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ON A CLASS OF SZÁSZ-MIRAKYAN TYPE OPERATORS

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Abstract. The actual construction of the Szász-Mirakyan operators and its various modifications require estimations of infinite series which in a certain sense restrict their usefulness from the computational point of view. Thus the question arises whether the Szász-Mirakyan operators and their generalizations cannot be replaced by a finite sum. In connection with this question we propose a new family of linear positive operators.

Keywords: linear positive operator, polynomial weighted space

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1. Introduction

Approximation properties of the Szász-Mirakyan operators

$$(1)\qquad S_n(f;x) := \mathrm{e}^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\Big(\frac{k}{n}\Big), \quad x\in\mathbb{R}_0=[0,+\infty),\ \ n\in\mathbb{N}:=\{1,2,\ldots\},$$

in polynomial weighted spaces C_p were examined in [2]. The space C_p, $p\in\mathbb{N}_0 := \{0,1,2,\ldots\}$, considered in [2] is associated with the weight function

$$(2)\qquad w_0(x):=1, \quad w_p(x):=(1+x^p)^{-1} \quad \text{if } p\geqslant 1,$$

and consists of all real-valued functions f continuous on $\mathbb{R}_0$ and such that w_pf is uniformly continuous and bounded on $\mathbb{R}_0$. The norm on C_p is defined by

$$(3)\qquad \|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x\in\mathbb{R}_0} w_p(x)|f(x)|.$$

These operators are very interesting approximation processes and have many nice properties.

For f the results on the degree of approximation were studied in [2] for the usual Szász-Mirakyan operators (1). From these theorems it was deduced that

$$\lim_{n\to\infty} S_n(f;x) = f(x) \tag{4}$$

for every $f \in C_p$, $p \in \mathbb{N}_0$ and $x \in \mathbb{R}_0$. Moreover, the above convergence is uniform on every interval $[x_1, x_2]$, $x_1 \geqslant 0$.

Recently, in many papers various modifications of S_n were introduced and examined. They have been studied intensively in connection with different branches of analysis such as convex and numerical analysis. We refer the reader to P. Gupta and V. Gupta [9], V. Gupta [10], N. Ispir and C. Atakut [1], [18], V. Gupta, V. Vasishtha and M. K. Gupta [13], G. Feng [7], [8], A. Ciupa [5], N. Ispir [17], S. Li [21], X. Linsen and Z. Xiaoping [22]. Their results improve other related results in literature.

The actual construction of the Szász-Mirakyan operators (the Baskakov operators, the Favard operators, the Meyer-König and Zeller operators) and its many various modifications (see, for example, the works cited above) requires estimations of infinite series which in a certain sense restrict their usefulness from the computational point of view. Thus the question arises whether the Szász-Mirakyan operators and their generalizations cannot be replaced by a finite sum. In connection with this question, in [33] certain positive linear operators were considered, namely

$$L_n(f;x) := \frac{1}{(1+(x+n^{-1})^2)^n} \sum_{k=0}^{n} \binom{n}{k} (x+n^{-1})^{2k} f\Big(\frac{k}{n}\cdot\frac{1+(x+n^{-1})^2}{x+n^{-1}}\Big), \tag{5}$$

$x \in \mathbb{R}_0$, $n \in \mathbb{N}$, for a function of one variable.

In [33] it was proved that if $f \in C_p$, $p \in \mathbb{N}_0$, then

$$\lim_{n\to\infty} \|L_n(f;\cdot) - f(\cdot)\|_p = 0. \tag{6}$$

The operators L_n are defined in terms of a sample of the given function f on the points $(k/n)\cdot(1+(x+n^{-1})^2)/(x+n^{-1})$ for $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$.

Thus new questions arise whether the knots $(k/n)\cdot(1+(x+n^{-1})^2)/(x+n^{-1})$ cannot be replaced by a given subset of points which are independent of x, provided this would not change essentially the approximation properties.

In connection to these questions we propose a new family of linear positive operators. This together with the simple form of the operator makes the results given in the present paper more helpful from the computational point of view.

