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ON THE DETERMINATION OF THE POTENTIAL FUNCTION FROM GIVEN ORBITS

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Abstract. The paper deals with the problem of finding the field of force that generates a given $(N-1)$-parametric family of orbits for a mechanical system with N degrees of freedom. This problem is usually referred to as the inverse problem of dynamics. We study this problem in relation to the problems of celestial mechanics. We state and solve a generalization of the Dainelli and Joukovski problem and propose a new approach to solve the inverse Suslov's problem. We apply the obtained results to generalize the theorem enunciated by Joukovski in 1890, solve the inverse Stäckel problem and solve the problem of constructing the potential-energy function U that is capable of generating a bi-parametric family of orbits for a particle in space. We determine the equations for the sought-for function U and show that on the basis of these equations we can define a system of two linear partial differential equations with respect to U which contains as a particular case the Szebehely equation. We solve completely a special case of the inverse dynamics problem of constructing U that generates a given family of conics known as Bertrand's problem. At the end we establish the relation between Bertrand's problem and the solutions to the Heun differential equation. We illustrate our results by several examples.

Keywords: ordinary differential equations, mechanical system, potential-energy function, inverse problem of dynamics, orbit, Riemann metric, Stäckel system, Heun equation

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1. Introduction

One of the fundamental classical problems in celestial mechanics is to determine the potential-energy function U such that every curve from a given family of curves

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will be a possible trajectory of a particle moving under the action of potential forces F admitting U; i.e. $F = \operatorname{grad} U$.

The importance of this problem was already acknowledged by Szebehely and Bozis [20], [3].

The first inverse problem in celestial mechanics was stated and solved by Newton [14] and concerns the determination of the potential field of force that ensures the planetary motion in accordance to the observed properties, namely to Kepler's laws.

Bertrand [2] proved that the expression for Newton's force of attraction can be obtained directly from Kepler's first law to within a constant multiplier.

Bertrand stated also a more general problem of determining a positional force under which a particle describes a conic section under any initial conditions.

If we denote by c the constant from Kepler's second law (angular momentum, sometimes referred to as the area integral) and consider the motion of the particle in a circle as a relative equilibrium in accordance with V. Arnold [1], we have the following Bertrand theorem:

Theorem 1.1 ([2]). *Suppose that for some $c \neq 0$ there exists a stable relative equilibrium and that the effective potential has the form*

$$U_c = U(x, y) + \frac{mc^2}{2r^2}$$

where m is the mass of the particle, $r = \sqrt{x^2 + y^2}$ and U is analytic for $r > 0$.

If all orbits that are sufficiently close to the given circular orbit are closed, then either $U = \gamma r^2$ or $U = -\gamma/r$ where $\gamma > 0$.

In the former case the system represents a harmonic oscillator and its orbits are ellipses centered at the point $r = 0$. The latter case is that of gravitational attraction. The problem of the motion of a point in a conservative force field with potential $U = -\gamma/r$ is usually called Kepler's problem.

The ideas of Bertrand were developed by Dainelli [5], Suslov [19], Joukovski [9], Ermakov [7], and Galiullin [8].

Dainelli in [5] essentially states a more general problem of how to determine the most general field of force (the force being supposed to depend only on the position of the particle on which it acts) under which a given family of planar curves is a family of orbits of a particle.

The solution proposed by Dainelli is the following [22].

The most general field of force $\mathbf{F} = (F_x, F_y)$ capable of generating a family of planar orbits $f(x, y) = \text{const}$ can be determine as follows [5], [22], [18]:

$$
\begin{cases} F_x = -\lambda^2\{f, \partial_y f\} - \lambda\{f,\, \lambda\}\partial_y f, \\ F_y = \lambda^2\{f, \partial_x f\} + \lambda\{f,\, \lambda\}\partial_x f \end{cases} \tag{1.1}
$$

where λ is an arbitrary function which depends on the velocity with which the given orbits are described. Considering that the components F_x and F_y are to be functions of the position of the particle, we can take λ to be an arbitrary function of x and y.

The above expressions for the field of force under which the curves of the given family are orbits were first given by Dainelli [5].

In [19], Suslov stated and solved a problem which was a further development of Bertrand's problem. He showed that, given an $(N-1)$-parametric family of orbits in the configuration space of a holonomic system with N degrees of freedom and kinetic energy $T = \frac{1}{2}\sum\limits_{j,k=1}^{N} G_{jk}(x)\dot{x}^j\dot{x}^k$, it is necessary to determine the potential field of force under which any trajectory of the family can be traced by the representative point of the system.

Suslov deduced the following system of linear partial differential equations with respect to the required potential function:

$$
\begin{gathered}
\frac{\partial\theta}{\partial\Delta_k}\frac{\partial U}{\partial x^N} - \frac{\partial\theta}{\partial\Delta_N}\frac{\partial U}{\partial x^k} \\
= \frac{U+h}{\theta}\Big(\frac{\partial\theta}{\partial\Delta_N}\frac{\partial\theta}{\partial x^k} - \frac{\partial\theta}{\partial\Delta_k}\frac{\partial\theta}{\partial x^N} + \sum_{m=1}^{N}\Delta^m\Big(\frac{\partial\theta}{\partial\Delta_k}\frac{\partial^2\theta}{\partial\Delta_N\partial x^m} - \frac{\partial\theta}{\partial\Delta_N}\frac{\partial^2\theta}{\partial\Delta_k\partial x^m}\Big)\Big), \\
k = 1, 2, \dots, N-1
\end{gathered}
$$

where $\theta, \Delta^1, \Delta^2, \dots, \Delta^N$ are defined by

$$
\sum_{k^1}^{N}\frac{\partial f_\alpha}{\partial x^k}\Delta^k = 0, \quad \Delta_k = \sum_{j=1}^{N} G_{jk}(x)\Delta^j, \quad k = 1, 2, \dots, N, \quad \alpha = 1, 2, \dots, N-1,
$$

$$
\theta = \frac{1}{2}\sum_{k,j=1}^{N} G_{kj}(x)\Delta^k\Delta^j \equiv \theta(x^1, x^2, \dots, x^N, \Delta_1, \Delta_2, \dots, \Delta_N)
$$

and proved that these equations represent necessary and sufficient conditions under which the equations of motion of the studied mechanical system admit the given $N-1$ partial integrals.

Assuming that the given trajectories admit a family of orthogonal surfaces, Joukovski in [9] constructed the potential-energy functions in explicit form for systems with two and three degrees of freedom.

