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SURGERY ON PAIRS OF CLOSED MANIFOLDS

Alberto Cavicchioli, Modena, Yuri V. Muranov, Vitebsk,
Fulvia Spaggiari, Modena

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Abstract. To apply surgery theory to the problem of classifying pairs of closed manifolds, it is necessary to know the subgroup of the group LP_* generated by those elements which are realized by normal maps to a pair of closed manifolds. This closely relates to the surgery problem for a closed manifold and to the computation of the assembly map. In this paper we completely determine such subgroups for many cases of Browder-Livesay pairs of closed manifolds. Moreover, very explicit results are obtained in the case of an elementary fundamental group. Then we generalize them, and obtain several further results about the realization of elements in the Browder-Quinn surgery obstruction groups by means of normal maps to a closed manifold filtered by closed submanifolds.

Keywords: surgery on manifolds, surgery obstruction groups for a manifold pair, assembly map, splitting problem, Browder-Livesay groups, Browder-Quinn surgery obstruction groups, splitting obstruction groups, manifolds with filtration

MSC 2010: 57R67, 19J25, 55T99, 58A35, 18F25

1. Introduction

The representation of an element x in a Wall group $L_n(\pi)$ as a surgery obstruction of a degree-one normal map $(f, b)\colon M^n \to X^n$ between closed manifolds with $\pi_1(X) = \pi$ is one of the basic problems in Surgery Theory. This is equivalent to the fact that the element x belongs to the image of the assembly map

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$A\colon H_n(B\pi, \mathbf{L}_\bullet) \to L_n(\pi)$, where $B\pi = K(\pi, 1)$ is the classifying space of the group π (see [29]).

Let $Y^{n-q} \subset X^n$ be a locally flat submanifold of codimension q in a closed connected topological n-manifold X, $n - q \geqslant 5$, and M^n a closed connected topological n-manifold. A simple homotopy equivalence $g\colon M^n \to X^n$ which is transversal to Y is said to be *splitted along the submanifold* Y if the restrictions

$$g\big|_N\colon N \to Y \quad \text{and} \quad g\big|_{(M\setminus N)}\colon (M \setminus N) \to (X \setminus Y) \tag{1.1}$$

are simple homotopy equivalences, where $N = g^{-1}(Y)$. Let

$$(f, b)\colon (M^n, \nu_M) \to (X^n, \nu_X)$$

be a degree-one normal map. It is well known that there is an obstruction $\sigma(f, b)$ to the existence of a splitted simple homotopy equivalence

$$g\colon M'^n \to X^n$$

in the normal cobordism class of (f, b) (see [25] and [29]). This obstruction lies in the surgery obstruction group $LP_{n-q}(F)$, that is, the surgery obstruction group for the manifold pair (X, Y), where F is the square of fundamental groups

$$F = \begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(X \setminus Y) \\ \big\downarrow & & \big\downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(X) \end{pmatrix} \tag{1.2}$$

together with the natural maps. Here U is a tubular neighborhood of Y in X with boundary ∂U. The groups LP_* depend on F and on the dimension $n - q \pmod 4$ (see [25] and [29]).

Let $f\colon M \to X$ be a simple homotopy equivalence. Then there is an obstruction to the splitting of the map f along Y, that is, an obstruction to the existence in its homotopy class of a map g satisfying (1.1) and transversal to Y. This obstruction lies in the splitting obstruction group $LS_{n-q}(F)$. These groups also depend on F and on the dimension $n - q \pmod 4$ [29].

It follows from the geometrical definitions of such groups that there are obvious maps [29]

$$\begin{aligned} &LS_{n-q}(F) \xrightarrow{s} LP_{n-q}(F), \qquad && LP_{n-q}(F) \xrightarrow{p_0} L_n(\pi_1(X)), \\ &LP_{n-q}(F) \xrightarrow{p_1} L_{n-q}(\pi_1(Y)), \qquad && LS_{n-q}(F) \to L_{n-q}(\pi_1(Y)). \end{aligned} \tag{1.3}$$

Note that all elements in the groups $LS_*(F)$ and $LP_*(F)$ are realized by means of maps between manifolds with boundary (see [25] and [29]).

