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POSITIVE UNBOUNDED SOLUTIONS OF SECOND ORDER QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS AND THEIR APPLICATION TO ELLIPTIC PROBLEMS

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Abstract. In this paper we consider positive unbounded solutions of second order quasilinear ordinary differential equations. Our objective is to determine the asymptotic forms of unbounded solutions. An application to exterior Dirichlet problems is also given.

Keywords: quasilinear ordinary differential equation, asymptotic form, unbounded solution

MSC 2010: 34D05, 35D05

1. Introduction

In this paper we consider second order quasilinear ordinary differential equations of the form

$$(p(t)|u'|^{\alpha-1}u')' = q(t)|u|^{\lambda-1}u, \tag{1.1}$$

where α and λ are positive constants satisfying the super-homogeneity condition $\lambda > \alpha$, and p and q are positive continuous functions defined on $[t_0, \infty)$. Our objective is to search for the asymptotic properties, especially for asymptotic behaviors of positive solutions of (1.1). In what follows we always assume that

$$\int^{\infty} p(t)^{-1/\alpha}\,\mathrm{d}t < \infty. \tag{1.2}$$

The reason why we assume (1.2) is that if, on the contrary, p satisfies

$$\int^{\infty} p(t)^{-1/\alpha}\,\mathrm{d}t = \infty,$$

then the equation (1.1) can be reduced, by the change of variable $\tau = \int_{t_0}^{t} p(s)^{-1/\alpha}\,\mathrm{d}s$, to

$$(1.3) \qquad (|u_\tau|^{\alpha-1}u_\tau)_\tau = Q(\tau)|u|^{\lambda-1}u.$$

The leading term of this equation is simpler than that of (1.1). So most of researchers usually consider (1.1) after the change of variables when $\int^{\infty} p(t)^{-1/\alpha}\,\mathrm{d}t = \infty$. For the equation (1.3) there are many studies of the asymptotic properties. When $\alpha = 1$, the equation (1.3) is reduced to the well-known Emden-Fowler equations, which arise in many fields of applied sciences; see, for example, [8], [9]. The mathematical results for Emden-Fowler equations are summarized in the monographs [1], [6]. When $\alpha \neq 1$, the equation (1.3), which is of the so called quasilinear type, has been studied in many papers with respect to its asymptotic properties; see, for example, [4], [5], [10]. On the other hand, for the equation (1.1) under the assumption (1.2) recent papers [3], [11] are devoted to the study of qualitative properties.

By a solution u of (1.1) we mean a function u such that u and $p|u'|^{\alpha-1}u'$ are of class C^1, and u satisfies (1.1) near $+\infty$. Throughout this paper we shall confine ourselves to the study of those solutions which remain positive near $+\infty$. It was already shown in [11] that every positive solution u of (1.1) has exactly one of the four asymptotic behaviors listed below:

(i) rapidly decaying solution: $\lim_{t\to\infty} u(t)/\pi(t) = 0$;
(ii) slowly decaying solution: $\lim_{t\to\infty} u(t)/\pi(t) \in (0,\infty)$;
(iii) asymptotically constant solution: $\lim_{t\to\infty} u(t) \in (0,\infty)$;
(iv) unbounded solution: $\lim_{t\to\infty} u(t) = \infty$,

where $\pi(t)$ is the decreasing function defined by

$$\pi(t) = \int_t^{\infty} p(s)^{-1/\alpha}\,\mathrm{d}s.$$

Necessary and/or sufficient conditions for the existence of each type of solutions are obtained in [11].

We also know the exact asymptotic behaviors, more precisely the leading order terms for slowly decaying solutions (type (ii)) and asymptotically constant solutions (type (iii)). For example, the asymptotic form of a type (ii) solution u is $u(t) = c\{1 + o(1)\}\pi(t)$ as $t \to \infty$, i.e., the leading order term is $c\pi(t)$, where c is a positive constant. But it was not known how rapidly decaying solutions (type (i)) and unbounded solutions (type (iv)) behave near $+\infty$ exactly, except for what follows from the definitions. For rapidly decaying solutions, we obtained the asymptotic behavior in [3]. However, as far as the authors are aware, very little is known about asymptotic forms of unbounded solutions. This is our main motivation to try to consider asymptotic behavior for unbounded solutions of (1.1).