The following theorem was enunciated by Joukovsky in 1890: If $q = \text{const}$ is the equation of a family of curves on a surface and $p = \text{const}$ denotes the family of curves orthogonal to them, then the curves $q = \text{const}$ can be freely described by a particle under the influence of forces derived from the potential-energy function

$$V = \Delta_1(p)\left(g(p) + \int h(q)\frac{\partial}{\partial q}\Big(\frac{1}{\Delta_1(p)}\Big)\,\mathrm{d}q\right)$$

where h and g are arbitrary functions and Δ_1 denotes the first differential parameter.

A new approach to the problem of constructing the potential field of force was proposed by Ermakov in [7], who integrated the equations for the potential-energy function for several particular cases.

In the most general form the inverse problem in dynamics was studied in [18], [16]. Applying the results presented in that work we propose the following new results:

1. Generalization of the Dainelli problem of a mechanical system with N degrees of freedom.
2. Generalization of the Joukovski problem and extension of the Joukovski theorem to mechanical system with $N \geqslant 3$ degrees of freedom.
3. Complete solution of the inverse Stäckel and Bertrand problems.
4. The relation between the Bertrand problem and solutions to a particular class of the Heun equation.

2. Solution of the generalized Dainelli problem

First, we introduce the necessary notation and give a brief overview of the main results obtained in [18].

Let X be a smooth manifold of dimension N with local coordinates $x = (x^1, \ldots, x^N)$ and equipped with the Riemann metric $G = (G_{kj}(x))$.

By $\xi(X)$, $\Lambda(X)$, ∇ we denote respectively the Lie algebra of vector fields on X, the algebra of the 1-form on X, and the Levi-Civita connection

$$\begin{aligned}\nabla\colon\ \xi(X) \times \xi(X) &\longmapsto \xi(X),\\ (u, v) &\longmapsto \nabla_u v\end{aligned}$$

which is $\mathbb{R}$ lineal with respect to v and C^∞ lineal with respect to v and is compatible with the metric G, i.e., $\nabla_u G(v, w) = 0$, for all $u, v, w \in \xi(X)$.

The vector field $v \in \xi(X)$ is called an integral element of $\Omega \in \Lambda(X)$ if $\Omega(v) = 0$. We shall denote by $K(\Omega_1, \Omega_2, \ldots \Omega_M)$, $M \leqslant N$ the set of the integral elements of independent 1-forms $\Omega_1, \Omega_2, \ldots, \Omega_M$.

Proposition 2.1. *The most general element of $K(\Omega_1, \Omega_2, \ldots, \Omega_M)$ admits the representation*

$$
\mathbf{v} = \det \begin{pmatrix}
\Omega_1(\partial_1) & \Omega_1(\partial_2) & \ldots & \Omega_1(\partial_N) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Omega_M(\partial_1) & \Omega_M(\partial_2) & \ldots & \Omega_M(\partial_N) & 0 \\
\Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \ldots & \Omega_{M+1}(\partial_N) & \lambda_{M+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Omega_N(\partial_1) & \Omega_N(\partial_2) & \ldots & \Omega_N(\partial_N) & \lambda_N \\
\partial_1 & \partial_2 & \ldots & \partial_N & 0
\end{pmatrix}, \tag{2.1}
$$

where $\partial_k = \partial/\partial x^k$, $\Omega_1, \Omega_2, \ldots, \Omega_M$, $M \leqslant N-1$ are given 1-forms, and Ω_{M+1}, $\Omega_{M+2}, \ldots, \Omega_N$, are arbitrary 1-forms on X. Furthermore, we assume that they are pointwise independent, i.e.

$$
\Upsilon \equiv \Omega_1 \wedge \Omega_2 \ldots \wedge \Omega_N(\partial_1, \partial_2, \ldots, \partial_N) \neq 0.
$$

It is important to observe that the arbitrary 1-forms can be determined only from the above condition. The functions λ_j, $j = M+1, \ldots, N$ are arbitrary functions on X.

Let σ be the 1-form associated with the vector field $\mathbf{v}$, i.e.,

$$
\sigma = (\mathbf{v}(x), \mathrm{d}x) \equiv \sum_{j,k=1}^{N} G_{jk}(x) v^j(x)\, \mathrm{d}x^k \equiv \sum_{k=1}^{N} v_k\, \mathrm{d}x^k.
$$

Then the 2-form $\mathrm{d}\sigma$ is $\mathrm{d}\sigma = \frac{1}{2} \sum\limits_{j,k=1}^{N} a_{jk}(x) \Omega_j \wedge \Omega_k$, where $A = (a_{jk})$ is a matrix such that

$$
a_{jk} = (-1)^{j+k-1} \frac{1}{\Upsilon} \mathrm{d}\sigma \wedge \Omega_1 \wedge \ldots \wedge \widehat{\Omega}_k \ldots \wedge \widehat{\Omega}_j \ldots \wedge \Omega_N(\partial_1, \partial_2, \ldots, \partial_N); \tag{2}
$$

$\widehat{\Omega}_j$, $\widehat{\Omega}_k$ means that these elements are omitted.

It is clear that the contraction of $\mathrm{d}\sigma$ along $\mathbf{v}$ is

$$
\iota_{\mathbf{v}}\, \mathrm{d}\sigma = \sum_{j=1}^{N} \Lambda_j \Omega_j, \quad \text{where } \boldsymbol{\Lambda} \equiv \mathrm{col}(\Lambda_1, \Lambda_2, \ldots, \Lambda_N) = A^T \boldsymbol{\lambda}. \tag{2.3}
$$

We shall analyze the differential equations generated by the vector field $\mathbf{v}$

$$
\dot{\mathbf{x}} = \mathbf{v}(x), \quad x \in X \tag{2.4}
$$

under the conditions

$$\begin{cases} \Lambda_j = 0, \quad j = M+1, \ldots, N, \\ \Upsilon = \Omega_1 \wedge \Omega_2 \ldots \wedge \Omega_N(\partial_1, \partial_2, \ldots, \partial_N) \neq 0. \end{cases} \tag{2.5}$$

We denote by

$$\mathcal{M} = \left\langle X,\, T = \frac{1}{2}\|\dot{x}\|^2,\, \omega = \sum_{k=1}^{N} F_k(x, \dot{x})\,\mathrm{d}x^k \right\rangle$$

a mechanical system [10] with the configuration space X, whose dimension is N, and with local coordinates $x = (x^1, \ldots x^N)$. Consequently, the kinetic energy is expressed as

$$T = \frac{1}{2}\|\dot{x}\|^2 \equiv \frac{1}{2}\sum_{k,j=1}^{N} G_{kj}(x)\dot{x}^k\dot{x}^j$$

and the field of force as

$$\omega = \sum_{k=1}^{N} F_k\,\mathrm{d}x^k.$$

The equations of motion of $\mathcal{M}$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}^k} - \frac{\partial T}{\partial x^k} = \omega\Big(\frac{\partial}{\partial x^k}\Big). \tag{2.6}$$