This note was inspired by the results obtained in our previous papers. In the papers [34], [29] and [36], for each function f of polynomial type defined on $\mathbb{R}_0$, the following operators were considered:

$$A_n^{\{1\}}(f;r;x) = \frac{1}{g(nx;r)} \sum_{k=0}^{\infty} \frac{(nx)^k}{(k+r)!} f\Big(\frac{k+r}{n}\Big), \quad x \in \mathbb{R}_0, \tag{7}$$

$$A_n^{\{2\}}(f;r;x) = \frac{1}{g((nx+1)^2;r)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+r)!} f\Big(\frac{k+r}{n(nx+1)}\Big), \tag{8}$$

and

$$A_n^{\{3\}}(f;p,r,s;x) = \frac{1}{g(n^s x;r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \sum_{j=0}^{p} \frac{f^{(j)}\big(\frac{k+r}{n^s}\big)}{j!} \Big(x - \frac{k+r}{n^s}\Big)^j, \tag{9}$$

where $n, r, p \in \mathbb{N}$, $s > 0$ and

$$g(t;r) = \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, \quad t \in \mathbb{R}_0, \tag{10}$$

i.e.

$$g(0;r) = \frac{1}{r!}, \quad g(t,r) = \frac{1}{t^r}\Big(\mathrm{e}^t - \sum_{j=0}^{r-1} \frac{t^j}{j!}\Big) \quad \text{if } t > 0.$$

Similar results in exponential weighted spaces can be found in [28], [30].

In this paper we will use the same method to obtain a new operator.

We introduce the following class of operators in C_p, $p \in \mathbb{N}$.

Definition 1. We introduce a class of new operators defined by

$$B_n(f;r;a_n;x) := \frac{1}{g(nx;r)} \sum_{k=0}^{[n(x+a_n)]} \frac{(nx)^k}{(k+r)!} f\Big(\frac{k+r}{n}\Big), \quad x \in \mathbb{R}_0, \ n \in \mathbb{N}, \tag{11}$$

where r is a fixed natural number and $(a_n)_1^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} n^{1/2} a_n = \infty$ and $[n(x + a_n)]$ denotes the integral part of $n(x + a_n)$.

Observe that the operator B_n is linear and positive.

Moreover, we will introduce certain linear positive operators in polynomial weighted spaces of functions of two variables.

Let $p, q \in \mathbb{N}_0$ and let

$$w_{p,q}(x,y) := w_p(x) w_q(y), \quad (x,y) \in \mathbb{R}_0^2 := \mathbb{R}_0 \times \mathbb{R}_0, \tag{12}$$

where $w_p(\cdot)$ is defined by (2). Denote by $C_{p,q}$ the weighted space of all real-valued functions f continuous on $\mathbb{R}_0^2$ for which $w_{p,q}f$ is uniformly continuous and bounded on $\mathbb{R}_0^2$. The norm on $C_{p,q}$ is defined by

$$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup_{(x,y)\in\mathbb{R}_0^2} w_{p,q}(x,y)|f(x,y)|. \tag{13}$$

Approximation properties of linear positive operators

$$\begin{aligned} L_{m,n}(f;x,y) &= \frac{1}{(1+(x+m^{-1})^2)^m(1+(y+n^{-1})^2)^n}\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k} \\ &\quad\times(x+m^{-1})^{2j}(y+n^{-1})^{2k} f\Big(\frac{j(1+(x+m^{-1})^2)}{m(x+m^{-1})}, \frac{k(1+(y+n^{-1})^2)}{n(y+n^{-1})}\Big), \\ A_{m,n}^{\{1\}}(f;r,s;x,y) &= \frac{1}{g(mx;r)g(ny;s)} \\ &\quad\times\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(mx)^j}{(j+r)!}\frac{(ny)^k}{(k+s)!}f\Big(\frac{j+r}{m}, \frac{k+s}{n}\Big), \quad r,s\in\mathbb{N}, \end{aligned} \tag{14}$$

in polynomial weighted spaces of functions of two variables were examined in [31] and [34].

In this paper we will give some properties of the following operators.