Remark. By Theorem 4.7 in [11] the equation (1.1) has an unbounded solution if

$$\int^{\infty} q(t)\,\mathrm{d}t < \infty.$$

As an example let us consider the following equation which is a prototype of (1.1):

$$(t^{\beta}|u'|^{\alpha-1}u')' = ct^{\sigma}|u|^{\lambda-1}u, \tag{1.4}$$

where $c > 0$ is a constant, and $\beta > 0$ and $\sigma \in \mathbb{R}$ are constants satisfying

$$\beta > \alpha \quad \text{and} \quad \beta - \sigma - \alpha - 1 > 0. \tag{1.5}$$

We can see easily that the equation (1.4) has an unbounded solution u_0 explicitly given by

$$u_0(t) = \hat{c}t^{k} \tag{1.6}$$

with

$$k = \frac{\beta - \sigma - \alpha - 1}{\lambda - \alpha}(> 0) \quad \text{and} \quad \hat{c}^{\lambda-\alpha} = \frac{k^{\alpha}\{\beta + \alpha(k-1)\}}{c}. \tag{1.7}$$

This fact leads us to conjecture that unbounded solutions of (1.1) behave like $u_0(t)$ if $p(t)$ and $q(t)$, respectively, behave like t^{β} and ct^{σ} under the condition (1.5). We will show that this conjecture is true as seen from the following theorem, which is the main result in this paper.

Theorem 1.1. *Assume that* (1.5) *holds. Suppose that*

$$p(t) \sim t^{\beta} \quad \text{and} \quad q(t) \sim ct^{\sigma} \quad \text{as } t \to \infty, \tag{1.8}$$

p is of class C^1, and

$$\lim_{t\to\infty} t\Big(\frac{p(t)}{t^{\beta}}\Big)' = 0. \tag{1.9}$$

Then every unbounded solutions u of (1.1) *has the asymptotic form*

$$u \sim u_0(t) \quad \text{as } t \to \infty.$$

Henceforth the notation $f(t) \sim g(t)$ means that $\lim\limits_{t\to\infty} f(t)/g(t) = 1$. The proof of this theorem will be given in Section 3.

The present paper is organized in the following way. In Section 2 we state some properties and growth estimates for unbounded solutions of (1.1). In Section 3 we prove Theorem 1.1. In Section 4 we consider exterior Dirichlet problems for elliptic equations as an application of our main result.

2. Some properties for unbounded solutions

The objective in this section is to obtain some properties and estimates for unbounded solutions of (1.1). We start by establishing the fundamental properties for unbounded solutions.

Lemma 2.1. *Let u be an unbounded solution of* (1.1). *Then $u' > 0$ near ∞ and*

$$\lim_{t\to\infty} p(t)(u')^{\alpha} = +\infty.$$

To prove this lemma we note that $p(t)|u'|^{\alpha-1}u'$ is monotone increasing. If we assume that $p(t)(u')^{\alpha}$ has a finite positive limit as $t \to \infty$, then we get a contradiction to the condition (1.2) immediately.

Next, we obtain some estimates for unbounded solutions of (1.1).

Theorem 2.1 (Upper-estimate for unbounded solutions)**.** *Let u be an unbounded solution of* (1.1). *Then there exist a positive constant c and sufficiently large t_0 such that*

$$u(t) \leqslant c\biggl[\int_{t_0}^{t} p(s)^{-1/\alpha}\biggl\{\int_{s}^{\infty}\Bigl(\frac{q(r)^{\alpha+1}}{p(r)^{\lambda+1}}\Bigr)^{1/(\lambda\alpha+2\alpha+1)}\,\mathrm{d}r\biggr\}^{-(\lambda\alpha+2\alpha+1)/\{\alpha(\lambda-\alpha)\}}\,\mathrm{d}s\biggr]^{\alpha/(\alpha+1)}$$

near $+\infty$.