Proposition 2.2 ([16]). *The differential equations* (2.4)+(2.5) *are invariant relationship of the equations* (2.6) *with*

$$\omega = \mathrm{d}\frac{1}{2}\|\mathbf{v}\|^2 + \sum_{j=1}^{M}\Lambda_j\Omega_j. \tag{2.7}$$

Clearly, the differential equations (2.6)+(2.7) can be interpreted as the equations of motion of nonholonomic mechanical systems with an active potential field of force with potential $\frac{1}{2}\|\mathbf{v}(x)\|^2$ and with the reactive forces which have components

$$\Big(\sum_{j=1}^{M}\Lambda_j\Omega_j(\partial_1), \sum_{j=1}^{M}\Lambda_j\Omega_j(\partial_2), \ldots, \sum_{j=1}^{M}\Lambda_j\Omega_j(\partial_N)\Big) \tag{2.8}$$

generated by the constraints

$$\sum_{k=1}^{N}\Omega_j(\partial_k)\dot{x}^k = 0, \quad j = 1, 2, \ldots, M.$$

Corollary 2.1. *Let us suppose that* $\Omega_j = \mathrm{d}f_j$, $j = 1, 2, \ldots, N-1$. *Then the* 1*-form* (2.7) *takes on the form*

$$\omega = \mathrm{d}\frac{1}{2}\|\mathbf{v}\|^2 + \sum_{j=1}^{N-1} \lambda a_{Nj}\,\mathrm{d}f_j \tag{2.9}$$

where

$$\begin{cases} a_{Nj} = (-1)^{N+j-1}\mathrm{d}\sigma \wedge \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \ldots \wedge \mathrm{d}f_{j-1} \wedge \mathrm{d}f_{j+1} \wedge \ldots \wedge \mathrm{d}f_{N-1}(\partial_1, \ldots, \partial_N) \\ a_{N1} = (-1)^N \mathrm{d}\sigma \wedge \mathrm{d}f_2 \wedge \ldots \wedge \mathrm{d}f_{N-1}(\partial_1, \ldots, \partial_N). \end{cases}$$

Definition (Generalized Dainelli's problem)**.** Given an $(N-1)$-parametric family of orbits in the configuration space of a holonomic system with N degrees of freedom and kinetic energy T, the generalized Dainelli problem is the problem of determining the most general field of force that depends only on the position of the system under which any trajectory of the family can be traced by a representative point of the system.

Proposition 2.3 (Solution of the generalized Dainelli problem)**.** *Given a mechanical system* $\mathcal{M}$ *with a configuration space* X *and kinetic energy* T*, then the most general field of force that depends only on the position of the system and is capable of generating the given orbits*

$$f_j(x) = c_j, \quad j = 1, \ldots, N-1$$

is described by the equation (2.9).

Here $f_1, \ldots, f_{N-1}$ are independent functions of class $C^r(\tilde{X} \subseteq X)$, $r \geqslant 2$, $\mathbf{v}$ is the vector field

$$\mathbf{v} = \lambda \begin{vmatrix} \mathrm{d}f_1(\partial_1) & \ldots & \mathrm{d}f_1(\partial_N) \\ \vdots & \ddots & \vdots \\ \mathrm{d}f_{N-1}(\partial_1) & \ldots & \mathrm{d}f_{N-1}(\partial_N) \\ \partial_1 & \ldots & \partial_N \end{vmatrix} \equiv \lambda\{f_1, \ldots, f_{N-1}, *\}, \tag{2.10}$$

λ is an arbitrary function, $\partial_j \equiv \partial/\partial x_j$. Clearly, this result represents a generalization of the ideas given by Dainelli in [5].

Proof of this proposition follows from Corollary 2.1. □

Corollary 2.2. *The field of force* (2.9) *assumes for a particle in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$ *respectively the following forms:*

$$\omega = \mathrm{d}\frac{1}{2}\lambda^2\big((\partial_x f)^2 + (\partial_y f)^2\big) + \lambda\big(\partial_x(\lambda f_x) + \partial_y(\lambda f_y)\big)\,\mathrm{d}f, \tag{2.11}$$

$$\begin{cases} \omega = \mathrm{d}\dfrac{\|\mathbf{v}\|^2}{2} + \lambda\imath_{\operatorname{rot}\mathbf{v}}(\mathrm{d}f_1 \wedge f_2) \\ \mathbf{v} = \lambda(x, y, z)\operatorname{grad} f_1 \times \operatorname{grad} f_2. \end{cases} \tag{2.12}$$

It is possible to show that (2.11) coincides with (1.1) [18].

In the next section we make use of the solution of the generalized Dainelli inverse problem for studying particular cases of the Suslov and of the generalized Joukovski problems.

3. Solution of the Suslov and generalized Joukovski problems

Definition (Suslov's problem)**.** Given an $(N-1)$-parametric family of orbits in the configuration space of a holonomic system with N degrees of freedom and kinetic energy T, Suslov's problem is the problem of determining the potential field of force under which any trajectory of the family can be traced by a representative point of the system.

If we assume that the field of force (2.9) is potential, then we obtain from Proposition 2.3 the solution of Suslov's problem [19].

Solution to Suslov's problem

Proposition 3.1. *The field of force* (2.9) *is potential, i.e.,* $\omega = -\mathrm{d}U(x)$ *if and only if*

$$\lambda \sum_{j=1}^{N-1} a_{Nj}(x)\,\mathrm{d}f_j = -\mathrm{d}h(f_1, f_2, \dots, f_{N-1}). \tag{3.1}$$

The function U *is given by*

$$U(x) = \frac{1}{2}\|\mathbf{v}(x)\|^2 - h(f_1, f_2, \dots, f_{N-1}). \tag{3.2}$$

Another interesting application of the solution to the generalized Dainelli problem is the determination of the solution of the generalized Joukovski problem.

Definition (Generalized Joukovski problem)**.** The generalized Joukovski problem is a particular case of the Suslov problem, which is obtained by assuming that the vector field (2.10) has the form

$$\{f_1, f_2, \ldots, f_{N-1}, *\} = \nu(x)\nabla f_N, \tag{3.3}$$

where

$$\nabla f_N = \sum_{j=1}^{N} G^{jk}(x)\partial_j f_N \partial_k,$$

$G^{-1} = (G^{jk})$ is the inverse matrix of the Riemann metric G and ν is a function such that

$$\operatorname{div}(\nu(x)\nabla f_N) = 0.$$

Clearly, from (3.3) we obtain that the 1-form associated with the vector field $\mathbf{v}$ is such that

$$\sigma = \Gamma(x)\,\mathrm{d}f_N(x), \quad \Gamma = \nu\lambda.$$

Taking into account the expression for the scalar product in the Riemann space with the metric G, we obtain that

$$(\mathbf{grad}\, f_N,\, \mathbf{grad}\, f_\alpha) \equiv \sum_{j,k} G^{jk}\partial_j f_N \partial_k f_\alpha.$$

On the other hand, in view of the equalities

$$(\mathbf{grad}\, f_N, \mathbf{grad}\, f_k) = \{f_1, f_2, \ldots, f_{N-1}, f_k\} = 0, \quad k = 1, \ldots, N-1$$

we deduce that the function f_N is orthogonal to the given functions $f_1, f_2, \ldots, f_{N-1}$.

