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Remarks on discretely absolutely star-Lindelöf spaces

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REMARKS ON DISCRETELY ABSOLUTELY  
STAR-LINDELÖF SPACES

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*Abstract.* In this paper, we prove the following statements:

- (1) There exists a Hausdorff Lindelöf space  $X$  such that the Alexandroff duplicate  $A(X)$  of  $X$  is not discretely absolutely star-Lindelöf.
- (2) If  $X$  is a regular Lindelöf space, then  $A(X)$  is discretely absolutely star-Lindelöf.
- (3) If  $X$  is a normal discretely star-Lindelöf space with  $e(X) < \omega_1$ , then  $A(X)$  is discretely absolutely star-Lindelöf.

*Keywords:* countably compact space, star-Lindelöf space, absolutely star-Lindelöf space, discretely absolutely star-Lindelöf

*MSC 2010:* 54D20, 54B10, 54D55

1. INTRODUCTION

By a space, we mean a  $T_1$  topological space. Recall that a space  $X$  is *countably compact* if every countable open cover of  $X$  has a finite subcover. Matveev defined in [5] a space  $X$  to be *absolutely countably compact* (= acc) if for every open cover  $\mathcal{U}$  of  $X$  and every dense subspace  $D$  of  $X$ , there exists a finite subset  $F$  of  $D$  such that  $\text{St}(F, \mathcal{U}) = X$ , where  $\text{St}(F, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$ . He also proved that every Hausdorff acc space is countably compact (see [5]).

A space  $X$  is *star-Lindelöf* (see [3], [6] under different names) (*discretely star-Lindelöf*) (see [9], [15]) if for every open cover  $\mathcal{U}$  of  $X$ , there exists a countable subset (a countable discrete closed subset, respectively)  $F \subseteq X$  such that  $\text{St}(F, \mathcal{U}) = X$ . It is clear that every separable space and every discretely star-Lindelöf space are star-

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Lindelöf as well as every space of countable extent (in particular, every countably compact space or every Lindelöf space).

In [2], a star-Lindelöf space is called  $*$  Lindelöf; in [3], a star-Lindelöf space is called strongly star-Lindelöf, and in [15], a discretely star-Lindelöf space is called space in countable web.

A space  $X$  is *absolutely star-Lindelöf* (see [1], [6]) (*discretely absolutely star-Lindelöf*) (see [10], [11]) if for every open cover  $\mathcal{U}$  of  $X$  and every dense subset  $D$  of  $X$ , there exists a countable subset  $F$  of  $D$  such that  $\text{St}(F, \mathcal{U}) = X$  ( $F$  is discrete and closed in  $X$  and  $\text{St}(F, \mathcal{U}) = X$ , respectively).

From the above definitions, it is not difficult to see that every acc space is absolutely star-Lindelöf, every absolutely star-Lindelöf space is star-Lindelöf, every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf and every discretely absolutely star-Lindelöf space is discretely star-Lindelöf.

Throughout the paper, the cardinality of a set  $A$  is denoted by  $|A|$ . The extent  $e(X)$  of a space  $X$  is the smallest infinite cardinal  $\kappa$  such that every discrete closed subset of  $X$  has cardinality at most  $\kappa$ . For a cardinal  $\kappa$ , let  $\kappa^+$  denote the smallest cardinal greater than  $\kappa$ . Let  $\mathfrak{c}$  denote the cardinality of the continuum,  $\omega_1$  the first uncountable cardinal and  $\omega$  the first infinite cardinal. For a pair of ordinals  $\alpha, \beta$  with  $\alpha < \beta$ , we write  $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ . Other terms and symbols that we do not define will be used as in [4].

## 2. SOME RESULTS ON DISCRETELY ABSOLUTELY STAR-LINDELÖF SPACES

For a space  $X$ , recall that the Alexandroff duplicate  $A(X)$  of  $X$  is constructed in the following way: The underlying set of  $A(X)$  is  $X \times \{0, 1\}$  and each point of  $X \times \{1\}$  is isolated; a basic neighbourhood of a point  $\langle x, 0 \rangle \in X \times \{0\}$  is a set of the form  $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1 \rangle\})$ , where  $U$  is a neighborhood of  $x$  of  $X$ . It is well-known that  $A(X)$  is compact (countably compact, Lindelöf) iff so is  $X$  and  $A(X)$  is Hausdorff (regular, Tychonoff, normal) iff so is  $X$ . Moreover, Vaughan [12] proved that if  $X$  is countably compact, then  $A(X)$  is acc. In this section, we show the statements stated in abstract.