P r o o f. Let $z = p(t)(u')^{\alpha}u$. Then

$$z' = \{p(t)(u')^{\alpha}\}'u + p(t)(u')^{\alpha+1} = q(t)u^{\lambda+1} + p(t)(u')^{\alpha+1}.$$

Young's inequality implies that there exists a positive constant c_1 such that for $\delta = (\alpha+1)/(\alpha\lambda+2\alpha+1) \in (0,1)$

$$z' \geqslant c_1 p(t)^{1-\delta} q(t)^{\delta} u^{(\lambda+1)\delta} (u')^{(\alpha+1)(1-\delta)} = c_1 z^{(\lambda+1)\delta} p(t)^{1-2\delta-\delta\lambda} q(t)^{\delta},$$

that is,

$$z' z^{-(\lambda+1)\delta} \geqslant c_1 p(t)^{1-2\delta-\delta\lambda} q(t)^{\delta}.$$

Note that $-(\lambda+1)\delta < -1$ and $\lim\limits_{t\to\infty} z(t) = \infty$ from Lemma 2.1, so we obtain by integrating over $[t,\infty)$ that

$$z^{1-(\lambda+1)\delta} \geqslant c_2 \int_t^{\infty} p(s)^{1-2\delta-\delta\lambda} q(s)^{\delta}\, \mathrm{d}s$$

and

$$u' u^{1/\alpha} \leqslant c_3 p(t)^{-1/\alpha} \biggl\{ \int_t^{\infty} p(s)^{1-2\delta-\delta\lambda} q(s)^{\delta}\, \mathrm{d}s \biggr\}^{1/(\alpha-\alpha\delta-\alpha\delta\lambda)},$$

where c_2 and c_3 are positive constants. Integrating this inequality over $[t_0, t]$, we find the conclusion. This completes the proof. □

Corollary 2.1. *Assume that*

$$0 < \liminf_{t\to\infty} \frac{p(t)}{t^{\beta}} \leqslant \limsup_{t\to\infty} \frac{p(t)}{t^{\beta}} < \infty \tag{2.1}$$

and

$$0 < \liminf_{t\to\infty} \frac{q(t)}{t^{\sigma}} \leqslant \limsup_{t\to\infty} \frac{q(t)}{t^{\sigma}} < \infty$$

holds for some β and σ satisfying (1.5). *Then we know by Theorem* 2.1 *that unbounded solutions u of* (1.1) *are estimated as $u(t) = O(t^k)$ as $t \to \infty$, where k is given by* (1.7).

Theorem 2.2 (Lower-estimate for unbounded solutions)**.** *Let u be an unbounded solution of* (1.1). *If there exists a function $\xi(t) \in C^1$ satisfying*

$$\xi(t) \geqslant p(t)^{1/\alpha} q(t), \quad \xi'(t) \geqslant 0, \quad \int^{\infty} p(t)^{-1/\alpha} \xi(t)^{1/(\alpha+1)}\, \mathrm{d}t < \infty,$$

then there exist a positive constant c and sufficiently large t_0 such that

$$u(t) \geqslant c \biggl\{ \int_t^{\infty} p(s)^{-1/\alpha} \xi(s)^{1/(\alpha+1)}\, \mathrm{d}s \biggr\}^{-(\alpha+1)/(\lambda-\alpha)} \quad \textit{for } t \geqslant t_0.$$

P r o o f. Let u be an unbounded solution of (1.1). Multiplying (1.1) by $p(t)^{1/\alpha}u'$, we have

$$\{p(t)(u')^{\alpha}\}'p(t)^{1/\alpha}u' = p(t)^{1/\alpha}q(t)u^{\lambda}u' \leqslant \xi(t)u^{\lambda}u',$$

since $u' > 0$. This inequality is equivalent to

$$\{p(t)(u')^{\alpha}\}'\{p(t)(u')^{\alpha}\}^{1/\alpha} \leqslant \xi(t)u^{\lambda}u';$$

that is

$$\frac{\alpha(\lambda+1)}{\alpha+1}\big[\{p(t)(u')^{\alpha}\}^{(\alpha+1)/\alpha}\big]' \leqslant \xi(t)(u^{\lambda+1})'.$$