The stated problem coincides with the Joukovski problem when $N = 3$ [22], [9].

Solution of the generalized Joukovski problem

Proposition 3.2. *The field of force expressed by equations* (2.9) *is potential if and only if*

$$\Gamma\imath_{\mathbf{grad}\, f_N}\big(\mathrm{d}(\Gamma \mathrm{d}f_N)\big) = -\mathrm{d}h(f_1, f_2, \ldots, f_{N-1}). \tag{3.4}$$

Clearly, if $\Gamma = \Gamma(f_N)$ then $\mathrm{d}h = 0$ and the required potential-energy function U is

$$U = \frac{1}{2}\Gamma(f_N)\|\nabla f_N\|^2 - h_0. \tag{3.5}$$

Let us illustrate this result by determining a solution of the inverse problem which we will call the inverse *Stäckel problem.*

Definition (Stäckel system). The Stäckel system is the triplet

$$\mathcal{M} = \left\langle X,\, T = \frac{1}{2}\sum_{k=1}^{N} \frac{\dot{x}_k^2}{A^k(x)},\, \omega = \mathrm{d}U(x) \right\rangle,$$

where $A^1, \ldots, A^N, U$ are the functions [4]

$$A^k(x) = \frac{1}{\Delta}\frac{\partial \Delta}{\partial \varphi_{k1}(x^k)}, \tag{3.6}$$

$$U(x) = \sum_{k=1}^{N} \Psi_k(x^k) A^k, \tag{3.7}$$

$$\Delta = \det \begin{pmatrix} \mathrm{d}\varphi_1(\partial_1) & \ldots & \mathrm{d}\varphi_1(\partial_N) \\ \vdots & \ddots & \vdots \\ \mathrm{d}\varphi_N(\partial_1) & \ldots & \mathrm{d}\varphi_N(\partial_N) \end{pmatrix} = \mathrm{d}\varphi_1 \wedge \ldots \wedge \mathrm{d}\varphi_N(\partial_1, \ldots, \partial_N),$$

$$\mathrm{d}\varphi_\alpha = \sum_{k=1}^{N} \varphi_{k\alpha}(x^k)\,\mathrm{d}x^k,$$

$\varphi_{k\alpha}$, Ψ_k are arbitrary functions, $k = 1, \ldots, N$, $\alpha = 2, \ldots, N$.

It is clear that the functions $A^1, \ldots, A^N$ can be represented as

$$A^k(x) = \frac{(-1)^{k+1}}{\Delta}\mathrm{d}\varphi_2 \wedge \ldots \wedge \mathrm{d}\varphi_N(\partial_1, \ldots, \widehat{\partial_k}, \ldots, \partial_N), \quad k = 1, \ldots, N,$$

where $\widehat{\partial_k}$ means that ∂_k is omitted.

By using this notation we can prove that

$$U(x) = \frac{1}{\Delta}\det \begin{pmatrix} \mathrm{d}\Psi(\partial_1) & \ldots & \mathrm{d}\Psi_1(\partial_N) \\ \mathrm{d}\varphi_2(\partial_1) & \ldots & \mathrm{d}\varphi_2(\partial_N) \\ \vdots & \ddots & \vdots \\ \mathrm{d}\varphi_N(\partial_1) & \ldots & \mathrm{d}\varphi_N(\partial_N) \end{pmatrix} = \frac{\mathrm{d}\Psi \wedge \ldots \wedge \mathrm{d}\varphi_N(\partial_1, \ldots, \partial_N)}{\mathrm{d}\varphi_1 \wedge \ldots \wedge \mathrm{d}\varphi_N(\partial_1, \ldots, \partial_N)},$$

where $\mathrm{d}\Psi = \sum\limits_{k=1}^{N} \Psi_k(x^k)\mathrm{d}x^k$.

The trajectories of the Stäckel system are [4]

$$f_{\mu-1}(x) \equiv \sum_{k=1}^{n} \int \frac{\varphi_{k\mu}(x^k)}{\sqrt{K_k(x^k)}}\mathrm{d}x^k = c_\mu, \quad \mu = 2, \ldots, N, \tag{3.8}$$

where $K_k(x^k) = 2\Psi_k(x^k) + 2\sum\limits_{j=1}^{N} \alpha_j \varphi_{kj}(x^k)$, α_j, $k = 1, 2, \ldots, N$, are constants.

We define the *inverse Stäckel problem* as follows [18].

Definition (Inverse Stäckel problem)**.** Let $\mathcal{M}$ be a mechanical system with a configuration space X and kinetic energy

$$T = \frac{1}{2}\sum_{k=1}^{N} \frac{\dot{x}_k^2}{A^k(x)},$$

where $A^1, \ldots, A^N$ are functions determined by (3.6).

The problem of constructing the potential field of force

$$\omega = \mathrm{d}U(x, y)$$

which is capable of generating the orbits (3.8) is called the inverse Stäckel problem.

In this case the field of force (2.9) takes on the form [16]

$$\omega = \mathrm{d}\left(\Gamma^2(x)\left(\sum_{n=1}^{N} A^n \Psi(x^n) + \alpha_1\right)\right) + \mathrm{d}\Gamma(\mathbf{v})\,\mathrm{d}f_N - \left(\sum_{n=1}^{N} A^n \Psi(x^n) + \alpha_1\right)\mathrm{d}\Gamma^2,$$

where $\mathbf{v} = \Gamma \nabla f_N = \sum\limits_{k=1}^{N} \Gamma A^k \sqrt{K_k(x^k)}\partial_k$.

It is clear that ω is potential if $\Gamma = \Gamma(f_N)$. Under this restriction we obtain the following expression for the potential-energy function:

$$U(x) = \Gamma^2(f_N)\left(\sum_{n=1}^{N} A^n \Psi_n(x^n) + \alpha_1\right) - h_0.$$

By choosing

$$\Gamma^2(f_N) = 1, \quad h_0 = \alpha_1,$$

we deduce exactly the potential function (3.7).