The groups LP_* and LS_* closely relate to the surgery exact sequence for the manifold X by means of the braid of exact sequences (see [24], [25], and [29])

$$
(1.4)\qquad
\begin{array}{ccccccccccc}
\to & \mathscr{S}^s_{n+1}(X,Y,\xi) & & \longrightarrow & & [X,G/TOP] & & \longrightarrow & & L_n(\pi) & \to \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
 & & & \mathscr{S}^s_{n+1}(X) & & & & LP_{n-q}(F) & & & \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & L_{n+1}(\pi) & & \longrightarrow & & LS_{n-q}(F) & & \longrightarrow & & \mathscr{S}^{BQ}_{n+1}(\mathscr{X}) & \to
\end{array}
$$

where $\pi = \pi_1(X)$, $\mathscr{S}^s_{n+1}(X)$ is the set of the s-cobordism classes of closed connected topological n-dimensional manifolds which are simple homotopy equivalent to X, $[X, G/TOP]$ is the set of the normal invariants, $\mathscr{S}_n(X,Y,\xi)$ is the set of the concordance classes of s-triangulations for the pair of manifolds (X,Y), and ξ denotes the normal bundle of Y in X [25].

General results about several structure sets that arise naturally for a manifold pair were obtained in [3] and [9]. Some results about the realization of the splitting obstructions for specified classes of groups were obtained in [1], [10], and [17].

In the present paper we compute the subgroups of LP_*-groups generated by those elements that are realized by normal maps to a pair of closed topological manifolds in the case of a Browder-Livesay pair of manifolds. In the case of an elementary fundamental group we obtain very explicit results. Then we prove several theorems about the realization of elements in the Browder-Quinn surgery obstruction groups by means of normal maps to a closed manifold filtered by closed submanifolds.

We use definitions and results in surgery theory from [2], [8], [9], [12], [19], [24], [25], and [29], and many applications of them to surgery on stratified manifolds (see [4], [6], [10], [17], [22], [23], and [32]). In Section 2 we prove several general technical results, and then use them to study the closed manifold surgery problem for LP_*-groups in the case of a Browder-Livesay pair with an abelian 2-group of the ambient manifold. In Section 3 we obtain several results about the surgery obstructions for closed manifolds with a filtration, and use them to study the closed manifold surgery problem for L^{BQ}-groups of triples of closed manifolds. The main results of the paper are given in Theorems 1–6.

Here we consider only topological manifolds and locally flat submanifolds. All pairs of manifolds are topological manifold pairs in the sense of Ranicki, and all obstruction groups are considered up to simple homotopy equivalence.

In this section we apply some results about the closed manifold surgery problem from [15] and [16] to investigating the surgery obstruction groups for manifold pairs. The main results relate to Browder-Livesay pairs of manifolds.

Let Y^{n-1} be a codimension 1 one-sided submanifold of a manifold X^n. Let $\pi = \pi_1(X)$ be the fundamental group of X equipped with the orientation homomorphism $w\colon \pi \to \{\pm 1\}$ given by the first Stiefel-Whitney character. The pair of manifolds (X, Y) is a Browder-Livesay pair if $n \geqslant 6$ and the square in (1.2) has the form (see [5], [7], [11], [18], and [19])

$$(2.1) \qquad F = \begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(X) \end{pmatrix} = \begin{pmatrix} \varrho & \xrightarrow{\cong} & \varrho \\ \downarrow{\scriptstyle i_-} & & \downarrow{\scriptstyle i_+} \\ \pi^- & \xrightarrow{\cong} & \pi^+ \end{pmatrix}$$

where the horizontal maps are isomorphisms, the vertical maps are inclusions of index 2, and the orientation homomorphisms on the groups $\pi_1(X) = \pi^+$ and $\pi_1(Y) = \pi^-$ coincide on the images of the vertical maps and differ outside these images. We denote such orientations of the groups by the symbols "+" and "−" and omit "+" if this does not lead to any confusion.