Integrating this inequality over $[t_0, t]$, we obtain

$$c_1 p(t)^{(\alpha+1)/\alpha}(u')^{\alpha+1} \leqslant \xi(t)u^{\lambda+1} - \int_{t_0}^{t}\xi'(s)u^{\lambda+1}\,\mathrm{d}s \leqslant \xi(t)u^{\lambda+1},$$

since $p(t)^{(\alpha+1)/\alpha}(u')^{\alpha+1} \to \infty$ as $t \to \infty$ from Lemma 2.1, where $c_1 > 0$ is a constant. Noting that $\lambda > \alpha$ and $\lim\limits_{t\to\infty} u(t) = \infty$, we immediately obtain the conclusion of this theorem by integrating the inequality

$$u'u^{-(\lambda+1)/(\alpha+1)} \leqslant c_2 p(t)^{-1/\alpha}\xi(t)^{1/(\alpha+1)}$$

over $[t, \infty)$, where $c_2 > 0$ is a constant. This completes the proof. $\square$

Corollary 2.2. *Let* (2.1) *hold for some β and σ satisfying* (1.5). *Assume further that $\beta + \alpha\sigma \geqslant 0$. Then Theorem* 2.2 *gives a lower-estimate for unbounded solutions of* (1.1), *that is $u(t) \geqslant c_1 t^k$ as $t \to \infty$ for some constant $c_1 > 0$, where k is given by* (1.7).

In fact by putting $\xi(t) \equiv ct^{\sigma+\beta/\alpha}$ for sufficiently large c, we obtain Corollary 2.2 immediately. Moreover we can see that $0 < \liminf\limits_{t\to\infty} u(t)/(t^k) \leqslant \limsup\limits_{t\to\infty} u(t)/(t^k) < \infty$ when (2.1) holds for some β and σ satisfying (1.5) and $\beta + \alpha\sigma \geqslant 0$.

These estimates play a very important role in determining the asymptotic behavior of unbounded solutions of (1.1). At the end of this section we will show a comparison lemma for unbounded solutions of (1.1). Let us consider the two differential equations of the same form

$$\Big(\frac{1}{f(t)}|y'|^{a-1}y'\Big)' = g(t)|y|^{b-1}y, \quad t \geqslant t_0, \tag{2.2}$$

and

$$\Big(\frac{1}{F(t)}|Y'|^{a-1}Y'\Big)' = G(t)|Y|^{b-1}Y, \quad t \geqslant t_0, \tag{2.3}$$

where a and b are positive constants, and f, g, F and G are positive continuous functions on $[t_0, \infty)$.

Lemma 2.2 (Comparison lemma for unbounded solutions). *Suppose that* $f(t) \leqslant F(t)$, $g(t) \leqslant G(t)$ *for* $t \geqslant t_0$. *Let* y *and* Y *be unbounded solutions on* $[t_0, \infty)$ *of* (2.2) *and* (2.3), *respectively. If* $0 < y(t_0) \leqslant Y(t_0)$, $0 \leqslant y'(t_0) < Y'(t_0)$ *and* $f(t)F(t_0)y'(t_0) \leqslant f(t_0)F(t)Y'(t_0)$ *for* $t \geqslant t_0$, *then* $y(t) < Y(t)$ *for* $t > t_0$.

Proof. For $t \geqslant t_0$ we have

$$y(t) = y(t_0) + \int_{t_0}^{t} \left\{ f(s)\frac{y'(t_0)}{f(t_0)} + \int_{t_0}^{s} g(r)y(r)^b \,\mathrm{d}r \right\}^{1/a} \mathrm{d}s$$

and

$$Y(t) = Y(t_0) + \int_{t_0}^{t} \left\{ F(s)\frac{Y'(t_0)}{F(t_0)} + \int_{t_0}^{s} G(r)Y(r)^b \,\mathrm{d}r \right\}^{1/a} \mathrm{d}s.$$

Taking the difference of the two equations, we obtain

$$\tag{2.4} Y(t) - y(t) = Y(t_0) - y(t_0) - \int_{t_0}^{t} \left\{ f(s)\frac{y'(t_0)}{f(t_0)} + \int_{t_0}^{s} g(r)y(r)^b \,\mathrm{d}r \right\}^{1/a} \mathrm{d}s + \int_{t_0}^{t} \left\{ F(s)\frac{Y'(t_0)}{F(t_0)} + \int_{t_0}^{s} G(r)Y(r)^b \,\mathrm{d}r \right\}^{1/a} \mathrm{d}s.$$