**Example 2.1.** There exists a Hausdorff Lindelöf space  $X$  such that  $A(X)$  is not discretely absolutely star-Lindelöf.

**Proof.** Let

$$A = \{a_n : n \in \omega\} \quad \text{and} \quad B = \{b_m : m \in \omega\},$$

$$A_n = \{\langle a_n, b_m \rangle : m \in \omega\} \quad \text{and} \quad Y = \bigcup_{n \in \omega} A_n$$

and let

$$X = Y \cup A \cup \{a\} \quad \text{where } a \notin Y \cup A.$$

We topologize  $X$  as follows: every point of  $Y$  is isolated; a basic neighborhood of a point  $a_n \in A$  for each  $n \in \omega$  takes the form

$$U_{a_n}(m) = \{a_n\} \cup \{\langle a_n, b_i \rangle : i > m\} \quad \text{for } m \in \omega$$

and a basic neighborhood of  $a$  takes the form

$$U_a(F) = \{a\} \cup \bigcup \{\langle a_n, b_m \rangle : a_n \in A \setminus F, m \in \omega\} \quad \text{for a finite subset } F \text{ of } A.$$

Clearly,  $X$  is a Hausdorff space by the construction of the topology on  $X$ ,  $X$  is not regular, since the point  $a$  can not be separated from the closed subset  $A$  by disjoint open subsets of  $X$ . Moreover,  $X$  is Lindelöf, since  $|X| = \omega$ .

We show that  $A(X)$  is not discretely absolutely star-Lindelöf.

Let us consider the open cover

$$\mathcal{U} = \{\langle a_n, 1 \rangle : n \in \omega\} \cup \{\langle a, 1 \rangle\} \cup \{\langle a, 0 \rangle \cup A(Y)\} \cup \{\langle a_n, 0 \rangle \cup A(A_n) : n \in \omega\}$$

and the dense subset

$$D = \bigcup \{A(A_n) : n \in \omega\} \cup (A \times \{1\}) \cup \{\langle a, 1 \rangle\}$$

of  $A(X)$ . Let  $F$  be any countable subset of  $D$  and  $F$  discrete closed in  $X$ . Then,  $\{n : F \cap A(A_n) \neq \emptyset\}$  is finite. In fact, if  $\{n : F \cap A(A_n) \neq \emptyset\}$  is not finite, then  $\langle a, 0 \rangle$  is a limit point of  $F \cap A(Y)$  by the definition of the topology of  $X$ . This is a contradiction, since  $F$  is discrete closed in  $A(X)$ . Hence, there exists a  $n_0 \in \omega$  such that  $F \cap A(A_{n_0}) = \emptyset$ . Thus,

$$\langle a_{n_0}, 0 \rangle \notin \text{St}(F, \mathcal{U}),$$

since  $\langle a_{n_0}, 0 \rangle \cup A(A_{n_0})$  is the only element of  $\mathcal{U}$  containing  $\langle a_{n_0}, 0 \rangle$ , which completes the proof.  $\square$

Recall from [8] that a space  $X$  has *property* (a) if for every open cover  $\mathcal{U}$  of  $X$  and every dense subset  $D$  of  $X$ , there exists a subset  $F$  of  $D$  such that  $F$  is discrete and closed in  $X$  and  $\text{St}(F, \mathcal{U}) = X$ . For a regular Lindelöf space  $X$ , we show that  $A(X)$  is discretely absolutely star-Lindelöf. For the proof of the statement, we need two lemmas:

**Lemma 2.2.** *Every  $T_1$  paracompact space has property (a).*

*Proof.* Let  $X$  be  $T_1$  paracompact, and let  $\mathcal{U}$  be an open cover of  $X$  and  $D$  a dense subset of  $X$ . Then, there exists a locally finite open refinement  $\mathcal{V}$  of  $\mathcal{U}$ . Thus, it is sufficient to show that there exists a subset  $F$  of  $D$  such that  $F$  is discrete closed in  $X$  and  $X = \text{St}(F, \mathcal{V})$ . By transfinite induction, we define a sequence  $x_\alpha$  of  $X$  and a sequence  $d_\alpha$  of  $D$  satisfying the following conditions (1) and (2) for each  $\alpha$ .