Definition 2. Fix $r,s\in\mathbb{N}$ and $p,q\in\mathbb{N}$. We define the class of operators $B_{m,n}$ by

$$\begin{aligned} B_{m,n}(f;r,s;a_m,b_n;x,y) &:= \frac{1}{g(mx;r)g(ny;s)} \\ \times\sum_{j=0}^{[m(x+a_m)]}\sum_{k=0}^{[n(y+b_n)]}&\frac{(mx)^j}{(j+r)!}\frac{(ny)^k}{(k+s)!}f\Big(\frac{j+r}{m},\frac{k+s}{n}\Big), \quad f\in C_{p,q},\ (x,y)\in\mathbb{R}_0^2, \end{aligned} \tag{15}$$

where $(a_m)_1^\infty$ and $(b_n)_1^\infty$ are given sequences of positive numbers such that

$$\lim_{m\to\infty} m^{1/2}a_m = \infty \quad \text{and} \quad \lim_{n\to\infty} n^{1/2}b_n = \infty.$$

Observe that the operator $B_{m,n}$ is linear and positive.

In this paper, by $K_i(\alpha,\beta)$, $i=1,2,\ldots$, we denote suitable positive constants depending only on the parameters α and β.

2. Preliminaries

In this section we will give some properties of the above operators which we will apply to the proofs of the main theorems.

It is known ([2]) that

$$S_n(1;x) = 1, \quad S_n(t-x;x) = 0, \tag{16}$$

$$S_n((t-x)^{q+1};x) = \frac{x}{n}\{S_n'((t-x)^q;x) + qS_n((t-x)^{q-1};x)\}, \tag{17}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $q \in \mathbb{N}$.

Using (16), (17) and mathematical induction on $q \in \mathbb{N}$ we can prove the following lemma.

Lemma 1 ([25])**.** *For every* $2 \leqslant q \in \mathbb{N}$ *we have*

$$S_n((t-x)^q;x) = \sum_{j=1}^{[q/2]} c_{j,q} \frac{x^j}{n^{q-j}}, \quad x \in \mathbb{R}_0, \ n \in \mathbb{N},$$

where $c_{j,q}$ *are positive numerical coefficients depending only on* j *and* q *(*$[y]$ *denotes the integral part of* $y \in \mathbb{R}_0$*).*

In the paper [34] the following results were proved for $A_n^{\{1\}}(f)$ defined by (7).

Lemma 2. $A_n^{\{1\}}$ *defines a positive linear operator* $C_p \to C_p$*.*

Theorem 1. *For every fixed* $r \in \mathbb{N}$ *and* $f \in C_p$, $p \in \mathbb{N}_0$, *we have*

$$\lim_{n\to\infty} \{A_n^{\{1\}}(f;r;x) - f(x)\} = 0, \quad x \in \mathbb{R}_0. \tag{18}$$

Moreover, (18) *holds uniformly on every interval* $[x_1, x_2]$, $x_2 > x_1 \geqslant 0$.

In [34] it was proved that if $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$, then $A_{m,n}^{\{1\}}$ is a positive linear operator $C_{p,q} \to C_{p,q}$. Moreover, we derived

Theorem 2. *Suppose that* $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$. *Then there exists a positive constant* $K_3(p,q,r,s)$ *such that for all* $(x,y) \in \mathbb{R}_0^2$

$$\begin{aligned} &w_{p,q}(x,y)|A_{m,n}^{\{1\}}(f;r,s;x,y) - f(x,y)| \\ &\qquad \leqslant K_3(p,q,r,s)\,\omega\Big(f, C_{p,q}; \sqrt{\frac{x+1}{m}}, \sqrt{\frac{y+1}{n}}\Big), \end{aligned} \tag{19}$$

$m, n \in \mathbb{N}$, $r, s \in \mathbb{N}$, *where*

$$\omega(f, C_{p,q}; t, s) := \sup_{0 \leqslant h \leqslant t,\, 0 \leqslant \delta \leqslant s} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{p,q}, \quad t, s \geqslant 0, \tag{20}$$

$\Delta_{h,\delta} f(x,y) := f(x+h, y+\delta) - f(x,y)$, $(x+h, y+\delta) \in \mathbb{R}_0^2$ *is the modulus of continuity of* $f \in C_{p,q}$.

