}}{yB_y} = \frac{z_i c_{z_i}}{c}. \tag{39}$$

Then $h_{z_i} = 0$ $(i = 1, \ldots, m)$ with regard to $F_y = 0$ and equations (39) are solved in Observation 5. As a result,

$$B = \frac{1}{J}p(x)|y|^J \mathrm{e}^{\hat{I}(F)} \prod_{j \in \mathscr{J}} |z_j|^{I^j(F)} + h(x), \tag{40}$$

$\hat{I}(F) = \int (I(F)/F)\,\mathrm{d}F$, $I^j(F) = k_j I(F)$ for $k_j \neq 0$, $j \in \mathscr{J}$ if $k_j = 0$. The invariance requirements become

$$F = \overline{F}, \quad p = \bar{p}(\varphi)\varphi'|L|^J \prod_{j \in \mathscr{J}} |L_j|^{I^j(F)}, \tag{41}$$

$$L'/L = \bar{h}(\varphi)\varphi' - h, \quad \omega_i = \overline{\omega}_i, \; i = 1, \ldots, m.$$

□

Theorem 3. *Let $F = F(x, y, z_1, \ldots, z_m)$ be a nonconstant invariant ($F = \overline{F}$) and let all invariants be functions of F only, $F_y \neq 0$. Then*

(ι) *$F = a(x)y \prod |z_i|^{k_i}$, $k_i \in \mathbb{R}$ and $a = \bar{a}(\varphi)L \prod |L_i|^{k_i}$ is satisfied for $\overline{F} = \bar{a}(\bar{x})\overline{y} \prod |\overline{z}_i|^{k_i}$ and the transformation (2).*

($\iota\iota$) *$B = A(F)q(x) \prod |z_i|^{r_i} + p(x)$ holds for $r_i \in \mathbb{R}$ and nonzero functions $p(x), q(x)$.*

($\iota\iota\iota$) *The symmetry equivalence problem (relations (16)–(18)) is expressed by $f = yB$ and by the invariance requirements*

$$\begin{aligned} q &= \bar{q}(\varphi)\varphi' \prod |L_i|^{r_i}, \\ L' &= (\bar{p}(\varphi)\varphi' - p)L, \\ L_i' &= (\overline{B}^i\varphi' - B^i)L_i; \end{aligned}$$

parameters B^i are not specified, $i = 1, \ldots, m$.

P r o o f. Now we consider the conditions (31) with $F_y \neq 0$. We get $\mathrm{d}F/G_0(F) = \mathrm{d}y/y$, i.e., $H_0(F) = \ln |C(x, z_1, \ldots, z_m)y|$ and a new invariant $F := C(x, z_1, \ldots, z_m)y$ solving the equation $yF_y = G_0(F)$ for $x, z_1, \ldots, z_m$ fixed. Then $z_iF_{z_i} = z_iC_{z_i}y = G_i(F)$, i.e.,

$$z_iC_{z_i} = \frac{G_i(F)}{y} = \frac{G_i(Cy)}{y} = k_iC, \quad k_i \in \mathbb{R}, \ i = 1, \ldots, m. \tag{42}$$

For every $i \in \{1, \ldots, m\}$, the function $G_i(F) = k^iF$ is linear because the left hand side $z_iC_{z_i}$ of the equation is independent of y. Solving the equations (42) we obtain $C(x, z_1, \ldots, z_m) = a(x)|z_1|^{k_1} \ldots |z_m|^{k_m} = a(x) \prod |z_i|^{k_i}$ in accordance with Observation 1, hence

$$F = F(x, y, z_1, \ldots, z_m) = a(x)y|z_1|^{k_1} \ldots |z_m|^{k_m} = a(x)y \prod |z_i|^{k_i}, \tag{43}$$

$k_i \in \mathbb{R}$ $(i = 1, \ldots, m)$, $a(x)$ being an arbitrary nonzero function.

The invariant $J = (yB_y)_y/B_y = J(F)$ and we have

$$\frac{(yB_y)_y}{yB_y} = \frac{J(F)}{y} = \frac{J(F)}{yF_y}F_y = \frac{J(F)}{F}F_y = \tilde{J}'(F)F_y,$$

where we define

$$\tilde{J}'(F) = \frac{J(F)}{F}.$$

Then

$$yB_y = G(F)\tilde{c}(x, z_1, \ldots, z_m) \tag{44}$$

where $\tilde{c}(x, z_1, \ldots, z_m)$ is an integrating factor and

$$J(F) = \frac{G'(F)F}{G(F)} \tag{45}$$

is satisfied for the function $G(F)$.

The invariants

$$J^i = \frac{z_i B_{yz_i}}{B_y} = \frac{z_i (yB_y)_{z_i}}{yB_y} = k_i \frac{G'(F)F}{G(F)} + \frac{z_i \tilde{c}_{z_i}}{\tilde{c}} = k_i J(F) + \frac{z_i \tilde{c}_{z_i}}{\tilde{c}} = J^i(F)$$

are functions of the invariant $F = a(x)y \prod |z_i|^{k_i}$. Then

$$\frac{z_i \tilde{c}_{z_i}}{\tilde{c}} = \tilde{J}^i(F) = r_i \in \mathbb{R},\ i = 1, \ldots, m$$

(the left-hand side is independent of y, thus $(\tilde{J}^i)'(F) = 0$) and

$$\tilde{c}(x, z_1, \ldots, z_m) = q(x)|z_1|^{r_1} \ldots |z_m|^{r_m} = q(x) \prod |z_i|^{r_i},\ r_i \in \mathbb{R}\ (i = 1, \ldots, m) \tag{46}$$

with an arbitrary nonzero function $q(x)$ in accordance with Observation 1 and the assumption $yB_y \neq 0$. We get

$$B = B(x, y, z_1, \ldots, z_m) = R(F)\tilde{c} + b = R(F)q(x) \prod |z_i|^{r_i} + b(x, z_1, \ldots, z_m) \tag{47}$$

with $G(F) = R'(F)F$ by using (44), (46).