- (1)  $x_\alpha \notin \text{St}(\{d_\beta: \beta < \alpha\}, \mathcal{V})$ , and
- (2)  $d_\alpha \in D \cap \bigcap \{V \in \mathcal{V}: x_\alpha \in V\}$ .

Pick  $d_0 \in D$ , if  $\text{St}(d_0, \mathcal{V}) = X$ , then we finish the transfinite induction. If not, we pick  $x_0 \in X \setminus \text{St}(d_0, \mathcal{V})$ . Assume that we have defined  $x_\gamma$  and  $d_\gamma$  for  $\gamma < \alpha$ . If  $\text{St}(\{d_\gamma: \gamma < \alpha\}, \mathcal{V}) = X$ , then we finish the induction. If not, we pick  $x_\alpha \in X \setminus \text{St}(\{d_\gamma: \gamma < \alpha\}, \mathcal{V})$  and  $d_\alpha \in D \cap \bigcap \{V \in \mathcal{V}: x_\alpha \in V\}$ . We finish the induction at some  $\alpha$  such that

$$\text{St}(\{d_\beta: \beta < \alpha\}, \mathcal{V}) = X.$$

Put  $F = \{d_\beta: \beta < \alpha\}$ . Since  $X$  is  $T_1$ , then  $\{d_\beta\}$  is closed for each  $\beta < \alpha$ . By the choice of the sequences  $x_\alpha$  and  $d_\alpha$ , clearly,

$$\text{St}(\{x_\gamma: \gamma < \beta\}, \mathcal{V}) \subseteq \text{St}(\{d_\gamma: \gamma < \beta\}, \mathcal{V}) \quad \text{for each } \gamma < \alpha.$$

Thus,

$$x_\beta \notin \text{St}(\{x_\gamma: \gamma < \beta\}, \mathcal{V}) \quad \text{for each } \beta < \alpha.$$

Hence, there is no element of  $\mathcal{V}$  containing two distinct elements of  $\{x_\beta: \beta < \alpha\}$ . By our construction, for  $V \in \mathcal{V}$ , if there is some  $\beta < \alpha$  such that  $x_\beta \in V$ , then  $V$  contains the only element  $d_\beta$  of  $\{d_\gamma: \gamma < \alpha\}$ . Thus  $F = \{d_\beta: \beta < \alpha\}$  is discrete in  $X$ . Since all one point subsets of  $F$  are closed and  $\mathcal{V}$  is locally finite,  $F$  is closed in  $X$ , which completes the proof.  $\square$

Since every regular Lindelöf space is paracompact, then we have the following lemma by Lemma 2.2:

**Lemma 2.3.** *Every regular Lindelöf space is discretely absolutely star-Lindelöf.*

Since  $A(X)$  is regular Lindelöf iff  $X$  regular Lindelöf, we have the following theorem by Lemma 2.3.

**Theorem 2.4.** *If  $X$  is a regular Lindelöf space  $X$ , then  $A(X)$  is discretely absolutely star-Lindelöf.*

In the following, we give an example showing that  $A(X)$  need not be discretely absolutely star-Lindelöf for a Tychonoff discretely absolutely star-Lindelöf space  $X$ .

**Example 2.5.** There exists a Tychonoff discretely absolutely star-Lindelöf space  $X$  such that  $A(X)$  is not star-Lindelöf (hence, is not discretely absolutely star-Lindelöf).