Example. Let us consider a mechanical system $\mathcal{M}$ with kinetic energy $T = \frac{1}{2}\sum\limits_{j,k=1}^{N} G_{jk}(x)\dot{x}^j\dot{x}^k$ and a configuration space X. Let us also suppose that the given $N-1$-parametric family of orbits is

$$x^j = C_j, \quad j = 1, 2, \ldots, N-1.$$

We are required to solve the Suslov problem under these conditions.

Noticing that for the given orbits the vector field $\mathbf{v}$ is

$$\mathbf{v} = \lambda \partial_{x^N}$$

we can easily see that (3.1) takes on the form

$$\sum_{j=1}^{N-1} \lambda(\partial_N(\lambda G_{Nj}) - \partial_j(\lambda G_{NN}))\, \mathrm{d}x^j = -\mathrm{d}h(x^1, x^2, \ldots, x^{N-1}). \tag{3.9}$$

By determining the function λ_N as a solution of the equation

$$\sum_{j=1}^{N-1} \mathrm{d}(\lambda(\partial_N(\lambda\, G_{Nj}) - \partial_j(\lambda G_{NN})) \wedge\, \mathrm{d}x^j = 0 \tag{3.10}$$

we obtain the following form of the required function U:

$$\begin{aligned} U(x^1, x^2, \ldots, x^N) = \frac{1}{2}\lambda^2(x^1, x^2, \ldots, x^N) G_{NN}(x^1, x^2, \ldots, x^N) \\ - h(x^1, x^2, \ldots, x^{N-1}). \end{aligned} \tag{3.11}$$

In particular, if the Riemann metric G is such that

$$G_{Nj} = 0, \quad j = 1, 2, \ldots, N-1$$

then from (3.9)+(3.10) we obtain that the function λ_N can be determined as

$$\begin{aligned} \lambda^2 = \frac{2}{G_{NN}^2} \Bigg(& g(x^N) + h(x^1, x^2, \ldots, x^{N-1}) \\ & - \sum_{j=1}^{N-1} \int h(x^1, x^2, \ldots, x^{N-1}) \frac{\partial G_{NN}(x^1, x^2, \ldots, x^N)}{\partial x^j}\, \mathrm{d}x^j \Bigg) \end{aligned} \tag{3.12}$$

where h and g are arbitrary functions.

By inserting (3.12) in (3.11) we prove the following proposition which represents an extension of the Joukovski theorem for a mechanical system with N degrees of freedom.

Proposition 3.3. *If $x^j = C_j = \mathrm{const}$, $j = 1, 2, \ldots, N-1$ are the equations of an $N-1$ parametric family of curves on X, and $x^N = \mathrm{const}$ denotes the family of curves orthogonal to them, then the curves $x^j = C_j = \mathrm{const}$ can be freely described by a particle under the influence of forces derived from the potential-energy function*

$$U = \frac{1}{G_{NN}(x^1, x^2, \ldots, x^N)} \times \left(g(x^N) + \sum_{j=1}^{N-1} \int h(x^1, x^2, \ldots, x^{N-1}) \frac{\partial G_{NN}(x^1, x^2, \ldots, x^N)}{\partial x^j} \, \mathrm{d}x^j \right)$$

where h and g are arbitrary functions.

Clearly, for $N = 2$ we obtain exactly the Joukovski result given in the introduction.

4. The Suslov problem for a particle in $\mathbb{R}^3$

In this section we study the Suslov problem for a particle in space.

Proposition 4.1. *The field of force* (2.11) *for a particle in $\mathbb{R}^2$ is potential if and only if*

$$\lambda\big(\partial_x(\lambda f_x) + \partial_y(\lambda f_y)\big)\, \mathrm{d}f = -\mathrm{d}h(f), \tag{4.1}$$

and in $\mathbb{R}^3$ if and only if

$$\lambda \imath_{\mathrm{rot}\,\mathbf{v}}(\mathrm{d}f_1 \wedge f_2) = -\mathrm{d}h(f_1, f_2) \tag{4.2}$$

where

$$\mathbf{v} = \lambda(x, y, z)\, \mathbf{grad}\, f_1 \times \mathbf{grad}\, f_2. \tag{4.3}$$

Proof is easily obtained from Corollary 2.2 and Proposition 3.1.1. □

In 1974 Szebehely obtained a linear first-order partial differential equation for the potential function U which gives rise to a one-parameter family of planar orbits with a given total energy h. This result initiated many works on inverse problems (see for instance [3]). The equation of Szebehely was generalized to a two-parameter family of three-dimensional orbits by Erdi (1982), Bozis (1983) and Puel (1984).

Below we show that the results, presented in those works, can be obtained from the solutions of the Suslov problem.

Corollary 4.1. *Equation* (4.1) *is equivalent to the equation*

$$\{\partial_x f \partial_x U + \partial_y f \partial_y U + 2(U+h)\|\operatorname{grad} f\| K(x,y) = 0, \tag{4.4}$$
$$K = \operatorname{div}\Big(\frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}\Big), \quad \|\operatorname{grad} f\|^2 = f_x^2 + f_y^2.$$

The equation (4.4) coincides with the Szebehely equation [21].

Proposition 4.2. *The system of equations* (4.2) *and* (4.3) *is equivalent to the system of partial differential equations for the potential-energy function* U

$$\mathrm{d}U(\operatorname{grad} f_j) = \frac{2(U+h(f_1,f_2))}{\|\mathbf{grad}\, f_1 \times \mathbf{grad}\, f_2\|}(\mathrm{d}f_1 \wedge \mathrm{d}f_2(\operatorname{rot}\mathbf{t}, \mathbf{grad}\, f_j)), \quad j=1,2 \tag{4.5}$$

where

$$\mathbf{t} = \frac{(\mathbf{grad}\, f_1 \times \mathbf{grad}\, f_2)}{\|\mathbf{grad}, f_1 \times \mathbf{grad}\, f_2\|}.$$

Introducing the notation

$$W_{f_j} = \mathbf{grad}\, f_j \cdot (\mathbf{t} \times \operatorname{rot}\mathbf{t}), \quad j=1,2$$

we obtain from (4.5) the equations

$$\mathbf{grad}\, f_j \cdot \mathbf{grad}\, U = 2(U+h)W_{f_j}, \quad j=1,2.$$

In particular, these equations were deduced in [15].

It is possible to determine the potential field of force for a particle in $\mathbb{R}^3$ with a complementary condition that (4.2) is such that $\mathrm{d}h=0$. This condition means that

$$\operatorname{rot}\mathbf{v} = \nu(x,y,z)\mathbf{v}.$$

5. The inverse Bertrand problem

In this section we study the problem of constructing the potential field of force which is capable to generate given conics.

To solve this problem, we start with the following example, which represents a particular case of Suslov's problem:

Example (Particular case of Suslov's problem)**.** Let a particle with a configuration space $X=\mathbb{R}^3$ and kinetic energy $T=\frac{1}{2}(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)$ be given.