The invariants

$$I^i = \frac{z_i B_{z_i}}{yB_y} = k_i + r_i \frac{R(F)}{G(F)} + \frac{z_i b_{z_i}}{yB_y} = I^i(F),$$

thus (see (44))

$$z_i b_{z_i} = yB_y \tilde{I}^i(F) G(F) q(x) \prod |z_i|^{r_i} = \hat{I}^i(F)\tilde{c}(x, z_1, \ldots, z_m) = s_i \tilde{c} \tag{48}$$

$(i = 1, \ldots, m)$ where $\tilde{I}^i(F) = (I(F) - k_i)G(F) - r_i R(F)$, the left-hand side is independent of y and $F_y \neq 0$. A solution of the system (48) is

$$b = b(x, z_1, \ldots, z_m) = k\tilde{c} + p(x),\ k \in \mathbb{R} - \{0\}$$

with an arbitrary function $p(x)$ according to Observation 2. Then

$$B = R(F)\tilde{c} + k\tilde{c} + p(x) = A(F)\tilde{c} + p(x) = A(F)q(x) \prod |z_i|^{r_i} + p(x) \tag{49}$$

and

$$f = yB = A(F)y\tilde{c} + p(x) = A(F)q(x)y \prod |z_i|^{r_i} + p(x)y \tag{50}$$

where

$$A(F) = R(F) + k.$$

We express the invariants I^i, J, J^i in terms of the invariant F:

$$\begin{gathered} I^i = k_i + \frac{A(F)}{A'(F)F} r_i = k_i + \frac{A(F)}{G(F)} r_i \qquad (\text{here } A'(F)F = R'(F)F = G(F)), \\ J = \frac{A''(F)F}{A'(F)} + 1 = J(F), \\ J^i = k_i J(F) + r_i = J^i(F), \end{gathered} \tag{51}$$

$i = 1, \ldots, m$. Then the Bianchi identities (14), (15) are identically satisfied. For example,

$$\begin{aligned} \frac{\partial I^i}{\partial \omega_0} &= y I^i_y = y\Big(k_i + \frac{A(F)}{G(F)} r_i\Big)_y = y r_i \Big(\frac{A'(F)F_y}{G(F)} - \frac{A(F)G'(F)F_y}{G(F)^2}\Big) \\ &= r_i \Big(\frac{A'(F)F}{G(F)} - \frac{A(F)}{G(F)}\frac{G'(F)F}{G(F)}\Big) = r_i \Big(1 - \frac{A(F)}{G(F)} J(F)\Big) = J^i - J I^i \end{aligned}$$

$(i = 1, \ldots, m)$ by virtue of (45), (51). □

4.4. A generalization.

Theorem 4. *For arbitrary $p \in \mathbb{N} = \{1, 2, \ldots\}$ fixed, let*

$$F^j = F^j(x, y, z_1, \ldots, z_m) = a_j(x) y |z_1|^{k_1^j} \ldots |z_m|^{k_m^j} = a_j(x) y \prod |z_i|^{k_i^j}$$

(where $k_i^j \in \mathbb{R}$, $i = 1, \ldots, m$, $a_j(x)$ are arbitrary nonzero functions, $j = 1, \ldots, p$) be functionally independent invariants ($F^j = \overline{F}^j$, $j = 1, \ldots, p$) and let all invariants be of the form $G(F^1, \ldots, F^p)$. Then

(ι) $a_j = \bar{a}_j(x) L |L_1|^{k_1^j} \ldots |L_m|^{k_m^j}$ is satisfied for $\overline{F}^j = \bar{a}_j(\bar{x}) \overline{y} \prod |\overline{z}_i|^{k_i^j}$ and the transformation (2), $i = 1, \ldots, m$, $j = 1, \ldots, p$.

($\iota\iota$) $B = A(F^1, \ldots, F^p) q(x) \prod |z_i|^{r_i} + p(x)$ is satisfied for $r_i \in \mathbb{R}$ and arbitrary nonzero functions $A(F^1, \ldots, F^p), p(x), q(x)$, $i = 1, \ldots, m$.

($\iota\iota\iota$) *The symmetry equivalence problem* (*relations* (16)–(18)) *is expressed by* $f = yB$ *and by the invariance requirements*

$$q = \bar{q}(\varphi)\varphi' \prod |L_i|^{r_i},$$
$$L' = (\bar{p}(\varphi)\varphi' - p)L,$$
$$L_i' = (\overline{B}^i\varphi' - B^i)L_i;$$

the parameters B^i *are not specified,* $i = 1, \dots, m$.

P r o o f. The assertion follows for $p = 1$ from Theorems 2, 3. We consider only the case $F_y \neq 0$. For an arbitrary fixed $p \in \{2, \dots\}$ there exist functionally independent invariants

$$F^j = F^j(x, y, z_1, \dots, z_m) = a_j(x)y|z_1|^{k_1^j} \dots |z_m|^{k_m^j} = a_j(x)y \prod |z_i|^{k_i^j} \tag{52}$$

such that

$$yF_y^j = F^j, \quad z_i F_{z_i}^j = k_i^j F^j, \tag{53}$$

$k_i^j \in \mathbb{R},\ i = 1, \dots, m,\ j = 2, \dots, p$. Then

$$\begin{aligned} y\frac{\partial}{\partial y}G(F^1, \dots, F^p) = \sum G_{F^j}F^j = \tilde{G}(F^1, \dots, F^p),\ z_i\frac{\partial}{\partial z_i}G(F^1, \dots, F^p) \\ = \sum k_i^j G_{F^j}F^j, \end{aligned} \tag{54}$$

i being fixed, for every function $G(F^1, \dots, F^p)$ with partial derivatives G_{F^j}.