From (20) it follows that

$$\lim_{t,s \to 0+} \omega(f, C_{p,q}; t, s) = 0 \tag{21}$$

for every $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$. This implies that

$$\lim_{m,n \to \infty} A_{m,n}^{\{1\}}(f; r, s; x, y) = f(x,y), \quad (x,y) \in \mathbb{R}_0^2 \tag{22}$$

uniformly on every rectangle $0 \leqslant x \leqslant x_0$, $0 \leqslant y \leqslant y_0$.

3. Main results

Now we give an approximation theorem for B_n.

Theorem 3. *Fix* $p, r \in \mathbb{N}$. *Then for* B_n *defined by* (11) *we have*

$$\lim_{n \to \infty} \{B_n(f; r; a_n; x) - f(x)\} = 0, \quad f \in C_p \tag{23}$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geqslant 0$.

Proof. We first suppose that $f \in C_p$, $p \in \mathbb{N}$. From (11) and (7) we obtain

$$\begin{aligned} B_n(f; r; a_n; x) - f(x) &= \frac{1}{g(nx; r)} \sum_{k=0}^{[n(x+a_n)]} \frac{(nx)^k}{(k+r)!} f\Big(\frac{k+r}{n}\Big) - f(x) \\ &= \frac{1}{g(nx; r)} \sum_{k=0}^{\infty} \frac{(nx)^k}{(k+r)!} f\Big(\frac{k+r}{n}\Big) - f(x) \\ &\quad - \frac{1}{g(nx; r)} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{(k+r)!} f\Big(\frac{k+r}{n}\Big) \\ &= A_n^{\{1\}}(f; r; x) - f(x) - M_n(f; r; x), \quad x \in \mathbb{R}_0, \ n, r \in \mathbb{N}. \end{aligned}$$

By our assumption, using the elementary inequality $(a+b)^k \leqslant 2^{k-1}(a^k + b^k)$, $a, b > 0$, $k \in \mathbb{N}_0$, we get

$$|f(t)| \leqslant K_1(1 + t^p) \leqslant K_1(1 + (|t-x| + x)^p) \leqslant K_1(1 + 2^{p-1}(|t-x|^p + x^p)). \tag{24}$$

Observe that

$$
\begin{aligned}
|M_n(f;r;x)| &\leqslant \frac{1}{g(nx;r)} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{(k+r)!}\Big|f\Big(\frac{k+r}{n}\Big)\Big| \\
&\leqslant \frac{1}{g(nx;r)} \sum_{k=[n(x+a_n)]+r+1}^{\infty} \frac{(nx)^{k-r}}{k!}\Big|f\Big(\frac{k}{n}\Big)\Big| \\
&\leqslant \frac{1}{(nx)^r g(nx;r)} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!}\Big|f\Big(\frac{k}{n}\Big)\Big|.
\end{aligned}
$$

This together with (10), (24) and (1) yields

$$
\begin{aligned}
|M_n(f;r;x)| &\leqslant \frac{1}{(nx)^r g(nx;r)} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} K_1\Big(1+2^{p-1}\Big(\Big|\frac{k}{n}-x\Big|^p + x^p\Big)\Big) \\
&\leqslant \frac{\mathrm{e}^{nx}}{\mathrm{e}^{nx} - \sum\limits_{j=0}^{r-1} (nx)^j/j!} K_1 \\
&\quad \times \Big((1+2^{p-1}x^p)\mathrm{e}^{-nx} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} + 2^{p-1}\mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!}\Big|\frac{k}{n}-x\Big|^p\Big) \\
&= \frac{\mathrm{e}^{nx}}{\mathrm{e}^{nx} - \sum\limits_{j=0}^{r-1} (nx)^j/j!} K_1 \\
&\quad \times \Big((1+2^{p-1}x^p)\mathrm{e}^{-nx} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} + 2^{p-1} S_n(|t-x|^p;x)\Big).
\end{aligned}
$$