The splitting obstruction groups for a Browder-Livesay pair (X, Y) are denoted by

$$LN_{n-1}(\pi_1(X \setminus Y) \to \pi_1(X)) = LN_{n-1}(\varrho \to \pi) = LS_{n-1}(F)$$

and are called the *Browder-Livesay groups* (see [5], [7], [11], [16], [18], [19], [25, § 7.2], [26] and [29, § 11]). In the case of a Browder-Livesay pair there is an isomorphism $LP_n(F) \cong L_{n+1}(i^!_-)$, where $i^!_-$ is the surgery transfer map for the inclusion i_- in the square F shown in (2.1).

Note that the LP_*-groups and the Browder-Livesay groups are defined for any inclusion $i\colon\ \varrho \to \pi$ of index 2 between oriented groups (see [11], [25], and [26]).

Recall that for an arbitrary manifold pair $Y^{n-q} \subset X^n$ the maps in (1.3) fit in the braid of exact sequences [29, p. 264]

$$(2.2) \qquad \begin{array}{ccccccccc} \to & L_n(\pi_1(X\setminus Y)) & \longrightarrow & & L_n(\pi_1(X)) & & \longrightarrow & LS_{n-q-1}(F) & \longrightarrow \\ \nearrow & & \searrow & \nearrow{\scriptstyle p_0} & & \searrow & \nearrow & & \searrow \\ & & & LP_{n-q}(F) & & & L_n^{\mathrm{rel}} & & \\ \searrow & & \nearrow & \searrow{\scriptstyle p_1} & & \nearrow & \searrow & & \nearrow \\ \to & LS_{n-q}(F) & \xrightarrow{\ s\ } & & L_{n-q}(\pi_1(Y)) & & \longrightarrow & L_{n-1}(\pi_1(X\setminus Y)) & \longrightarrow \end{array}$$

where $L_n^{\mathrm{rel}} = L_n(\pi_1(X \setminus Y) \to \pi_1(X))$.

Lemma 1. *Let $x \in LP_{n-q}(F)$ be an element such that either $y = p_0(x) \in L_n(\pi_1(X))$ or $z = p_1(x) \in L_{n-q}(\pi_1(Y))$ is not realized by a normal map of closed manifolds. Then x cannot be the obstruction to surgery on a closed manifold pair.*

P r o o f. By definition (see [25] and [29]) the maps p_0 and p_1 in Diagram (2.2) are the natural forgetful maps. Thus, if an element $x \in LP_{n-q}(F)$ is realized by a normal map $f\colon (M, N) \to (X, Y)$ of manifold pairs, then the restriction $f\big|_M$ represents the element $p_0(x)$, and the restriction $f\big|_N$ represents the element $p_1(x)$. Then the result immediately follows. □

Now let us consider the case of a Browder-Livesay pair (X, Y) such that $\pi_1(X) = \pi$ is an elementary abelian 2-group of rank $r + 1 \geqslant 1$ with a trivial orientation. Up to isomorphism of inclusions, we have only one inclusion $i\colon \varrho \to \pi$ of index 2. Note that ϱ is an elementary 2-group of rank r with a trivial orientation. Let π^- be the group π equipped with a nontrivial orientation. Up to isomorphism of inclusions, we have two oriented subgroups of index 2 in the group π. The first is given by an inclusion

$$i\colon \varrho \to \pi^- \tag{2.3}$$

where the subgroup ϱ has a trivial orientation. In this case, $\pi^- \cong \varrho \oplus \mathbb{Z}/2^-$. The second (for $r \geqslant 1$) is given by an inclusion