Let all invariants be of the form $G(F^1, \dots, F^p)$. The invariant J satisfies $J = (yB_y)_y/B_y = J(F^1, \dots, F^p)$ and

$$\frac{(yB_y)_y}{yB_y} = \frac{J(F^1, \dots, F^p)}{y} = \frac{\partial}{\partial y}\tilde{J}(F^1, \dots, F^p)$$

for $J(F^1, \dots, F^p) = y\frac{\partial}{\partial y}\tilde{J} = \sum \tilde{J}_{F^j}F^j$. Then

$$yB_y = G(F^1, \dots, F^p)\tilde{c}(x, z_1, \dots, z_m) \tag{55}$$

with the integrating factor $\tilde{c}$ and

$$J = \frac{y}{G}\frac{\partial}{\partial y}G = \sum \frac{G_{F^j}F^j}{G} \tag{56}$$

is satisfied for functions $J(F^1,\dots,F^p), G(F^1,\dots,F^p)$. The invariants

$$J^i = \frac{z_iB_{yz_i}}{B_y} = \frac{z_i(yB_y)_{z_i}}{yB_y} = z_i\frac{1}{G}y\frac{\partial}{\partial y}G + z_i\frac{\tilde{c}_{z_i}}{\tilde{c}} = \sum k_i^j\frac{G_{F^j}F^j}{G} = J^i(F^1,\dots,F^p)$$

are functions of the invariants $F^1,\dots,F^p$. Then

$$z_i\tilde{c}_{z_i}/\tilde{c} = \tilde{J}^i(F^1,\dots,F^p)$$

and we have

$$0 = y\frac{\partial}{\partial y}\tilde{J}^i = \sum \tilde{J}^i_{F^j}F^j$$

because the left-hand side of $z_i\tilde{c}_{z_i}/\tilde{c}$ is independent of y. Thus $(\tilde{J}^i)'(F^j) = 0$ $(j = 1,\dots,p)$ since $F^1,\dots,F^p$ are functionally independent invariants. The solution of the equations

$$z_i\tilde{c}_{z_i}/\tilde{c} = r_i \in \mathbb{R},\ i = 1,\dots,m$$

is of the form

$$\tilde{c}(x,z_1,\dots,z_m) = q(x)|z_1|^{r_1}\dots|z_m|^{r_m} = q(x)\prod|z_i|^{r_i},\ r_i \in \mathbb{R}\ (i = 1,\dots,m) \tag{57}$$

with an arbitrary function $q(x)$ in accordance with Observation 1. We get

$$\begin{aligned} B = B(x,y,z_1,\dots,z_m) &= R(F^1,\dots,F^p)\tilde{c} + b \\ &= R(F^1,\dots,F^p)q(x)\prod|z_i|^{r_i} + b(x,z_1,\dots,z_m) \end{aligned} \tag{58}$$

with

$$G = y\frac{\partial R}{\partial y} = \sum R_{F^j}F^j,\quad G_{F^j} = \sum R_{F^jF^l}F^l + R_{F^j} \tag{59}$$

by using (55)–(57).

The invariants are of the form

$$\begin{aligned} I^i = \frac{z_iB_{z_i}}{yB_y} &= \frac{1}{G\tilde{c}}\Big(z_i\frac{\partial R}{\partial z_i}\tilde{c} + r_iR\tilde{c} + z_ib_{z_i}\Big) \\ &= \sum k_i^jR_{F^j}F^j/G + r_iR/G + \frac{z_ib_{z_i}}{yB_y} = I^i(F^1,\dots,F^p) \end{aligned} \tag{60}$$

$(i = 1,\dots,m)$ where

$$z_ib_{z_i} = yB_y\tilde{I}^i(F^1,\dots,F^p) = \tilde{I}^iG\tilde{c} = \hat{I}^i(F^1,\dots,F^p)\tilde{c} = s_i\tilde{c},\ s_i \in \mathbb{R} - \{0\}, \tag{61}$$

$i = 1, \ldots, m$ (the left-hand side is independent of y and $F^1, \ldots, F^p$ are functionally independent). A solution of the system (61) is of the form

$$b(x, z_1, \ldots, z_m) = k\tilde{c} + p(x) \quad (k = \text{const.} \neq 0) \tag{62}$$

with an arbitrary function $p(x)$ due to Observation 2. We have

$$\begin{aligned} B &= A(F^1, \ldots, F^p)\tilde{c} + p(x) = A(F^1, \ldots, F^p)\tilde{c} + p(x) \\ &= A(F^1, \ldots, F^p)q(x) \prod |z_i|^{r_i} + p(x) \end{aligned} \tag{63}$$

and

$$f = yB = A(F^1, \ldots, F^p)q(x)y \prod |z_i|^{r_i} + p(x)y \tag{64}$$

with

$$A(F^1, \ldots, F^p) = R(F^1, \ldots, F^p) + k \tag{65}$$

owing to (58), (62).

We express the invariants I^i, J, J^i in terms of the invariants $F^1, \ldots, F^p$:

$$\begin{aligned} J &= J(F^1, \ldots, F^p), \\ J^i &= \frac{z_i(yB_y)_{z_i}}{yB_y} = r_i + \frac{1}{G} \sum k_i^j G_{F^j} F^j, \\ I^i &= \frac{z_i B_{z_i}}{yB_y} = \frac{1}{G\tilde{c}}(A\tilde{c})_{z_i} = r_i \frac{A}{G} + \frac{1}{G} \sum k_i^j R_{F^j} F^j, \end{aligned}$$

$j = 1, \ldots, m$. Then the Bianchi identities (14), (15) are identically satisfied. For example,

$$\begin{aligned} \frac{\partial I^i}{\partial \omega_0} &= yI_y^i = r_i y(A/G)_y + y\Big(\frac{1}{G} \sum k_i^j R_{F^j} F^j\Big)_y \\ &= r_i(1 - AJ/G) + \frac{1}{G} \sum k_i^j R_{F^j} F^j (1 - J) + \frac{1}{G} \sum k_i^j R_{F^j F^l} F^j F^l \end{aligned}$$

and at the same time

$$\begin{aligned} J^i - JI^i &= r_i + \frac{1}{G} \sum k_i^j G_{F^j} F^j - Jr_i A/G - \frac{J}{G} \sum k_i^j R_{F^j} F^j \\ &= r_i(1 - JA/G) + \sum k_i^j (G_{F^j} - JR_{F^j}) F^j = \frac{\partial I^i}{\partial \omega_0} \end{aligned}$$

in accordance with (56), (59). □

5. Main results—the determined system

Suppose $B^i = B^i(x, y, z_1, \ldots, z_m)$ are certain *given functions* (and not uncertain parameters), then the Pfaffian system $\omega_i = 0$ provides the differential equations $z_i' = z_i B_i(x, y, z_1, \ldots, z_m)$ for the functions $z_1, \ldots, z_m$ and altogether we deal with transformations of the *determined system*