Since $y(t_0) \leqslant Y(t_0)$ and $y'(t_0) < Y'(t_0)$, there exists $\delta > 0$ such that $Y(t) - y(t) > 0$ for $t_0 < t < t_0 + \delta$. Suppose to the contrary that there exists $t_1 \in (t_0, \infty)$ such that $Y(t_1) < y(t_1)$. Then there exists $t_2 \in (t_0, t_1)$ satisfying $y(t) < Y(t)$ for $t_0 < t < t_2$ and $y(t_2) = Y(t_2)$. Putting $t = t_2$ in (2.4) we find that the left-hand side is zero and the right-hand side is positive, which in a contradiction. So $y(t) \leqslant Y(t)$ for $t \in (t_0, \infty)$. This completes the proof. □

3. Proof of Theorem 1.1

The objective in this section is to prove Theorem 1.1., i.e., to determine the asymptotic behavior of unbounded solutions of (1.1). Regrettably we can not determine the asymptotic behavior of unbounded solutions in general situations. So we impose the assumption on the asymptotic behavior of functions $p(t)$ and $q(t)$ to be power-like, as in (1.8). When (1.8) holds we put

$$\tag{3.1} p(t) = \{1 + \varepsilon_1(t)\}t^{\beta}, \quad q(t) = c\{1 + \varepsilon_2(t)\}t^{\sigma},$$

where ε_1 and ε_2 are continuous functions satisfying $\lim\limits_{t\to\infty} \varepsilon_1(t) = \lim\limits_{t\to\infty} \varepsilon_2(t) = 0$. The change of variables introduced in the next lemma is a typical method to research the asymptotic behavior of positive solutions [1], [2], [3], [4].

Lemma 3.1 (Reduction to semi-autonomous form). *Let u be an unbounded solution of* (1.1). *Put $v = u/u_0$ and $t = \mathrm{e}^s$. Then v solves*

$$\ddot{v} + \left[\frac{\beta}{\alpha} + 2k - 1 + \frac{\dot{\tilde{\varepsilon}}_1(s)}{\alpha\{1+\tilde{\varepsilon}_1(s)\}}\right]\dot{v} + \frac{k}{\alpha}\left\{\frac{\dot{\tilde{\varepsilon}}_1(s)}{1+\tilde{\varepsilon}_1(s)} + \beta + \alpha(k-1)\right\}v = \frac{k^\alpha\{\beta+\alpha(k-1)\}\{1+\tilde{\varepsilon}_2(s)\}(\dot{v}+kv)^{1-\alpha}v^\lambda}{\alpha\{1+\tilde{\varepsilon}_1(s)\}}, \tag{3.2}$$

where $\dot{} = \mathrm{d}/\mathrm{d}s$, *$u_0$ is given by* (1.6), $\tilde{\varepsilon}_1(s) = \varepsilon_1(\mathrm{e}^s)$ *and* $\tilde{\varepsilon}_2(s) = \varepsilon_2(\mathrm{e}^s)$.

Next we introduce a lemma which is used when we apply the comparison lemma for unbounded solutions (Lemma 2.2).

Lemma 3.2. *Let $\mu > 0$ be a constant. Then the equation*

$$\alpha\varrho^{\alpha+1} + (\beta-\alpha)\varrho^\alpha = 2\mu \tag{3.3}$$

has only one positive root ϱ_μ. Moreover $\varrho_\mu \downarrow 0$ as $\mu \downarrow 0$.

Let us prove Theorem 1.1.

P r o o f of Theorem 1.1. Put $v = u/u_0$ and $t = \mathrm{e}^s$. Then we can see that v solves (3.2) from Lemma 3.1 and that $\limsup\limits_{t\to\infty} v < \infty$ from Corollary 2.1. Define an auxiliary function φ by

$$\varphi(s) = \left\{\frac{\dot{\tilde{\varepsilon}}_1(s)}{(\beta+\alpha(k-1))(1+\tilde{\varepsilon}_2(s))} + \frac{1+\tilde{\varepsilon}_1(s)}{1+\tilde{\varepsilon}_2(s)}\right\}^{1/(\lambda-\alpha)}$$

for sufficiently large s. By noting the fact that assumption (1.9) is equivalent to $\dot{\tilde{\varepsilon}}_1(s) \to 0$ as $s \to \infty$, we can see that $\lim\limits_{s\to\infty} \varphi(s) = 1$. If $\dot{v} = 0$ and $v > \varphi(s)$, then $\ddot{v} > 0$ there. This means that only minima can occur in the region $v > \varphi(s)$. Similarly, only maxima can occur in the region $0 < v < \varphi(s)$. Using the same method as in Theorem 1 of [4], we see that v is eventually monotone or that $v(s) \to 1$ as $s \to \infty$. Hence v has a nonnegative finite limit.