$$j\colon \varrho^- \to \pi^- \tag{2.4}$$

where the subgroup ϱ^- has a nontrivial orientation, and we have

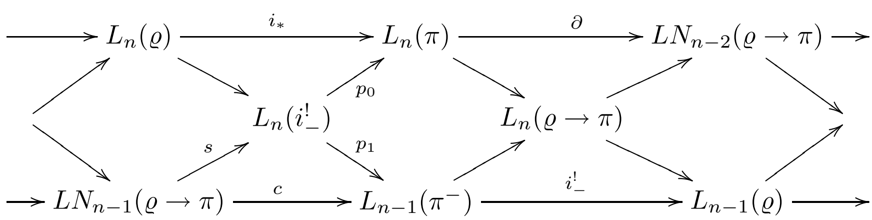
$$\pi^- \cong \varrho^- \oplus \mathbb{Z}/2^- \cong \varrho^- \oplus \mathbb{Z}/2. \tag{2.5}$$

There are the following isomorphisms [29]:

$$\begin{aligned} LN_n(\varrho \to \pi^-) &\cong LN_{n+2}(\varrho \to \pi) \cong L_n(\varrho), \\ LN_n(\varrho^- \to \pi^-) &\cong LN_{n+2}(\varrho^- \to \pi^-) \cong L_n(\varrho^-). \end{aligned} \tag{2.6}$$

For the square in (2.1), Diagram (2.2) has the form (see [11], [19], [21], [25, § 7], [28], and [29, § 11])

(2.7)



For an elementary group π with an arbitrary orientation, the splitting obstruction groups and the surgery obstruction groups from Diagram (2.7) are known (see [17], [21], [29]):

$$L_n(\varrho) = \mathbb{Z}^{2^r} \oplus (\mathbb{Z}/2)^{2^r-r-1-\binom{r}{2}},\ (\mathbb{Z}/2)^{2^r-r-1-\binom{r}{2}},\ \mathbb{Z}/2,\ (\mathbb{Z}/2)^{2^r-1} \tag{2.8}$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively, and

$$\begin{aligned} L_n(\pi^-) &= (\mathbb{Z}/2)^{2^r-r}, && (\mathbb{Z}/2)^{2^r-r-1} \\ L_n(\varrho^-) &= (\mathbb{Z}/2)^{2^{r-1}-r+1}, && (\mathbb{Z}/2)^{2^{r-1}-r} \end{aligned} \tag{2.9}$$

for $n = 0, 1 \pmod 2$, respectively.

Note that the determination of the subgroup of $LP_{n-q}(F)$ generated by the normal maps of closed pairs is necessary for computing the map $[X, G/TOP] \to LP_{n-q}(F)$ and the structure set $\mathscr{S}^s_{n+1}(X, Y, \xi)$ from Diagram (1.4).

Lemma 2. *Let $\varrho^- \to \pi^{\pm} = \varrho^- \oplus \mathbb{Z}/2^{\pm}$ be an inclusion of index 2 between elementary 2-groups, and $r = \operatorname{rank} \varrho \geqslant 1$. Let*

$$F^{\pm} = \left(\begin{array}{ccc} \varrho^- & \longrightarrow & \varrho^- \\ \downarrow & & \downarrow \\ \pi^{\mp} & \longrightarrow & \pi^{\pm} \end{array}\right) \tag{2.10}$$

denote the various squares in the corresponding splitting problems. Then there are natural isomorphisms $\pi^- \cong \pi^+$ and $F^- \cong F^+$ of oriented objects.

P r o o f. The proof of Lemma 2 is immediate. □

Proposition 1. *Under the assumptions of Lemma 2, there are isomorphisms*

$$LP_n(F^{\pm}) \cong (\mathbb{Z}/2)^{2^r+2^{r-1}-2r}$$

for every $n = 0, 1, 2, 3 \pmod 4$.