Construct the potential field of force capable of generating the two-parametric family of trajectories defined as intersections of two families of surfaces

$$\begin{cases} \zeta = c_1, \\ H(\xi, \eta, \zeta) = c_2. \end{cases} \tag{5.1}$$

The solution of this problem can easily be derived from Corollary 4.3. The vector field $\mathbf{v}$, $\operatorname{curl} \mathbf{v}$ and the field of force ω in this case are

$$\begin{cases} \mathbf{v} = \lambda\Big(\dfrac{\partial H}{\partial \eta}\dfrac{\partial}{\partial \xi} - \dfrac{\partial H}{\partial \xi}\dfrac{\partial}{\partial \eta}\Big), \\ \operatorname{rot} \mathbf{v} = \dfrac{\partial}{\partial \zeta}\Big(\lambda\dfrac{\partial H}{\partial \xi}\Big)\dfrac{\partial}{\partial \xi} + \dfrac{\partial}{\partial \zeta}\Big(\lambda\dfrac{\partial H}{\partial \eta}\Big)\dfrac{\partial}{\partial \eta} - \mu\dfrac{\partial}{\partial \zeta}, \\ \omega = \Big[\dfrac{\partial}{\partial \xi}\Big(\dfrac{\lambda^2}{2}\Big(\dfrac{\partial H}{\partial \xi}\Big)^2 + \Big(\dfrac{\partial H}{\partial \eta}\Big)^2\Big) - \lambda\mu\dfrac{\partial H}{\partial \xi}\Big]\,\mathrm{d}\xi \\ \qquad + \Big[\dfrac{\partial}{\partial \eta}\Big(\dfrac{\lambda^2}{2}\Big(\dfrac{\partial H}{\partial \xi}\Big)^2 + \Big(\dfrac{\partial H}{\partial \eta}\Big)^2\Big) - \lambda\mu\dfrac{\partial H}{\partial \eta}\Big]\,\mathrm{d}\eta \end{cases} \tag{5.2}$$

where

$$\mu = \frac{\partial}{\partial \xi}\Big(\lambda\frac{\partial H}{\partial \xi}\Big) + \frac{\partial}{\partial \eta}\Big(\lambda\frac{\partial H}{\partial \eta}\Big). \tag{5.3}$$

Clearly, the field of force is potential if and only if

$$\begin{cases} \lambda\mu = -\dfrac{\partial h}{\partial H}, \\ \partial_\zeta\Big[\dfrac{\lambda^2}{2}\Big(\Big(\dfrac{\partial H}{\partial \xi}\Big)^2 + \Big(\dfrac{\partial H}{\partial \eta}\Big)^2\Big)\Big] = -\dfrac{\partial h}{\partial \zeta}. \end{cases} \tag{5.4}$$

We illustrate the above results by studying the following particular problem.

Definition (Bertrand's problem)**.** The problem of constructing the potential field of force capable of generating the bi-parametric family of conics

$$\begin{cases} f_1 \equiv \zeta = c_1, \\ H \equiv r + b\xi = p, \quad r = \sqrt{\xi^2 + \eta^2 + \zeta^2} \end{cases} \tag{5.5}$$

where b is a positive constant, is well known as Bertrand's problem.

First, we study this problem assuming that the function λ in (5.2), (5.3), (5.4) is

$$\lambda = \lambda(H).$$

Under this restriction the system (5.3) yields the differential equations

$$
\begin{cases}
\ddot{\xi} = -\lambda^2(H)H\dfrac{\xi}{r^3} - \lambda^2(H)\dfrac{\zeta^2}{r^3}\dfrac{\partial H}{\partial \xi}, \\
\ddot{\eta} = -\lambda^2(H)H\dfrac{\eta}{r^3} - \lambda^2(H)\dfrac{\zeta^2}{r^3}\dfrac{\partial H}{\partial \eta}, \\
\ddot{\zeta} = 0.
\end{cases} \tag{5.6}
$$

It is interesting to analyze this differential system in new coordinates x, y, z related to the coordinates ξ, η, ζ by the orthogonal transformation

$$\vec{r} = \mathbf{A}\vec{R},$$

where $\vec{r} = (x, y, z)$, $\vec{R} = (\xi, \eta, \zeta)$,

$$
\mathbf{A} = \begin{pmatrix}
\dfrac{b_1}{b} & \dfrac{Bb_3 - Cb_2}{ab} & \dfrac{A}{a} \\
\dfrac{b_2}{b} & \dfrac{Cb_1 - Ab_3}{ab} & \dfrac{B}{a} \\
\dfrac{b_1}{b} & \dfrac{Ab_2 - Bb_1}{ab} & \dfrac{C}{a}
\end{pmatrix},
$$

$$b = \sqrt{b_1^2 + b_1^2 + b_1^2}, \quad a = \sqrt{A^2 + B^2 + C^2}, \quad Ab_1 + Bb_2 + Cb_3 = 0.$$

Orbits (5.5) in the new coordinates are written as

$$
\begin{cases}
F_1 \equiv Ax + By + Cz = c_1, \\
F_2 \equiv \sqrt{x^2 + y^2 + z^2} + b_1 x + b_2 y + b_3 z = p.
\end{cases} \tag{5.7}
$$

Note that if $c_1 = 0$ then these families of conics appear in the unperturbed Kepler movement [6].

Differential equations (5.6) in the Cartesian coordinates (x, y, z) assume the form

$$\ddot{\vec{r}} = -\frac{\lambda^2(F_2)F_2}{r^3}\vec{r} - \frac{\lambda^2(F_2)F_1^2}{r^3}\overrightarrow{\operatorname{grad} F_2} - \frac{\lambda^2(F_2)F_1^3}{r^3}\overrightarrow{\operatorname{grad} F_1}$$

where $r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\xi^2 + \eta^2 + \zeta^2}$.