$$y' = f, \quad z_i' = z_i B_i \quad (i = 1, \ldots, m).$$

The results drastically change since we obtain also the *invariants*

$$M^i = \frac{B^i_y}{B_y}, \quad N^{j(i)} = \frac{z_j B^i_{z_j}}{y B_y}, \quad j = 1, \ldots, m, \ i \in \{1, \ldots, m\} \text{ being fixed,} \tag{66}$$

by using the exterior derivatives

$$d\omega_i = \mathrm{d}x \wedge dB^i = \omega \wedge \left(M^i \omega_0 + \sum N^{j(i)} \omega_j\right) \tag{67}$$

and the equalities $d\omega_i = \mathrm{d}\overline{\omega}_i, \ i \in \{1, \ldots, m\}$. Then

$$\begin{aligned} d^2\omega_j = \Big(J\omega_0 \wedge \omega + \sum J^i \omega_i \wedge \omega_0\Big) \wedge \Big(M^j \omega_0 + \sum N^{i(j)} \omega_i\Big) \\ + \Big(\mathrm{d}M^j \wedge \omega_0 + \sum \mathrm{d}N^{i(j)} \wedge \omega_i\Big) \wedge \omega = 0 \end{aligned}$$

is equivalent to the identities

$$\frac{\partial M^j}{\partial \omega_i} - \frac{\partial N^{i(j)}}{\partial \omega_0} = JN^{i(j)} - J^i M^j, \quad \frac{\partial N^{i(j)}}{\partial \omega_k} - \frac{\partial N^{k(j)}}{\partial \omega_i} = J^i N^{k(j)} - J^k N^{i(j)}, \tag{68}$$

$i, k \in \{1, \ldots, m\}$, $j \in \{1, \ldots, m\}$ being fixed.

Theorem 5. *Let all invariants be constant. Then:*
(ι) *For* $J \neq 0$,

$$B = \frac{1}{J} q(x) |y|^J \prod |z_i|^{JI^i} + b(x) \ (= f/y), \quad B^i = \frac{M^i}{J} q(x) |y|^J \prod |z_j|^{JI^j} + b_i(x)$$

($b(x)$, $b_i(x)$ are arbitrary functions) and the symmetry equivalence problem (relations (16)–(18)) is expressed by $f = yB$ and by the invariance requirements

$$q = \bar{q}(\varphi) \varphi' |L|^J \prod |L_j|^{JI^j}, \quad L'/L = \bar{b}(\varphi)\varphi' - b, \quad L_i'/L_i = \bar{b}_i(\varphi)\varphi' - b_i.$$

This symmetry subcase is governed by the $(m+2)$*-parameter Lie-group*

$$\overline{Y} = Y + c_0,\ \overline{Z}_i = Z_i + c_i,\ \overline{X} = \mathrm{e}^{-(c_0 J + \sum c_j J^j)} X + c \quad (c_0, c, c_i \in \mathbb{R})$$

where

$$X = \int q(x) \mathrm{e}^{\int (Jb(x) + \sum J^j b_j(x))\,\mathrm{d}x}\,\mathrm{d}x,$$
$$Y = \ln|y| - \int b(x)\,\mathrm{d}x, \quad Z_i = \ln|z_i| - \int b_i(x)\,\mathrm{d}x \quad (i = 1, \dots, m).$$

($\iota\iota$) *For* $J = 0$,

$$B = q(x)\Big(\ln|y| + \sum I^j \ln|z_j|\Big) + p(x),\ B^i = M^i q(x)\Big(\ln|y| + \sum I^j \ln|z_j|\Big) + p_i(x)$$

($p(x)$, $p_i(x)$ *are arbitrary functions) and the symmetry equivalence problem (relations* (16)–(18)*) is expressed by* $f = yB$ *and by the invariance requirements*

$$q = \bar{q}(\varphi)\varphi',\ \frac{L'}{L} - q(x)\Big(\ln|L| + \sum I^j \ln|L_j|\Big) = \bar{p}(\varphi)\varphi' - p,$$
$$\frac{L_i'}{L_i} - M^i \frac{L'}{L} = \bar{p}_i(\varphi)\varphi' - p_i - M^i(\bar{p}(\varphi)\varphi' - p), \quad i = 1, \dots, m.$$

Proof. The assertions of Theorem 1 are fulfiled and, moreover, $B^i = B^i(x, y, z_1, \dots, z_m)$. For constant invariants

(69) $$J,\ I^i,\ J^i = JI^i,\ M^i = \frac{B^i_y}{B_y},\quad N^{j(i)} = \frac{z_j B^i_{z_j}}{yB_y},\quad i, j = 1, \dots, m,$$

relations (68) are equivalent to

(70) $$N^{i(j)} = I^i M^j, \quad i, j = 1, \dots, m$$

and $J = \overline{J}$, $I^i = \overline{I}^i$ $M^i = \overline{M}^i$ are the same constants ($i = 1, \dots, m$).

We get the corresponding structural formulae

(71) $$\mathrm{d}\omega = \Big(J\omega_0 + \sum J^j \omega_j\Big) \wedge \omega = J\Big(\omega_0 + \sum I^j \omega_j\Big) \wedge \omega,$$
$$\mathrm{d}\omega_0 = \omega \wedge \Big(\omega_0 + \sum I^j \omega_j\Big),$$
$$\mathrm{d}\omega_i = \omega \wedge \Big(M^i \omega_0 + \sum N^{j(i)} \omega_j\Big) = M^i \omega \wedge \Big(\omega_0 + \sum I^j \omega_j\Big).$$

(ι) *Subcase* $J \neq 0$.