First, we will show that $\lim\limits_{s\to\infty} v(s) \neq 0$. For this purpose, we may show that there exists a positive constant c_1 satisfying $c_1 t^k \leqslant u$. If $\beta + \alpha\sigma \geqslant 0$, then this is clear from Corollary 2.2. So we consider the case $\beta + \alpha\sigma < 0$. We note that this means $\sigma < -1$ since $0 < \alpha < \beta$. Let us assume to the contrary that $v \to 0$ as $s \to \infty$. This implies that $u/t^k \to 0$ as $t \to \infty$. Note that the equation (1.1) can be rewritten in the form

$$(p(t)|u'|^{\alpha-1}u')' = \{q(t)u(t)^{\lambda-\alpha}\}u^\alpha.$$

Let $\mu > 0$ be a sufficiently small constant satisfying $\sigma + \lambda\varrho_\mu < -1$, where ϱ_μ is the positive root of the equation (3.3) in Lemma 3.2. Since $p(t) \sim t^\beta$ and $u/t^k \to 0$ as $t \to \infty$, there exists sufficiently large $t_1 > 0$ satisfying

$$q(t)u(t)^{\lambda-\alpha} \leqslant \mu t^{\beta-\alpha-1} \quad \text{and} \quad \frac{t^\beta}{2} \leqslant p(t) \quad \text{for } t \geqslant t_1.$$

We consider the equation

$$\Big(\frac{t^\beta}{2}|w'|^{\alpha-1}w'\Big)' = \mu t^{\beta-\alpha-1}w^\alpha, \quad t \geqslant t_1.$$

We know that this half-linear equation has a family of positive solutions $U(t;M) = Mt^{\varrho_\mu}$, $M > 0$. We can take M large enough so that

$$u(t_1) \leqslant Mt_1^{\varrho_\mu}, \quad u'(t_1) < M\varrho_\mu t_1^{\varrho_\mu-1} \quad \text{and} \quad \frac{\{1+\varepsilon_1(t_1)\}u'(t_1)}{1+\varepsilon_1(t)} \leqslant M\varrho_\mu t_1^{\varrho_\mu-1}.$$

From Lemma 2.2 we obtain that $u(t) \leqslant U(t;M)$, $t \geqslant t_1$. Substituting this estimate into the equation (1.1), we find that

$$(p(t)|u'|^{\alpha-1}u')' \leqslant M^\lambda q(t)t^{\lambda\varrho_\mu} \leqslant c_1 t^{\sigma+\lambda\varrho_\mu}, \quad t \geqslant t_1,$$

where c_1 is a positive constant. Since $\sigma + \lambda\varrho_\mu < -1$ by the choice of $\mu > 0$, an integration of this inequality shows that $p(t)(u')^\alpha = O(1)$ as $t \to \infty$. However, this is an obvious contradiction to Lemma 2.1. Hence v has a positive finite limit l as $s \to \infty$.

Next, we will show that $l = 1$. Employing l'Hospital's rule, we obtain

$$\begin{aligned} l = \lim_{t\to\infty}\frac{u(t)}{u_0(t)} = \lim_{t\to\infty}\frac{u'(t)}{u_0'(t)} &= \Big[\lim_{t\to\infty}\frac{\{p(t)u'(t)^\alpha\}'}{\{t^\beta u_0'(t)^\alpha\}'}\Big]^{1/\alpha} \\ &= \Big\{\lim_{t\to\infty}\frac{q(t)u(t)^\lambda}{ct^\sigma u_0(t)^\lambda}\Big\}^{1/\alpha} = l^{\lambda/\alpha}. \end{aligned}$$

Since $\alpha < \lambda$, this implies that $l = 1$, i.e., $u(t) \sim u_0(t)$. This completes the proof. $\square$

As an application of this theorem, we give the following result which will be used to show the existence of some kinds of solutions of elliptic equations. The details appear in Section 4.