*Proof.* Let  $\mathcal{R}$  be a maximal almost disjoint family of infinite subsets of  $\omega$  with  $|\mathcal{R}| = \mathfrak{c}$ . Let

$$X = (\mathfrak{c}^+ \times \omega) \cup \mathcal{R}.$$

We topologize  $X$  as follows:  $\mathfrak{c}^+ \times \omega$  has the usual product topology and is an open subspace of  $X$ , and a basic neighborhood of  $r \in \mathcal{R}$  takes the form

$$G_{\beta, K}(r) = ((\beta, \mathfrak{c}^+) \times (r \setminus K)) \cup \{r\} \quad \text{for } \beta < \mathfrak{c}^+ \text{ and a finite subset } K \text{ of } \omega.$$

Then,  $X$  is discretely absolutely star-Lindelöf and  $e(X) = \mathfrak{c}$  (see [8, Example 3.1]). We show that  $A(X)$  is not star-Lindelöf. Let us consider the open cover

$$\mathcal{U} = \{\langle r, 0 \rangle \cup A(\mathfrak{c}^+ \times r) : r \in \mathcal{R}\} \cup \{\langle r, 1 \rangle : r \in \mathcal{R}\} \cup \{A(\mathfrak{c}^+ \times \{n\}) : n \in \omega\}.$$

Let  $F$  be a countable subset of  $A(X)$ . Then there exists an  $r \in \mathcal{R}$  such that  $\langle r, 1 \rangle \notin F$ , since  $|\mathcal{R}| = \mathfrak{c}$ . Hence,  $\langle r, 1 \rangle \notin \text{St}(F, \mathcal{U})$ , since  $\{\langle r, 1 \rangle : r \in \mathcal{R}\}$  is open and closed in  $A(X)$  and  $\langle r, 1 \rangle$  is isolated for every  $r \in \mathcal{R}$ , which completes the proof.  $\square$

**Remark 1.** Since every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf and discretely star-Lindelöf, Example 2.5 shows that  $A(X)$  of an absolutely star-Lindelöf (discretely star-Lindelöf) space  $X$  need not be absolutely star-Lindelöf (discretely star-Lindelöf, respectively).

For a normal space, we have the following consistent example.

**Example 2.6.** Assuming  $2^{\aleph_0} = 2^{\aleph_1}$ , there exists a normal discretely absolutely star-Lindelöf space  $S_1$  with  $e(X) > \omega$  such that  $A(X)$  is not star-Lindelöf (hence, is not discretely absolutely star-Lindelöf).

*Proof.* Let  $S = L \cup \omega$  be the same space  $X$  (see [13, Example E]). Let  $\kappa$  be regular and  $cf(\kappa) \geq |S|$ . We define

$$X = L \cup (\kappa^+ \times \omega)$$

and topologize  $X$  as follows: a basic neighborhood of  $l \in L$  in  $X$  is a set of the form

$$G_{U,\alpha}(l) = (U \cap L) \cup ((\alpha, \kappa^+) \times (U \cap \omega))$$

for a neighborhood  $U$  of  $l$  in  $X$  and for  $\alpha < \omega_1$ , and a basic neighborhood of  $\langle \alpha, x \rangle \in \kappa^+ \times \omega$  in  $S_1$  is a set of the form

$$G_V(\langle \alpha, x \rangle) = V \times \{x\},$$

where  $V$  is a neighborhood of  $\alpha$  in  $\kappa^+$ . Then,  $S_1$  is normal and discretely absolutely star-Lindelöf (see [12, Example 2.2]). Similarly to the proof of Example 2.5, it is not difficult to prove that  $X$  is not star-Lindelöf.  $\square$

**Remark 2.** The author does not know if there exists a normal discretely absolutely star-Lindelöf space  $X$  such that  $A(X)$  is not discretely absolutely star-Lindelöf in ZFC.

In Example 2.5, we note that the  $e(X) = \mathfrak{c}$ . In the following, we give an example showing that  $A(X)$  of a discretely star-Lindelöf (or absolutely star-Lindelöf) space  $X$  with  $e(X) = \omega$  need not be discretely absolutely star-Lindelöf.

**Example 2.7.** There exist both a Tychonoff absolutely star-Lindelöf space and a Tychonoff discretely star-Lindelöf space  $X$  with  $e(X) = \omega$  such that  $A(X)$  is not discretely absolutely star-Lindelöf.

*Proof.* Let  $X = ((\omega_1 + 1) \times (\omega + 1)) \setminus \{(\omega_1, \omega)\}$  be the Tychonoff plank.