$$B = \frac{1}{J} q(x)|y|^J \prod |z_i|^{J^i} + b(x) \ (= f/y). \tag{72}$$

Moreover, $B^i_y = M^i B_y$ ($i \in \{1, \dots, m\}$ being fixed) gives

$$B^i = M^i B + \tilde{b}^i(x, z_1, \dots, z_m)$$

and we have $N^{j(i)} y B_y = z_j B^i_{z_j} = z_j(M^i B_{z_j} + \tilde{b}^i_{z_j}) = M^i I^j y B_y + z_j b^i_{z_j}$, i.e., $z_j b^i_{z_j} = (N^{j(i)} - M^i I^j) y B_y = 0$ by virtue of (69). Hence $\tilde{b}^i = \tilde{b}^i(x)$ and

$$B^i = M^i B + \tilde{b}^i(x) = \frac{M^i}{J} q(x)|y|^J \prod |z_j|^{JI^j} + b_i(x) \quad (b_i(x) = M^i b(x) + \tilde{b}^i(x)). \tag{73}$$

All forms $\omega, \omega_0, \omega_i$ are determined through functions B, B^i and we are passing to the structural formulae (71). We introduce forms $\eta_0 = J\omega_0 + \omega$, $\eta_i = M^i \omega_0 - \omega_i$; then $d\eta_0 = d\eta_i = 0$, therefore η_0, η_i are total differentials and the system (22) can be replaced by the simpler system

$$\begin{aligned} &\omega = \overline{\omega}, \quad \omega = q(x)|y|^J \prod |z_j|^{J^j} \,\mathrm{d}x, \\ &\eta_0 = \overline{\eta}_0, \ \eta_0 = J(\mathrm{d}y/y - b(x)\,\mathrm{d}x), \\ &\eta_i = \overline{\eta}_i, \ \ \eta_i = M^i(\mathrm{d}y/y - b(x)\,\mathrm{d}x) - (\mathrm{d}z_i/z_i - b_i(x)\,\mathrm{d}x). \end{aligned} \tag{74}$$

We obtain the transformation relations

$$\begin{aligned} \omega = \overline{\omega} &\Longleftrightarrow q = \bar{q}(\varphi)\varphi'|L|^J \prod |L_j|^{J^j}, \\ \eta_0 = \overline{\eta}_0 &\Longleftrightarrow L'/L = \bar{b}(\varphi)\varphi' - b, \\ \eta_i = \overline{\eta}_i &\Longleftrightarrow L'_i/L_i = \bar{b}_i(\varphi)\varphi' - b_i \end{aligned} \tag{75}$$

for differential equations $y' = f$, $\overline{y}' = \overline{f}$ by inserting the transformation relations $\bar{x} = \varphi(x), \overline{y} = L(x)y, \overline{z}_i = L_i(x)z_i$ to (74), $i = 1, \dots, m$. Moreover, the relations (74) can be drastically simplified into the system

$$\begin{aligned} &\eta_0 = \overline{\eta}_0 \Leftrightarrow \mathrm{d}Y = J\,\mathrm{d}\overline{Y} \Leftrightarrow \overline{Y} = Y + c_0 \quad (c_0 \in \mathbb{R}), \\ &\eta_i = \overline{\eta}_i \Leftrightarrow M^i\,\mathrm{d}Y - \mathrm{d}z_i = M^i\,\mathrm{d}\overline{Y} - \mathrm{d}\overline{Z}_i \Leftrightarrow \overline{Z}_i = Z_i + c_i \quad (c_i \in \mathbb{R}), \\ &\omega = \overline{\omega} \Leftrightarrow \mathrm{d}x = \mathrm{e}^{c_0 J} \prod \mathrm{e}^{\sum c_j J^j}\,\mathrm{d}\overline{X} \Leftrightarrow \overline{X} = \mathrm{e}^{-(c_0 J + \sum c_j J^j)} X + c \end{aligned} \tag{76}$$

($c \in \mathbb{R}$) by means of transformations

$$X = \int q(x)\mathrm{e}^{\int (Jb(x) + \sum J^j b_j(x))\,\mathrm{d}x}\,\mathrm{d}x, \ Y = \ln|y| - \int b(x)\,\mathrm{d}x, \ Z_i = \ln|z_i| - \int b_i(x)\,\mathrm{d}x$$

$(i = 1, \ldots, m)$. So the higher symmetry subcase (with constant invariants) is governed by the $(m+2)$-parameter Lie-group (76).

$(\iota\iota)$ *Subcase* $J = 0$. In this subcase $J^i = 0$ $(i = 1, \ldots, m)$ and the resulting function is

$$B = q(x)\Big(\ln|y| + \sum I^j \ln|z_j|\Big) + p(x) \ (= f/y). \tag{77}$$

We can find factors B^i in terms of constants $M^i, N^{j(i)}$ $(i, j \in \{1, \ldots, m\})$. First,

$$B^i = q(x)\ln|y|^{M^i} + b^i(x, z_1, \ldots .z_m)$$

is a solution of the equation $B^i_y = M^i B_y = M^i q(x)/y$, $i \in \{1, \ldots, m\}$ being fixed. Second,

$$b^i = q(x)\sum I^j M^i \ln|z_j| + p_i(x)$$

follows from $z_j B^i_{z_j} = z_j b^i_{z_j} = I^j M^i q(x)$ $(= N^{j(i)} y B_y)$ and from Observation 3. As a result we have

$$B^i = q(x)\left(M^i \ln|y| + \sum I^j M^i \ln|z_j|\right) + p_i(x) \tag{78}$$

where $p_i(x)$ are arbitrary functions, $i = 1, \ldots, m$. The structural formulae (71) become

$$\begin{aligned} \mathrm{d}\omega &= 0,\\ \mathrm{d}\omega_0 &= \omega \wedge \left(\omega_0 + \sum I^j \omega_j\right),\\ \mathrm{d}\omega_i &= M^i \omega \wedge \left(\omega_0 + \sum I^j \omega_j\right) \end{aligned} \tag{79}$$

in this subcase. We introduce forms $\eta_i = \omega_i - M^i \omega_0$; then $\mathrm{d}\eta_i = 0$, η_i are total differentials and the system (22) is replaced by the system

$$\begin{aligned} \omega &= \overline{\omega}, \quad \omega = q(x)\,\mathrm{d}x,\\ \omega_0 &= \overline{\omega}_0, \quad \omega_0 = \mathrm{d}y/y - \left(q(x)\left(\ln|y| + \sum I^j \ln|z_j|\right) + p(x)\right)\mathrm{d}x,\\ \eta_i &= \overline{\eta}_i, \quad \eta_i = (\mathrm{d}z_i/z_i - p_i(x)\,\mathrm{d}x) - M^i(\mathrm{d}y/y - p(x)\,\mathrm{d}x). \end{aligned} \tag{80}$$