P r o o f. Let us consider Diagram (2.7) for the inclusion $i\colon\ \varrho^- \to \pi^- = \varrho^- \oplus \mathbb{Z}/2^+$ in the first direct summand. In this diagram all the maps

$$i_*\colon\ L_n(\varrho^-) \to L_n(\varrho^- \oplus \mathbb{Z}/2^+)$$

are monomorphisms. This easily follows from the functoriality of the L_*-groups and from the existence of the left inverse of the inclusion i. The same argument (using Lemma 2 and results of [29, p. 161]) implies that all the maps

$$c\colon\ LN_n(\varrho^- \to \varrho^- \oplus \mathbb{Z}/2^+) \to L_n(\varrho^- \oplus \mathbb{Z}/2^-)$$

in Diagram (2.7) are monomorphisms, too. Hence all the maps

$$\partial\colon\ L_n(\varrho^- \oplus \mathbb{Z}/2^+) \to LN_{n-2}(\varrho^- \to \varrho^- \oplus \mathbb{Z}/2^+)$$

and

$$i^!_-\colon\ L_n(\varrho^- \oplus \mathbb{Z}/2^-) \to L_n(\varrho^-)$$

are trivial, and this yields a natural decomposition

$$LP_n(F^+) \cong L_{n+1}(\varrho^-) \oplus L_n(\varrho^- \oplus \mathbb{Z}/2^-)$$

(see also [23]). The discussion for the square F^- is similar by using again Lemma 2. Now the result follows from (2.9). □

Theorem 1. *Under the assumptions of Lemma* 2, *for all* $n = 0, 1, 2, 3 \pmod 4$ *there exists a direct summand*

$$C = \mathbb{Z}/2 \subset LP_n(F^\pm) = (\mathbb{Z}/2)^{2^r + 2^{r-1} - 2r}$$

such that the elements that do not lie in C *cannot be realized as surgery obstructions of normal maps between closed manifold pairs.*

For all $n = 0, 1, 2, 3 \pmod 4$ *the nontrivial element of* $C \subset LP_n(F^\pm)$ *is realized as a surgery obstruction of a normal map* $f\colon\ (M^{n+1}, N^n) \to (X^{n+1}, Y^n)$ *between pairs of closed topological manifolds.*

P r o o f. First we consider the case $LP_1(F^+)$. Using Proposition 1, we can write down the following part of Diagram (2.7):

$$\begin{array}{ccccc} L_2(\varrho^-) & & \xrightarrow{\text{mono}} & & L_2(\varrho^- \oplus \mathbb{Z}/2^+) \\ & \searrow & & \nearrow{\scriptstyle \text{epi}} & \\ & & LP_1(F^+) & & \\ & \nearrow & & \searrow{\scriptstyle \text{epi}} & \\ LN_1(\varrho^- \to \varrho^- \oplus \mathbb{Z}/2^+) & & \xrightarrow{\text{mono}} & & L_1(\varrho^- \oplus \mathbb{Z}/2^-). \end{array} \tag{2.11}$$

By [16] there are no nontrivial elements in the group $L_1(\varrho^- \oplus \mathbb{Z}/2^-) = L_1(\pi^-)$ that are realized by normal maps of closed manifolds. Furthermore, there exists only one such element y in the group $L_2(\varrho^- \oplus \mathbb{Z}/2^+) = L_2(\pi^-)$. Now Proposition 1 and Diagram (2.11) imply the first statement of the theorem for the group $LP_1(F^+)$.

We can realize the element $y \in L_2(\pi^-)$ by a normal map $f\colon M^{4k+2} \to X^{4k+2}$ of closed manifolds, where $\pi_1(X) = \pi^-$. Using the multiplication of the surgery problem on $\mathbb{C}P^2$ (see [29, § 9]) we can always assume that $4k + 2 \geqslant 6$. Let us consider a map $\phi\colon X \to \mathbb{R}P^N$ from X to a real projective space of high dimension which induces an epimorphism of the fundamental groups $\pi \to \mathbb{Z}/2$ with kernel ϱ. Let $Y = \phi^{-1}(\mathbb{R}P^{N-1})$ be a transversal preimage. Let us consider a map g in the class of normal bordism of f which is transversal to Y and $g^{-1}(Y) = N$. Then we obtain a normal map

$$g\colon (M, N) \to (X, Y)$$

of manifold pairs with a surgery obstruction $x \in L_1(F^+)$. By our construction, the element $p_0(x)$ is represented by the normal map f. Hence the element x is nontrivial. The other odd dimensional cases can be treated in a similar way.