We remark that

$$
\frac{\mathrm{e}^{t}}{\mathrm{e}^{t} - \sum\limits_{j=0}^{r-1} t^j/j!} = O(1)
$$

and

$$
\begin{aligned}
\mathrm{e}^{-nx} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} &\leqslant \mathrm{e}^{-nx} \sum_{a_n<|k/n-x|}^{\infty} \frac{(nx)^k}{k!} \\
&\leqslant \mathrm{e}^{-nx} \sum_{a_n<|k/n-x|}^{\infty} \frac{(nx)^k}{k!}\Big|\frac{k}{n}-x\Big|^p \frac{1}{a_n^p} \\
&\leqslant \frac{1}{a_n^p}\mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!}\Big|\frac{k}{n}-x\Big|^p = \frac{1}{a_n^p} S_n(|t-x|^p;x).
\end{aligned}
$$

This implies that

$$|M_n(f;r;x)| \leqslant K_2\left(\frac{(1+2^{p-1}x^p)}{a_n^p}+2^{p-1}\right)S_n(|t-x|^p;x).$$

Consequently, in view of Lemma 1, the Hölder inequality and (16), we further have

$$\begin{aligned}|M_n(f;r;x)| &\leqslant K_2\left(\frac{(1+2^{p-1}x^p)}{a_n^p}+2^{p-1}\right)\{S_n((t-x)^{2p};x)S_n(1;x)\}^{1/2}\\ &= K_2\left(\frac{(1+2^{p-1}x^p)}{a_n^p}+2^{p-1}\right)\left\{\sum_{j=1}^{p} c_{j,2p}\frac{x^j}{n^{2p-j}}\right\}^{1/2}\\ &\leqslant \frac{K_2}{n^{p/2}}\left(\frac{(1+2^{p-1}x^p)}{a_n^p}+2^{p-1}\right)\left\{\sum_{j=1}^{p} c_{j,2p}x^j\right\}^{1/2}.\end{aligned}$$

The relation

$$\lim_{n\to\infty} n^{1/2}a_n=\infty$$

implies that

$$\lim_{n\to\infty} M_n(f;r;x)=0$$

uniformly on every interval $[x_1,x_2]$, $x_2>x_1\geqslant 0$. In view of Theorem 1 this yields

$$\lim_{n\to\infty}\{B_n(f;r;a_n;x)-f(x)\}=0,$$

uniformly on every interval $[x_1,x_2]$, $x_2>x_1\geqslant 0$. This completes the proof of (23). □

Applying Theorem 2, we can prove the basic property of $B_{m,n}$.

Theorem 4. *Fix* $p,q,r,s\in\mathbb{N}$. *Then for* $B_{m,n}$ *defined by* (15) *we have*

(25) $$\lim_{m,n\to\infty} B_{m,n}(f;r,s;a_m,b_n;x,y)=f(x,y),\quad f\in C_{p,q}.$$

Moreover, (25) *holds uniformly on every rectangle* $0\leqslant x\leqslant x_0$, $0\leqslant y\leqslant y_0$.

P r o o f. Suppose that $f\in C_{p,q}$, $p,q\in\mathbb{N}$ and $r,s\in\mathbb{N}$. This implies that

$$|f(t,z)|\leqslant K_4(1+t^p)(1+z^q)\leqslant K_4(1+2^{p-1}(|t-x|^p+x^p))(1+2^{q-1}(|z-y|^q+y^q)).$$

From (15) and (14) we have

$$B_{m,n}(f;r,s;a_m,b_n;x)-f(x,y)=A_{m,n}^{\{1\}}(f;r,s;x,y)-f(x,y)-M_{m,n}(f;r,s;x,y)$$

where

$$M_{m,n}(f;r,s;x,y) = \frac{1}{g(mx;r)g(ny;s)} \times \sum_{j=[m(x+a_m)]+1}^{\infty} \sum_{k=[n(y+b_n)]+1}^{\infty} \frac{(mx)^j}{(j+r)!} \frac{(ny)^k}{(k+s)!} f\Big(\frac{j+r}{m}, \frac{k+s}{n}\Big), \qquad (x,y) \in \mathbb{R}_0^2.$$