We obtain the transformation relations

$$\begin{aligned} \omega = \overline{\omega} &\iff q = \bar{q}(\varphi)\varphi',\\ \omega_0 = \overline{\omega}_0 &\iff L'/L - q(x)\left(\ln|L| + \sum I^j \ln|L_j|\right) = \bar{p}(\varphi)\varphi' - p,\\ \eta_i = \overline{\eta}_i &\iff L_i'/L_i - M^i L'/L = \bar{p}_i(\varphi)\varphi' - p_i - M^i(\bar{p}(\varphi)\varphi' - p) \end{aligned} \tag{81}$$

for differential equations $y' = f$, $\overline{y}' = \overline{f}$ by inserting the relations $\bar{x} = \varphi(x)$, $\overline{y} = L(x)y$, $\overline{z}_i = L_i(x)z_i$ into (80), $i = 1, \ldots, m$. The assertion is proved. $\square$

Corollary 1. *Let us assume Theorem 2 holds for the determined case. For every $i \in \{1, 2, \dots, m\}$, the corresponding function $B^i(x, y, z_1, \dots, z_m)$ and the invariance requirement $\omega_i = \overline{\omega}_i$ are given as follows:*

(ι) *For $J(F) = 0$, there exists $N^i(F)$ such that $N^{j(i)}(F) = k_j N^i(F)$ for $k_j \neq 0$,*

$$B^i = q(x)\Big(M^i \ln|y| + \hat{N}^i(F) + \sum_{j \in \mathscr{J}} N^{j(i)}(F) \ln|z_j|\Big) + p_i(x)$$

where $M^i(F) \equiv M^i = const.$, $F = a(x) \prod |z_i|^{k_i}$ $(k_i \in \mathbb{R})$, $j \in \mathscr{J}$ if $k_j = 0$. Moreover, $\omega_i = \overline{\omega}_i$ is equivalent to

$$L_i'/L_i = q\bigg(M^i \ln|L| + \sum_{j \in \mathscr{J}} N^{j(i)}(F) \ln|L_j|\bigg) + \bar{p}_i(\varphi)\varphi' - p_i.$$

$(\iota\iota)_1$ *For $J(F) \neq 0$,*

$$B^i = M^i(F) \frac{p(x)}{J(F)} |y|^{J(F)} \prod |z_i|^{J^i(F)} + h_i(x), \; F = a(x) \neq 0$$

with the invariance requirement

$$L_i'/L_i = \bar{h}_i(\varphi)\varphi' - h_i.$$

The invariants are connected by $J(F)N^{j(i)}(F) = J^j(F)M^i(F)$.

$(\iota\iota)_2$ *For $J(F) \equiv J = const. \neq 0$,*

$$B^i = K^i(F)\frac{1}{J}p(x)|y|^J \prod_{j \in \mathscr{J}} |z_j|^{J^j(F)} + h_i(x) \quad (K^i(F) = M^i(F)\mathrm{e}^{\hat{I}(F)}),$$

$F = a(x) \prod |z_i|^{k_i}$ $(k_i \in \mathbb{R})$, $j \in \mathscr{J}$ if $k_j = 0$. Moreover, $\omega_i = \overline{\omega}_i$ is equivalent to

$$L_i'/L_i = \bar{h}_i(\varphi)\varphi' - h_i.$$

Proof. We need to solve the conditions (66), hence

$$B_y^i = M^i(F)B_y, \; z_j B_{z_j}^i = N^{j(i)} y B_y, j = 1, \dots, m, \quad i \in \{1, \dots, m\} \text{ being fixed.}$$

(ι) We have $z_i F_{z_i} = k_i F, F_y = 0$ and $yB_y = q(x)$ according to (ι) of Theorem 2. Then $B_y^i = M^i(F)q(x)/y$ and $B^i = q(x)M^i(F)\ln|y| + \alpha^i(x, z_1, \dots, z_m)$. We get $z_j B_{z_j}^i = q(x)(M^i(F))' k_j F \ln|y| + z_j \alpha_{z_j}^i = q(x)N^{j(i)}$ and $M^i(F) \equiv M^i = \text{const.}$ because $F_y = 0$. There exists $N^i(F)$ such that $N^{j(i)} = k_j N^i(F)$ for j with $k_j \neq 0$ and

$B^i = q(x)(M^i \ln|y| + \hat{N}^i(F) + \sum\limits_{j\in\mathscr{J}} N^{j(i)}(F)\ln|z_j|) + p_i(x)$, $\overline{N}^i(F) = \int(N^i(F)/F)\,\mathrm{d}F$ in accordance with Observation 4. The equivalent requirement $\omega_i = \overline{\omega}_i$ is equivalent to $L_i'/L_i = \overline{B}^i(\varphi)\varphi' - B_i$, i.e., $L_i'/L_i = q(M^i \ln|L| + \sum\limits_{j\in\mathscr{J}} N^{j(i)}(F)\ln|L_j|) + \bar{p}_i(\varphi)\varphi' - p_i$.

$(\iota\iota)_1$ We analyze the case $J(F) \neq 0$, $F = a(x)$, $B = \frac{p(x)}{J(F)}|y|^{J(F)}\prod|z_i|^{J^i(F)} + h(x)$. We obtain $B^i = M^i(F)\frac{p(x)}{J(F)}|y|^{J(F)}\prod|z_i|^{J^i(F)} + h^i(x, z_1, \ldots, z_m)$ by using $B^i_y = M^i(F)B_y$. This implies $z_j h^i_{z_j} = p(x)|y|^{J(F)}\prod|z_i|^{J^i(F)}(N^{j(i)}(F) - M^i(F)J^i(F)\frac{1}{J(F)}) = 0$ because the left-hand side is independent of y, i.e., $h^i(x, z_1, \ldots, z_m) = h_i(x)$ and $\omega^i = \hat{\omega}^i$ is equivalent to $L_i'/L_i = \bar{h}_i(\varphi)\varphi' - h_i$ due to the equivalent condition $p = \bar{p}(\varphi)\varphi'|L|^{J(F)}\prod|L_i|^{J^i(F)}$.