First, we show that  $X$  is absolutely star-Lindelöf. To this end, let  $\mathcal{U}$  be an open cover of  $X$ . Let  $S$  be the set of all isolated points of  $\omega_1$  and let  $D = S \times \omega$ . Then,  $D$  is dense in  $X$  and every dense subspace of  $X$  includes  $D$ . Thus, it is sufficient to show that there exists a countable subset  $F$  of  $D$  such that  $\text{St}(F, \mathcal{U}) = X$ . Since  $\omega_1$  is countably compact, it follows from [5, Theorem 1.8] that  $\omega_1$  is acc. By [5, Theorem 2.3], we see that  $\omega_1 \times (\omega + 1)$  is acc. Hence, there exists a finite subset  $F_1$  of  $D$  such that

$$\omega_1 \times (\omega + 1) \subseteq \text{St}(F_1, \mathcal{U}).$$

On the other hand, for each  $n \in \omega$ , there exists a  $U_n \in \mathcal{U}$  such that  $\langle \omega_1, n \rangle \in U_n$ . Pick  $d_n \in U_n \cap D$  for each  $n \in \omega$ . Then,

$$\{\omega_1\} \times \omega \subseteq \text{St}(\{d_n : n \in \omega\}, \mathcal{U}).$$

If we put  $F = F_1 \cup \{d_n : n \in \omega\}$ , then  $F$  is a countable subset of  $D$  such that  $X = \text{St}(F, \mathcal{U})$ , which shows that  $X$  is absolutely star-Lindelöf.

Next, we show that  $X$  is discretely star-Lindelöf. To this end, let  $\mathcal{U}$  be an open cover of  $X$ . Since  $\omega_1 \times (\omega + 1)$  is countably compact, then there exists a finite subset  $F_1$  of  $X$  such that

$$\omega_1 \times (\omega + 1) \subseteq \text{St}(F_1, \mathcal{U}).$$

We put  $F = F_1 \cup \{\langle \omega_1, n \rangle : n \in \omega\}$ . Then,  $F$  is a countable discrete closed subset of  $X$  such that  $\text{St}(F, \mathcal{U}) = X$ , which completes the proof.

Finally, we show that  $A(X)$  is not discretely absolutely star-Lindelöf, let us consider the open cover

$$\mathcal{U} = \{A(\omega_1 \times (\omega + 1))\} \cup \{\langle \langle \omega_1, n \rangle, 0 \rangle \cup A(\omega_1 \times \{n\}) : n \in \omega\} \cup \{\langle \langle \omega_1, n \rangle, 1 \rangle : n \in \omega\}$$

and the dense subset

$$D = A(\omega_1 \times (\omega + 1)) \cup \{\langle \langle \omega_1, n \rangle, 1 \rangle : n \in \omega\}$$

of  $A(X)$ . Let  $F$  be a countable subset of  $D$  which is discrete closed in  $A(X)$ . Since  $A(\omega_1 \times (\omega + 1))$  is countably compact, then  $F \cap A(\omega_1 \times (\omega + 1))$  is finite, hence there exists an  $n_0 \in \omega$  such that  $F \cap A(\omega_1 \times \{n_0\}) = \emptyset$ , therefore

$$\langle \langle \omega_1, n_0 \rangle, 0 \rangle \notin \text{St}(F, \mathcal{U}),$$

since  $\langle \langle \omega_1, n_0 \rangle, 0 \rangle \cup A(\omega_1 \times \{n_0\})$  is the only element of  $\mathcal{U}$  containing  $\langle \langle \omega_1, n_0 \rangle, 0 \rangle$ , which completes the proof.  $\square$

**Remark 3.** In Example 2.7, it is not difficult to show that  $X$  is not discretely absolutely star-Lindelöf. Thus, the author does not know if there exists a Tychonoff discretely absolutely star-Lindelöf space  $X$  with  $e(X) = \omega$  such that  $A(X)$  is not discretely absolutely star-Lindelöf.

In the following, we give a positive result.