Hence, if

$$F_1(x, y, z) = 0, \quad \lambda^2(F_2)F_2 = \mu = \text{const}$$

then we obtain the classical equations of the unperturbed Kepler movement [6]:

$$\ddot{\vec{r}} = -\frac{\mu\vec{r}}{r^3}.$$

Now we will study the Bertrand problem under the conditions

$$\begin{cases} \lambda = \lambda(\xi, \eta), \\ \zeta = 0. \end{cases}$$

Introducing the notation

$$\begin{cases} f = H\big|_{\zeta=0} = \sqrt{\xi^2+\eta^2} + b\xi, \\ \mu_0 = \dfrac{\partial}{\partial \xi}\Big(\lambda(\xi,\eta)\dfrac{\partial f}{\partial \xi}\Big) + \dfrac{\partial}{\partial \eta}\Big(\lambda(\xi,\eta)\dfrac{\partial f}{\partial \eta}\Big) \end{cases}$$

we can see that the field of force (5.3) capable of generating the given family of conics

$$\sqrt{\xi^2+\eta^2} + b\xi = p \tag{5.8}$$

is potential if and only if

$$\begin{cases} \lambda\mu_0 = \dfrac{\partial h}{\partial f}, \\ 0 = \dfrac{\partial h}{\partial \zeta}. \end{cases} \tag{5.9}$$

Introducing the polar coordinates $\xi = r\cos\theta$, $\eta = r\sin\theta$, we find that condition (5.9) takes on the form

$$(1 + b\cos\theta)\frac{\partial \lambda^2}{\partial r} - \frac{b\sin\theta}{r}\frac{\partial \lambda^2}{\partial \theta} + \frac{2\lambda^2}{r} = 2\frac{\partial h}{\partial f} \tag{5.10}$$

or, equivalently,

$$\begin{cases} (1+b\tau)\dfrac{\partial \lambda^2}{\partial r} + \dfrac{b(1-\tau^2)}{r}\dfrac{\partial \lambda^2}{\partial \tau} + \dfrac{2\lambda^2}{r} = 2\dfrac{\partial h}{\partial f}, \\ f = r(1+b\tau), \quad \tau = \cos\theta. \end{cases} \tag{5.11}$$

We embark now on the study of the case when $b \neq 0$ and h is such that

$$h(f) = \nu_{-1}\ln|f| + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -1}} \nu_j \frac{f^{j+1}}{j+1}, \tag{5.12}$$

where ν_j, $j \in \mathbb{Z}$, are real constants and λ is determined in such a way that

$$\lambda^2 = \sum_{j\in\mathbb{Z}} \psi_j(r) H_j(\tau). \tag{5.13}$$

It is clear that the series (5.12), (5.13) are formal series.

By inserting (5.12), (5.13) into (5.11) we obtain

$$\sum_{j\in\mathbb{Z}}\Big[(1+b\tau)\frac{\mathrm{d}\psi_j(r)}{\mathrm{d}r}H_j(\tau)+b(1-\tau^2)\frac{\psi_j(r)}{r}\frac{\mathrm{d}H_j(\tau)}{\mathrm{d}\tau}+2\frac{\psi_j(r)}{r}-2\nu_j r^j(1+b\tau)^j\Big]=0.$$

This equation holds if

$$\begin{cases}\psi_j(r)=a_j r^{j+1}, & j\in\mathbb{Z},\\ \nu_j=-a_jK_j\end{cases}$$

and we determine H_j as a solution to the equation

$$\begin{cases}b(1-\tau^2)H_j'(\tau)+\big((j+1)b\tau+j+3\big)H_j(\tau)+2K_j(1+b\tau)^j=0,\\ j\in\mathbb{Z}.\end{cases}\tag{5.14}$$

The general solution of this equation is

$$\begin{cases}H_j(\tau)=\xi_j(\tau)\Big(C_j-\dfrac{2K_j}{b}\displaystyle\int\frac{(1+b\tau)^j}{(1-\tau^2)\xi_j(\tau)}\,\mathrm{d}\tau\Big),\\ \xi_j(\tau)=(1-\tau)^{(j+1)/2+(j+3)/2b}(\tau+1)^{(j+1)/2-(j+3)/2b}\end{cases}$$

where C_j, $j\in\mathbb{Z}$, are arbitrary constants.

Under these conditions, the potential-energy function U such that $\omega=\mathrm{d}U(r,\tau)$ results in the form

$$U(r,\tau)=\frac{1}{2}\lambda^2(1+b^2+2b\tau)-h(f)\equiv\sum_{j\in\mathbb{Z}}a_jU_j(r,\tau)$$

where

$$\begin{cases}U_j(r,\tau)=\dfrac{1}{2}r^{j+1}H_j(\tau)(1+b^2+2b\tau)+\dfrac{K_j}{j+1}f^{j+1} & \text{if } j\neq-1,\\[2mm] U_{-1}(r,\tau)=\dfrac{1}{2}H_{-1}(\tau)(1+b^2+2b\tau)+K_{-1}\ln|f|.\end{cases}$$

We will study the subcase when $b=1$ separately from the subcase when $b\neq1$.

If $b=1$, then

$$U(r,\tau)=\lambda^2(1+\tau)-h(f)=\sum_{j\in\mathbb{Z}}a_jU_j(r,\tau)$$

where

$$\begin{cases}U_j(r,\tau)=r^{j+1}(1-\tau)^{j+2}\Big(C_j-2K_j\displaystyle\int\frac{(1+\tau)^j}{(1-\tau)^{j+3}}\,\mathrm{d}\tau\Big)+\dfrac{K_j}{j+1}f^{j+1}, & \text{if } j\neq-1,\\[2mm] U_{-1}(r,\tau)=(1-\tau)\Big(C_{-1}-2K_{-1}\displaystyle\int\frac{\mathrm{d}\tau}{(1-\tau)^2(1+\tau)}\Big)+K_{-1}\ln|f|,\\[2mm] f=r(1+\tau).\end{cases}$$

Evidently,

$$U_{-2} = \frac{C_{-2}}{r} - 2\frac{K_{-2}}{r}\Big(\int \frac{\mathrm{d}\tau}{(1+\tau)^2(1-\tau)} + \frac{1}{1+\tau}\Big) \equiv \frac{C_{-2}}{r} + \frac{K_{-2}}{r}g(\tau)$$

where

$$g(\tau) = \ln\sqrt{\frac{1-\tau}{1+\tau}}.$$

Therefore, if $b=1$, then

$$U(r,\tau) = \frac{a_{-2}C_{-2}}{r} + \frac{a_{-2}K_{-2}g(\tau)}{r} + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -2}} a_j U_j(r,\tau).$$

If $b\neq 1$, $b\neq 0$, it is easy to prove that

$$\begin{cases} H_{-2}(\tau) = \dfrac{(1-\tau)^{(1-b)/2b}}{(1+\tau)^{(1+b)/2b}}C_{-2} - \dfrac{2K_{-2}}{(b\tau+1)(1-b^2)}, \\ U_{-2}(r,\tau) = \dfrac{H_{-2}}{2r}(1+b^2+2b\tau) - \dfrac{K_{-2}}{r(b\tau+1)} = \dfrac{2K_{-2}}{r(b^2-1)} + \dfrac{C_{-2}}{r}G(\tau) \end{cases}$$

where

$$G(\tau) = \frac{1}{2}\sqrt{\Big(\frac{1-\tau}{1+\tau}\Big)^{1/b}\frac{1}{1-\tau^2}}(1+b^2+2b\tau).$$