$(\iota\iota)_2$ For $J(F) \equiv J = \text{const.} \neq 0$ we get $B^i = \frac{1}{J}p(x)K^i(F)|y|^J \prod\limits_{j\in\mathscr{J}}|z_j|^{J^j(F)} + h^i(x, z_1, \ldots, z_m)$, $K^i(F) = M^i(F)\mathrm{e}^{\hat{I}(F)}$ and $B = \frac{1}{J}p(x)|y|^J\mathrm{e}^{\hat{I}(F)}\prod\limits_{j\in\mathscr{J}}|z_j|^{J^j(F)} + h(x)$ in a way analogous to (ι). Then $z_j h^i_{z_j} = p(x)|y|^J(N^{j(i)}(F)\mathrm{e}^{\hat{I}(F)}\prod\limits_{j'\in\mathscr{J}}|z_{j'}|^{J^{j'}(F)} - z_j(\frac{1}{J}K^i(F)\prod\limits_{j'\in\mathscr{J}}|z_{j'}|^{J^{j'}(F)})_{z_j}) = 0$, hence $h^i(x, z_1, \ldots, z_m) = h_i(x)$ because $h^i(x, z_1, \ldots, z_m)$ is independent of y. The invariance requirement $\omega^i = \overline{\omega}^i$ is equivalent to $L_i'/L_i = \frac{1}{J}K^i(F)|y|^J\prod\limits_{j\in\mathscr{J}}|z_j|^{J^j(F)}(\bar{p}(\varphi)\varphi'|L|^J\prod\limits_{j\in\mathscr{J}}|L_j|^{J^j(F)} - p) + \bar{h}_i(\varphi)\varphi' - h_i$ and $L_i'/L_i = \bar{h}_i(\varphi)\varphi' - h_i$ due to $p = \bar{p}(\varphi)\varphi'|L|^J\prod\limits_{j\in\mathscr{J}}|L_j|^{J^j(F)}$ in $(\iota\iota)_2$ of Theorem 2. $\square$

Corollary 2. *Let us assume Theorem* 3 *holds for the determined case. For every* $i \in \{1, 2, \ldots, m\}$, *the corresponding function* $B^i(x, y, z_1, \ldots, z_m)$ *and the invariance requirement* $\omega_i = \overline{\omega}_i$ *in* $(\iota\iota)$ *are given by*

$$B^i = K(F)q(x)\prod|z_i|^{r_i} + p_i(x), \quad \textit{where } K(F) = \int A'(F)M^i(F)\,\mathrm{d}F,$$
$$L_i' = (\bar{p}_i(\varphi)\varphi' - p_i)L_i.$$

Proof. The proof is similar to the above. We analyze the conditions $B^i_y = M^i(F)B_y$, $z_j B^i_{z_j} = N^{j(i)}yB_y$, $(j = 1, \ldots, m,\ i \in \{1, \ldots, m\}$ being fixed) for $F = a(x)y\prod|z_i|^{k_i}$, $k_i \in \mathbb{R}$, $B = A(F)q(x)\prod|z_i|^{r_i} + p(x)$ and $L_i' = (\overline{B}^i\varphi' - B^i)L_i$. $\square$

For $x \in \mathbf{i} \subset \mathbb{R}$, $\bar{x} \in \mathbf{j} \subset \mathbb{R}$ we consider equations

$$y'(x) = f(x, y(x), y(\xi_1(x)), \dots, y(\xi_m(x))), \tag{82}$$

$$\overline{y}'(\bar{x}) = \overline{f}(\bar{x}, \overline{y}(\bar{x}), \overline{y}(\bar{\xi}_1(\bar{x})), \dots, \overline{y}(\bar{\xi}_m(\bar{x}))). \tag{83}$$

We suppose that $\xi_i(x) \neq x$, $\xi_j(x) \neq \xi_i(x)$ on $\mathbf{i}$ for $j \neq i$, $i, j \in \{1. \dots, m\}$ and analogously for the deviations $\bar{\xi}_i$ on $\mathbf{j}$.

We say that (82) is *globally transformable* into (83) if there exist two functions φ, L such that the function L is of the class $C^1(\mathbf{i})$ and is nonvanishing on $\mathbf{i}$, the function φ is a C^1 diffeomorphism of the interval $\mathbf{i}$ onto $\mathbf{j}$ and the function $\overline{y}(\bar{x}) = \overline{y}(\varphi(x)) = L(x)y(x)$ is a solution of (83) whenever $y(x)$ is a solution of (82). Then (1) is called *a global transformation* (of (82) into (83)). If (82) is globally transformable into (83), then $\bar{\xi}_i(\bar{x}) = \bar{\xi}_i(\varphi(x)) = \varphi(\xi_i(x))$ is satisfied on $\mathbf{i}$ for some choice of deviations $\xi_i, \bar{\xi}_i; i = 1, 2, \dots, m$ (see (6)) and we say that (82),(83) are *equivalent* equations.