Example 3.1. Consider the equation

$$(t^{N-1}|u'|^{m-2}u')' = t^{N-1}h(t)|u|^{\lambda-1}u, \tag{3.4}$$

where m, λ and N are constants such that $0 < m - 1 < \lambda$ and $m < N$, and h is a positive continuous function satisfying $h(t) \sim ct^{\sigma_1}$ for some $c > 0$, $\sigma_1 \in \mathbb{R}$. If $m + \sigma_1 < 0$, then every unbounded solution of (3.4) has the asymptotic form

$$u(t) \sim \hat{c}_1 t^{k_1}, \tag{3.5}$$

where

$$k_1 = \frac{-m - \sigma_1}{\lambda - m + 1} > 0$$

and

$$\hat{c}_1^{\lambda - m + 1} = \frac{(-m - \sigma_1)^{m-1}\{N(\lambda - m + 1) + \sigma_1(1 - m) - m\lambda\}}{c(\lambda - m - 1)^m}. \tag{3.6}$$

Note that solutions of the equation (3.4) are radial solutions of the quasilinear elliptic equation

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) = h(|x|)|u|^{\lambda-1}u$$

in an exterior domain in $\mathbb{R}^N$.

4. Application to elliptic problems

In this section we show that the preceding result for the ordinary differential equation (1.1) can be applied to obtain the existence of some solutions of the exterior Dirichlet problem for the elliptic equation :

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) = f(x)|u|^{\lambda-1}u, \quad \text{in } \Omega, \tag{4.1}$$

$$u = g(x), \quad \text{on } \partial\Omega, \tag{4.2}$$

where Ω is an unbounded exterior domain in $\mathbb{R}^N$, $N \geqslant 2$, with boundary $\partial\Omega$ of class C^1, $0 < m - 1 < \lambda$, $f \in C^1(\mathbb{R}^N; (0,\infty))$ and $g \in C^1(\partial\Omega; (0,\infty))$. We assume throughout this section that for some constants $c_1 > 0$ and $\sigma_1 \in \mathbb{R}$

$$f(x) \sim c_1|x|^{\sigma_1} \quad \text{as } |x| \to \infty.$$

A function $u \in W^{1,m}_{\mathrm{loc}}(\Omega)$ is said to be a solution (subsolution, supersolution) of the equation (4.1) in Ω if

$$\int_\Omega (|\nabla u|^{m-2}\nabla u \cdot \nabla\varphi + f(x)|u|^{\lambda-1}u\varphi)\,\mathrm{d}x = 0 \ \ (\leqslant 0,\ \geqslant 0),$$

for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$ in Ω, and $u = g(x)$ ($\leqslant 0,\ \geqslant 0$) almost everywhere on $\partial\Omega$ in the trace sense. See [7] for details.

To find positive solutions of the problem (4.1)–(4.2) we use the supersolution-subsolution method which can be formulated, in our context, as follows:

Proposition 4.1 (Theorem 4.4 in [7]). *Let v and w be a subsolution and a supersolution of* (4.1) *in Ω, respectively, such that $v \leqslant w$ a.e. in Ω and $v \leqslant g \leqslant w$ a.e. on $\partial\Omega$. Then the problem* (4.1)-(4.2) *has a solution u such that $v \leqslant u \leqslant w$ a.e. in Ω.*

We introduce notations used here. We may assume without loss of generality that $0 \notin \bar{\Omega}$. Let

$$\begin{aligned} g_* &= \min_{\partial\Omega} g(x), & g^* &= \max_{\partial\Omega} g(x);\\ r_* &= \operatorname{dist}(0, \partial\Omega), & r^* &= \max\{|x|; x \in \partial\Omega\};\\ f_*(r) &= \min_{|x|=r} f(x), & f^*(r) &= \max_{|x|=r} f(x). \end{aligned}$$

Theorem 4.1. *If $m < N$ and $\sigma_1 + N < 0$, then the problem* (4.1)–(4.2) *has a positive unbounded solution u satisfying*

$$0 < \liminf_{|x|\to\infty} \frac{u(x)}{|x|^{k_1}} \leqslant \limsup_{|x|\to\infty} \frac{u(x)}{|x|^{k_1}} < \infty, \tag{4.3}$$

where k_1 is given by (3.6).