Under these conditions, the potential function U takes on the form

$$U(r,\tau) = \frac{a_{-2}C_{-2}}{r}G(\tau) + \frac{2a_{-2}K_{-2}}{r(b^2-1)} + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -2}} a_j U_j(r,\tau).$$

Summarizing the above computations, we deduce that if $b\neq 0$ the function U is represented as

$$U(r,\tau) = \frac{\alpha}{r} + \frac{\beta(\tau)}{r} + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -2}} a_j U_j(r,\tau). \tag{5.15}$$

If $b=0$, then $f=r$ and condition (5.11) assumes the form

$$\partial_r\lambda^2 + 2\frac{\lambda^2}{r} = 2\partial_f h(f).$$

Therefore,

$$r^2\lambda^2 = 2\int r^2\partial_r h(r)\,\mathrm{d}r + 2\Psi(\tau),$$

which rearranged results in the expression

$$\lambda^2 = \frac{2}{r^2}\int r^2\partial_r h(r)\,\mathrm{d}r + \frac{2\Psi(\tau)}{r^2} = 2h(r) - \frac{4}{r^2}\int h(r) r\,\mathrm{d}r + \frac{2\Psi(\tau)}{r^2}$$

where Ψ is an arbitrary function.

Hence,

$$U(r,\tau) = \frac{\Psi(\tau)}{r^2} - \frac{2}{r^2}\int h(r)\,\mathrm{d}r. \tag{5.16}$$

Finally, we obtain the following solution of the Bertrand problem:

Proposition 5.1. *The potential-energy function capable of generating the one-parameter family of conics* (5.8) *can be calculated by the formula* (5.15) *if the eccentricity $b \neq 0$ and by the formula* (5.16) *if $b = 0$.*

6. The Heun differential equation in mechanics

In this section we establish the relation between the solution of the Bertrand problem proposed above and the solution of the particular class of Heun's equation.

The canonical form of Heun's general equation will be taken as [17]

$$\frac{\mathrm{d}^2x}{\mathrm{d}z^2} + \Big(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\Big)\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{\alpha\beta z - B}{z(z-1)(z-a)}x = 0. \tag{6.1}$$

In equation (6.1), x and z are regarded as complex variables and α, β, γ, δ, ε, a, b are parameters, generally complex and arbitrary, with the only condition that $a \neq 0, 1$. The first five parameters are linked by the relation $\alpha + \beta + 1 = \gamma + \delta + \varepsilon$.

The equation is, therefore, of the Fuchsian type [11], with regular singularities at the points $z = 0, 1, a, \infty$. The exponents at these singularities are, respectively, $(0, 1-\gamma)$, $(0, 1-\varepsilon)$, $(0, 1-\delta)$, (α, β).

Now we establish the relation between equation (5.14) and Heun's equation.

By replacing

$$\tilde{z} = \frac{1}{2}(z+1)$$

we can easily obtain the following representation for (5.14):

$$\tilde{z}(\tilde{z}-1)\frac{\mathrm{d}H_j}{\mathrm{d}\tilde{z}} + \frac{1}{2b}\big((1+e-2b\tilde{z})(j+1)+2\big)H_j(\tilde{z}) - \frac{k_j}{b}(1+b-2b\tilde{z})^j = 0.$$

Corollary 6.1. *The function*

$$F_j(\tilde z) = \Big(\tilde z(\tilde z-1)\frac{\mathrm{d}H_j}{\mathrm{d}\tilde z} + \frac{1}{2b}\big((1+e-2b\tilde z)(j+1)+2\big)H_j(\tilde z)\Big)(1+b-2b\tilde z)^{-j} \quad (j\in\mathbb{Z})$$

is the first integral of the Heun equation.

By differentiating and performing some straightforward calculations, we deduce the equation

$$\frac{\mathrm{d}^2H_j}{\mathrm{d}\tilde z^2} + \Big(\frac{1-a(1+j)-1/b}{\tilde z} + \frac{a(1+j)+1/b-j}{\tilde z-1} + \frac{-j}{\tilde z-a}\Big)\frac{\mathrm{d}H_j}{\mathrm{d}\tilde z} + \frac{(j^2-1)\tilde z - j/b - a(j^2-1)}{\tilde z(\tilde z-1)(\tilde z-a)}H_j(\tilde z)=0 \tag{6.2}$$

where $a=(1+b)/2b$.

By comparison with (6.1) we obtain

$$\begin{cases} \gamma_j = 1-\dfrac{1+b}{2b}(1+j)-\dfrac{1}{b}, \quad \delta_j = \dfrac{1+b}{2b}(1+j)+\dfrac{1}{b}-j, \quad \varepsilon_j=-j, \\ \alpha_j\beta_j = j^2-1 = -(1+\varepsilon_j)(2-\gamma_j-\delta_j), \\ B_j = \dfrac{j}{b}+\dfrac{1+b}{2b}(j^2-1) = -a(2-\gamma_j-\delta_j)-(1-\gamma_j)\varepsilon_j. \end{cases}$$

Evidently, when the given conics are parabolas then in (6.2) we have the confluence of singularities. In fact, in this case $b=1$, so $a=1$, and as a consequence Heun's equation is transformed into the hypergeometric differential equation

$$\frac{\mathrm{d}^2H_j}{\mathrm{d}\tilde z^2} + \Big(\frac{-1-j}{\tilde z} + \frac{2-j}{\tilde z-1}\Big)\frac{\mathrm{d}H_j}{\mathrm{d}\tilde z} + \frac{(j^2-1)(\tilde z-1)-j}{\tilde z(\tilde z-1)^2}H_j(\tilde z)=0, \quad j\in\mathbb{Z}.$$

Concluding, from the results given in Sections 5 and 6, we obtain

Proposition 6.1. *The potential-energy function U capable of generating a one-parameter family of conics with eccentricity b is the function*

$$\begin{aligned} U &= a_{-1}H_{-1}(\cos\theta - K_{-1}\log r(1+b\cos\theta)) \\ &\quad + \sum_{j\in\mathbb{Z}\setminus\{-1\}} a_j r^{j+1}\Big(H_j(\cos\theta) - \frac{1+b\cos\theta}{j+1}\Big) \end{aligned}$$

where a_j, $j \in \mathbb{Z}$, K_1 are real constants and H_j, $j \in \mathbb{Z}$ are solutions of the Heun equations with singularities at the points

$$0,\ 1,\ \frac{1+b}{b},\ \infty$$

and with the exponents

$$\Big(0, \frac{j+3+b(j+1)}{2b}\Big);\ \Big(0, j-\frac{j+3+b(j+1)}{2b}\Big);\ (0, j+1);\ (-1-j, 1-j),$$

respectively.

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