6.1. An illustrative example of global transformations. Choosing

$$A(F^1, \dots, F^p) = \sum |F^j|^{s_j} = \sum a_j^{s_j}(x)|y|^{s_j} \prod |z_i|^{k_i^j s_j}, \ s_j \in \mathbb{R} - \{0\} \tag{84}$$

$(i = 1, \dots, m, \ j = 1, \dots, p)$ as a subcase in Theorem 4 we get

$$f = \sum a_j^{s_j}(x)y|y|^{s_j} \prod |z_i|^{k_i^j s_j + r_i} + p(x)y = \sum q_j(x)y|y|^{l_0^j} \prod |z_i|^{l_i^j} + p(x)y \tag{85}$$

with the invariance requirements

$$q_j = \bar{q}_j(\varphi)\varphi'|L|^{l_0^j} \prod |L_i|^{l_i^j}, \ L' = (\bar{p}(\varphi)\varphi' - p)L, \ L_i' = (\overline{B}^i\varphi' - B^i)L_i, \tag{86}$$

where $q_j = a_j^{s_j} q$, $l_0^j = s_j$, $l_i^j = k_i^j s_j + r_i$, $i = 1, \dots, m$, $j = 1, \dots, p$. Then any differential equation of the first order of the form

$$y'(x) = \sum q_j(x)y(x)|y(x)|^{l_0^j} \prod |y(\xi_i(x))|^{l_i^j} + p(x)y(x) \tag{87}$$

(for $z_i(x) = y(\xi_i(x))$, $x \in \mathbf{i} \subset \mathscr{D}(\varphi) \subset \mathbb{R}$) with deviating arguments $\xi_1, \dots, \xi_m$ is transformed into an equation

$$\overline{y}'(\bar{x}) = \sum \bar{q}_j(\bar{x})\overline{y}(\bar{x})|\overline{y}(\bar{x})|^{l_0^j} \prod |\overline{y}(\bar{\xi}_i(\bar{x}))|^{l_i^j} + \bar{p}(\bar{x})\overline{y}(\bar{x}) \tag{88}$$

(for $\bar{z}_i(\bar{x}) = \bar{y}(\bar{\xi}_i(\bar{x}))$, $\bar{x} \in \mathbf{j} \subset \mathbb{R}$) with deviations $\bar{\xi}_1, \dots, \bar{\xi}_m$ by means of transformation (2)

$$\bar{x} = \varphi(x), \quad \bar{y} = \bar{y}(\varphi(x)) = L(x)y(x),$$
$$\bar{z}_i = \bar{y}(\bar{\xi}_i(\bar{x})) = \bar{y}(\bar{\xi}_i(\varphi(x))) = \bar{y}(\varphi(\xi_i(x))) = L(\xi_i(x))y(\xi_i(x)) = L_i z_i \tag{89}$$

$(i = 1, \dots, m)$ if and only if (1) is a global transformation and the relations (86), (89) are satisfied. This statement follows from

$$\bar{y}'(\varphi)\varphi' = L'y + Ly',$$

i.e.,

$$\begin{aligned}\sum \bar{q}_j(\varphi)\varphi' Ly|L|^{l_0^j}|y|^{l_0^j} \prod |L_i|^{l_i^j} \prod |z_i|^{l_i^j} + \bar{p}(\varphi)\varphi' Ly \\ = L'y + L \sum q_j(x)y|y|^{l_0^j} \prod |z_i|^{l_i^j} + pLy,\end{aligned}$$

i.e.,

$$(L' + pL - \bar{p}(\varphi)\varphi' L)y + Ly \sum (q_j - \bar{q}_j(\varphi)\varphi' |L|^{l_0^j} \prod |L_i|^{l_i^j})|y|^{l_0^j} \prod |z_i|^{l_i^j} = 0$$

and (89). (The last invariance condition $L_i' = (\overline{B}^i\varphi' - B^i)L_i$ is always satisfied.)

Conjecture 1. *Let* (1) *be a global transformation of* (87) *into* (88). *Then* (87), (88) *are equivalent equations if and only if*

$$\bar{\xi}_j(\bar{x}) = \bar{\xi}_j(\varphi(x)) = \varphi(\xi_j(x))$$

is satisfied for some choice of deviations $\xi_j, \bar{\xi}_j$ *and*

$$q_j = \bar{q}_j(\varphi)\varphi' |L|^{l_0^j} \prod |L(\xi_i)|^{l_i^j}, \; L' = (\bar{p}(\varphi)\varphi' - p)L$$

$(i, j = 1, \dots, m)$ *on the interval* $\mathbf{i}$.

The symmetry equivalence problem for the equations (82), (83) and transformation (2) is solved in [18] by means of functional equations under the restricting conditions

$$\varphi' = g(x, \varphi, M, M_1, \dots, M_m), \; M' = h(x, \varphi, M, M_1, \dots, M_m), \tag{90}$$

$(M = M(x) = 1/L(x), \; M_i = M_i(x) = 1/L_i(x) = 1/L(\xi_i(x)), \; i = 1, \dots, m)$ in the class of point-continuous functions. The problem is resolved (see [18], Theorem 3) by means of functions

$$f(x,y,z_1,\dots,z_m)=\sum q_j(x)b_j(y)\prod \delta_{ij}(z_i)+p(x)y, \tag{91}$$

$$g(x,\varphi,M,M_1,\dots,M_m)=\frac{1}{\bar q_j(\varphi)M}q_j(x)b_j(M)\prod \delta_{ij}(M_j), \tag{92}$$

$$h(x,\varphi,M,M_1,\dots,M_m)=(p(x)-\bar p(\varphi)g(x,\varphi,M,M_1,\dots,M_m))M \tag{93}$$

($j=1,\dots,p,\ i=1,\dots,m$) where functions b_j, δ_{ij} are continuous solutions of *Cauchy's power equation* of the form $g(xy)=g(x)g(y)$, $g\colon \mathbb{R}^*\to\mathbb{R}$, $\mathbb{R}^*=\mathbb{R}-\{0\}$. The general solutions of Cauchy's power equation in the class of functions continuous at a point are given by $g(x)=0$, $g(x)=x^c$, $g(x)=|x|^c\operatorname{sgn}x$, $c\in\mathbb{R}$ being an arbitrary constant (see Aczél [1]). In our exposition,

$$b_j(y)=y|y|^{l_0^j},\ \delta_{ij}(z_i)=|z_i|^{l_i^j},\ i=1,\dots,m,\ j=1,\dots,p \tag{94}$$

are solutions of Cauchy's power equation. Substituting (90) and (94) into (91)-(93) we get (85), $(86)_{1,2}$ and Conjecture 1 can be applied to this exposition.

Remark 4. All results given in Theorems 1–3, Corollary 1 and 2 may be expressed in terms of the global transformations in a similar way. We need to guarantee the definition properties of global transformation (1) together with the commutativity requirements $\bar\xi_i(\varphi(x))=\varphi(\xi_i(x))$, $i=1,\dots,m$. The invariance requirements are the same both in the local and global approaches.

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