Observe that

$$\begin{aligned} |M_{m,n}(f;r,s;x,y)| &\leqslant \frac{1}{g(mx;r)g(ny;s)} \times \sum_{j=[m(x+a_m)]+1}^{\infty} \sum_{k=[n(y+b_n)]+1}^{\infty} \frac{(mx)^j}{(j+r)!} \frac{(ny)^k}{(k+s)!} \Big| f\Big(\frac{j+r}{m}, \frac{k+s}{n}\Big)\Big| \\ &\leqslant K_4 \frac{1}{g(mx;r)} \sum_{j=[m(x+a_m)]+r+1}^{\infty} \frac{(mx)^{j-r}}{j!} \Big(1 + 2^{p-1}\Big(\Big|\frac{j}{m} - x\Big|^p + x^p\Big)\Big) \\ &\quad \times \frac{1}{g(ny;s)} \sum_{k=[n(y+b_n)]+s+1}^{\infty} \frac{(ny)^{k-s}}{k!} \Big(1 + 2^{q-1}\Big(\Big|\frac{k}{n} - y\Big|^q + y^q\Big)\Big). \end{aligned}$$

Arguing as in the second part of Theorem 3 we derive

$$\begin{aligned} \frac{1}{g(mx;r)} \sum_{j=[m(x+a_m)]+r+1}^{\infty} \frac{(mx)^{j-r}}{j!} \Big(1 + 2^{p-1}\Big(\Big|\frac{j}{m} - x\Big|^p + x^p\Big)\Big) &\leqslant \frac{K_5}{m^{p/2}} \Big(\frac{(1 + 2^{p-1}x^p)}{a_m^p} + 2^{p-1}\Big) \Big\{\sum_{j=1}^{p} c_{j,2p} x^j\Big\}^{1/2}, \\ \frac{1}{g(ny;s)} \sum_{k=[n(y+b_n)]+s+1}^{\infty} \frac{(ny)^{k-s}}{k!} \Big(1 + 2^{q-1}\Big(\Big|\frac{k}{n} - y\Big|^q + y^q\Big)\Big) &\leqslant \frac{K_5}{n^{q/2}} \Big(\frac{(1 + 2^{q-1}y^q)}{b_n^q} + 2^{q-1}\Big) \Big\{\sum_{j=1}^{q} c_{j,2q} y^j\Big\}^{1/2}. \end{aligned}$$

This yields in view of Definition 2

$$\lim_{m,n\to\infty} M_{m,n}(f;r,s;x,y) = 0$$

uniformly on every rectangle $0 \leqslant x \leqslant x_0$, $0 \leqslant y \leqslant y_0$. Applying Theorem 2 and (26) we immediately obtain (25). □

It is similarly verified that analogous approximation properties hold for the two operators

$$T_n(f;r;x) := \frac{1}{g(nx;r)} \sum_{k=0}^{nr} \frac{(nx)^k}{(k+r)!} f\Big(\frac{k+r}{n}\Big),$$

$f \in C_{[0,2r]}$, $x \in [0,r)$, $n, r \in \mathbb{N}$,

$$T_{m,n}(f;r,s;x,y) := \frac{1}{g(mx;r)g(ny;s)} \times \sum_{j=0}^{mr} \sum_{k=0}^{ns} \frac{(mx)^j}{(j+r)!} \frac{(ny)^k}{(k+s)!} f\Big(\frac{j+r}{m}, \frac{k+s}{n}\Big),$$

$f \in C_{[0,2r],[0,2s]}$, $(x,y) \in [0,r) \times [0,s)$, $m,n,r,s \in \mathbb{N}$.

Observe that the operators T_n, $n \in \mathbb{N}$, are obtained from (11) for $a_n = r - x$, $x \in [0,r)$, $r \in \mathbb{N}$.

Analogously we obtain

$$B_{m,n}(f;r,s;r-x,s-y;x,y) = T_{m,n}(f;r,s;x,y), \quad (x,y) \in [0,r) \times [0,s), \ \ m,n,r,s \in \mathbb{N}.$$

The methods used to prove the Theorems are similar to those used in the construction of the modified Szász-Mirakyan operators [19], [31], [34], [38].

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