In even dimension, the same kind of argument provides the first statement of the theorem. Example 2.4 in [1] gives a simple homotopy equivalence

$$f\colon M^7 \to X^7 \tag{2.12}$$

to a Browder-Livesay pair (X^7, Y^6) of closed manifolds with the nontrivial splitting obstruction $\sigma(f)$ lying in the group

$$LN_2(\mathbb{Z}/2^- \to \mathbb{Z}/2^- \oplus \mathbb{Z}/2^+) = LN_2(\mathbb{Z}/2^- \to \mathbb{Z}/2^- \oplus \mathbb{Z}/2^+) = \mathbb{Z}/2.$$

Let us consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}/2 & & \mathbb{Z}/2 \\
\downarrow{=} & & \downarrow{=} \\
LN_2(\mathbb{Z}/2^- \to \mathbb{Z}/2^- \oplus \mathbb{Z}/2^+) & \xrightarrow{\cong} & L_2(\mathbb{Z}/2^- \oplus \mathbb{Z}/2^+) \\
\downarrow{\times(\mathbb{R}P^2)^{r-1}} & & \downarrow{\times(\mathbb{R}P^2)^{r-1}} \\
LN_{2k}(\varrho^- \to \varrho^- \oplus \mathbb{Z}/2^+) & \longrightarrow & L_{2k}(\varrho^- \oplus \mathbb{Z}/2^+)
\end{array} \tag{2.13}$$

where $k = 1$ for r odd and $k = 0$ for r even. The vertical maps are obtained by multiplying the corresponding splitting and surgery problems by $(\mathbb{R}P^2)^{r-1}$. The right vertical map is a monomorphism by [29, pp. 179–180]. Hence the left vertical map is a monomorphism, too. Thus multiplying the splitting problem (2.12) by $(\mathbb{R}P^2)^{r-1}$ provides a splitting problem

$$f \times \mathrm{Id}\colon M^7 \times (\mathbb{R}P^2)^{r-1} \to X^7 \times (\mathbb{R}P^2)^{r-1} \tag{2.14}$$

with nontrivial splitting obstruction

$$\sigma(f \times \mathrm{Id}) \in LN_{2k}(\varrho^- \to \varrho^- \oplus \mathbb{Z}/2^+).$$

The map $f \times \mathrm{Id}$ in (2.14) is a simple homotopy equivalence by our construction. Let us consider the following commutative triangle from Diagram (2.7):

$$\begin{array}{ccccc} & & LP_{2k}(F) & & \\ & \overset{s}{\nearrow} & & \overset{p_1}{\searrow} & \\ LN_{2k}(\varrho^- \to \varrho^- \oplus \mathbb{Z}/2^+) & & \xrightarrow{\quad c \quad} & & L_{2k}(\varrho^- \oplus \mathbb{Z}/2^-). \end{array} \tag{2.15}$$

The element $y = s(\sigma(f \times \mathrm{Id})) \in LP_{2k}(F)$ is represented by the map (2.14) as follows from the geometrical definition (see [25] and [29]). The element y is nontrivial since $p_1(y)$ coincides with $c(\sigma(f \times \mathrm{Id}))$ which is nontrivial by Diagram (2.13). So the theorem is completely proved. □

Let now ϱ be an elementary 2-group of rank $r \geqslant 0$ with a trivial orientation. Let $\varrho \to \pi^{\pm} = \varrho \oplus \mathbb{Z}/2^{\pm}$ be an inclusion of index 2 between elementary 2-groups, and let

$$F^{\pm} = \left(\begin{array}{ccc} \varrho & \longrightarrow & \varrho \\ \downarrow & & \downarrow \\ \pi^{\mp} & \longrightarrow & \pi^{\pm} \end{array} \right) \tag{2.16}$$

denote the various squares in the corresponding splitting problems.