**Theorem 2.8.** *If  $X$  is a normal discretely star-Lindelöf space  $X$  with  $e(X) < \omega_1$ , then  $A(X)$  is discretely absolutely star-Lindelöf.*

*Proof.* We prove that  $A(X)$  is discretely absolutely star-Lindelöf. To this end, let  $\mathcal{U}$  be an open cover of  $A(X)$ . Obviously every point of  $X \times \{1\}$  is isolated. Let  $B$  be the set of all isolated points of  $X$ , and let

$$D = (X \times \{1\}) \cup (B \times \{0\}).$$



Then,  $D$  is a dense subspace of  $A(X)$  and every dense subset of  $A(X)$  includes  $D$ . Thus, it is sufficient to show that there exists a countable subset  $F \subseteq D$  such that  $F$  is discrete closed in  $X$  and  $\text{St}(F, \mathcal{U}) = A(X)$ . For each  $x \in X$ , choose an open neighborhood  $W_x = (V_x \times \{0, 1\}) \setminus \{\langle x, 1 \rangle\}$  of  $\langle x, 0 \rangle$  satisfying that there exists a  $U \in \mathcal{U}$  such that  $W_x \subseteq U$ , where  $V_x$  is an open subset of  $X$  containing  $x$ . Put  $\mathcal{V} = \{V_x : x \in X\}$ . Then,  $\mathcal{V}$  is an open cover of  $X$ . Hence, there exists a countable subset  $E_0 \subseteq X$  such that  $E_0$  is discrete closed in  $X$  and  $X = \text{St}(E_0, \mathcal{V})$ , since  $X$  is discretely star-Lindelöf. For the collection  $\mathcal{V} = \{V_x : x \in E_0\}$  of  $X$ , since  $E_0$  is discrete closed, there exists a pairwise disjoint open family  $\{U_x : x \in E_0\}$  in  $X$  such that  $x \in U_x \subseteq V_x$  for each  $x \in E_0$ , since  $E_0$  is a discrete closed subset of a normal space  $X$ . By normality, there is an open subset  $U$  in  $X$  such that

$$E_0 \subseteq U \subseteq \bar{U} \subseteq \bigcup_{x \in F_0} U_x.$$

Clearly,  $\{U \cap U_x : x \in F_0\}$  is a discrete family of nonempty open subsets of  $X$ . Let

$$E'_1 = \{x \in E_0 : x \text{ is not isolated in } X\}.$$

For every  $x \in E'_1$ , pick  $y_x \in U \cap U_x$  such that  $x \neq y_x$ . Then,

$$\{\{x\} : x \in E\} \cup \{\{y_x\} : x \in E'_1\}$$

is discrete closed in  $X$  and  $\langle y_x, 1 \rangle \in W_x$  and  $\langle x, 0 \rangle \in W_x$ .

Put  $E_1 = E_0 \times \{1\}$ . For every  $x \in X \setminus (E_0 \cup \{V_x : x \in E'_1\})$ , there exists  $x' \in X$  such that  $x \in V_{x'}$  and  $V_{x'} \cap E_0 \neq \emptyset$ , hence  $W_{x'} \cap E_1 \neq \emptyset$ . Let

$$E_2 = E_1 \cup \{\langle y_x, 1 \rangle : x \in E'_1\} \cup ((E_0 \setminus E'_1) \times \{0\}).$$

Then,  $E_2$  is a countable discrete closed (in  $X$ ) subset of  $D$  and  $X \times \{0\} \subseteq \text{St}(E_2, \mathcal{U})$ . Let  $E_3 = A(X) \setminus \text{St}(E_2, \mathcal{U})$ . Then,  $E_3$  is a discrete and closed subset of  $A(X)$ . Since  $e(X) < \omega_1$ , then  $e(A(X)) < \omega_1$ . Thus we see that  $E_3$  is countable. If we put  $F = E_2 \cup E_3$ , then  $F$  is a countable discrete closed (in  $X$ ) subset of  $D$  and  $A(X) = \text{St}(F, \mathcal{U})$ , which completes the proof.  $\square$

We have the following corollary of Theorem 2.8.

**Corollary 2.9.** *Every normal discretely star-Lindelöf space  $X$  with  $e(X) < \omega_1$  can be embedded in a normal discretely absolutely star-Lindelöf space as a closed subspace.*

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