P r o o f. We construct an appropriate supersolution and a subsolution of (4.3) as radially symmetric functions, and investigate the asymptotic forms by Example 3.1. A function $\bar{u}$ satisfying

$$\operatorname{div}(|\nabla \bar{u}|^{m-2} \nabla \bar{u}) \leqslant f_*(|x|)\bar{u}^\lambda, \quad |x| \geqslant r_* \tag{4.4}$$

is a supersolution of the equation (4.1). Similarly a function $\underline{u}$ satisfying

$$\operatorname{div}(|\nabla \underline{u}|^{m-2} \nabla \underline{u}) \geqslant f^*(|x|)\underline{u}^\lambda, \quad |x| \geqslant r_* \tag{4.5}$$

is a subsolution of the equation (4.1). Since these inequalities have radial symmetry, it is natural to construct such $\bar{u}$ and $\underline{u}$ as radially symmetric functions. By putting $\bar{u}(x) = \overline{v}(r)$ and $\underline{u}(x) = \underline{v}(r)$, where $r = |x|$, (4.4) and (4.5) are reduced to

$$(r^{N-1}|\overline{v}'|^{m-2}\overline{v}')' \leqslant r^{N-1} f_*(r)\overline{v}^\lambda, \quad r \geqslant r_* \tag{4.6}$$

and

$$(r^{N-1}|\underline{v}'|^{m-2}\underline{v}')' \geqslant r^{N-1}f^*(r)\underline{v}^\lambda, \quad r \geqslant r_*, \tag{4.7}$$

respectively, where $' = \mathrm{d}/\mathrm{d}r$.

First we construct a supersolution $\bar{u}$. Consider the initial value problem for the ordinary differential equation

$$\begin{cases} (r^{N-1}|\overline{w}'|^{m-2}\overline{w}')' = r^{N-1}f_*(r)\overline{w}^\lambda, \quad r \geqslant r_*, \\ \overline{w}(r_*) = g^*. \end{cases}$$

From Theorem 4.7 in [11], this problem has at least one positive unbounded solution $\overline{w}$. Since $m < N$ and $\sigma_1 + N < 0$, we find from Example 3.1 that

$$\overline{w}(r) \sim c_1 r^{k_1} \quad \text{as } r \to \infty, \tag{4.8}$$

where c_1 and k_1 are given by (3.6). Hence, the function given by $\overline{v}(r) \equiv \overline{w}(r)$ satisfies (4.7) (with $\leqslant$ replaced by $=$). This means that the function $\bar{u}(x) \equiv \overline{v}(|x|)$ is a supersolution of (4.1) satisfying $\bar{u}(x) \geqslant g(x)$ on $\partial\Omega$.

Next we must construct a subsolution $\underline{u}$ so that $\underline{u} \leqslant \bar{u}$ in Ω and $\underline{u} \leqslant g$ on $\partial\Omega$. Let $\delta > 0$ be a sufficiently small constant satisfying $\delta\overline{w}(r^*) \leqslant g_*$ and $f^*(r) \leqslant \delta^{m-1-\lambda}f_*(r)$ for $r \geqslant r_*$. This choice of δ is possible because $f_*(r) \sim f^*(r)$ as $r \to \infty$. Then the function $\underline{w}(r) \equiv \delta\overline{w}(r)$ satisfies

$$(r^{N-1}|\underline{w}'|^{m-2}\underline{w}')' = \delta^{m-1-\lambda}r^{N-1}f_*(r)\underline{w}^\lambda \geqslant f^*(t)\underline{w}^\lambda, \quad r \geqslant r_*.$$

Therefore, the function $\underline{u}(x) \equiv \underline{w}(|x|)$ $(= \delta\bar{u}(x))$ is a subsolution of (4.1) satisfying $\underline{u} \leqslant \bar{u}$ in Ω and $\underline{u} \leqslant g_*$ on $\partial\Omega$. Hence Proposition 4.1 guarantees that the problem (4.1)–(4.2) has a positive solution u satisfying

$$\delta\bar{u}(x) \leqslant u(x) \leqslant \bar{u}(x), \quad \text{a.e. } x \in \Omega.$$

Since $\bar{u}(x) \equiv \overline{v}(t)$, $t = |x|$, satisfying (4.8), we find that (4.3) holds. This completes the proof. $\square$

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