Proposition 2. *Let $F^{\pm}$ be the square in* (2.16) *with* $\operatorname{rank} \varrho = r$. *Then there are isomorphisms*

$$LP_n(F^+) = (\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-1},\ (\mathbb{Z}/2)^{2^r-r},\ (\mathbb{Z}/2)^{2^{r+1}-r-1},$$
$$(\mathbb{Z})^{2^r} \oplus (\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-2}$$

for $n = 0, 1, 2, 3 \pmod 4$, *respectively, and there are isomorphisms*

$$LP_n(F^-) \cong LP_{n-1}(F)$$

for $n = 0, 1, 2, 3 \pmod 4$.

P r o o f. The proof is similar to that given in Proposition 1. Let us consider Diagram (2.7) for the inclusion $\varrho \to \pi$. By diagram chasing we obtain

$$LP_n(F) = L_{n+1}(\varrho) \oplus L_n(\pi^-)$$

for any n. Now the result follows from (2.8) and (2.9). The discussion of the case F^- is similar. Thus, we obtain a direct sum decomposition

$$LP_n(F^-) = LN_n(\varrho \to \pi^-) \oplus L_{n+1}(\pi^-) \cong L_n(\varrho) \oplus L_{n+1}(\pi^-).$$

□

Theorem 2. *Under the assumptions of Proposition 2 for the square F^+, we have the following results:*

i) *Only the elements of the direct summand*

$$\mathbb{Z}/2 = \mathrm{Im}\{\mathbb{Z}/2 = LN_0(\varrho \to \pi) \overset{\text{mono}}{\longrightarrow} LP_0(F)\} \subset LP_0(F) = (\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-1}$$

are realized as surgery obstructions of normal maps between closed manifold pairs.

ii) *Only the elements of the direct summand*

$$\mathbb{Z}/2 = \mathrm{Im}\{\mathbb{Z}/2 = L_2(\varrho) \overset{\text{mono}}{\longrightarrow} LP_1(F)\} \subset LP_1(F) = (\mathbb{Z}/2)^{2^r-r}$$

are realized as surgery obstructions of normal maps between closed manifold pairs.

iii) *Only the elements of the direct summand*

$$(\mathbb{Z}/2)^{r+1} = S' \subset LP_2(F) = (\mathbb{Z}/2)^{2^{r+1}-r-1}$$

are realized as surgery obstructions of normal maps between closed manifold pairs. The subgroup S' is the preimage $p_0^{-1}(S)$ of the subgroup $S \subset L_3(\pi)$ which is generated by the images of all the inclusions

$$\mathbb{Z}/2 = L_3(\mathbb{Z}/2) \overset{\text{mono}}{\longrightarrow} L_3(\pi)$$

induced by the inclusions $\mathbb{Z}/2 \to \pi$ on the direct summands.

iv) *Only the elements of the direct summand*

$$\mathbb{Z} = \mathrm{Im}\{\mathbb{Z} = L_0(1) \to LP_3(F)\} \subset LP_3(F) = (\mathbb{Z})^{2^r} \oplus (\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-2}$$

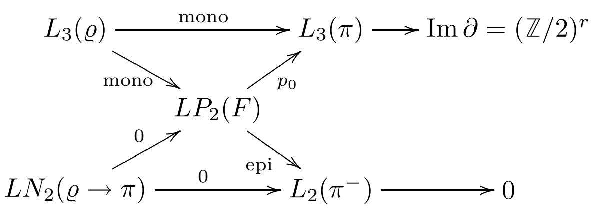
are realized as surgery obstructions of normal maps between closed manifold pairs.

P r o o f. For the case i) we consider a diagram that is similar to Diagram (2.11). No element of the group $L_1(\pi)$ can be realized by normal maps of closed manifolds [16]. The nontrivial element of the group $LN_0(1 \to \mathbb{Z}/2^+) = \mathbb{Z}/2$ is realized by a simple homotopy equivalence of closed manifolds [18, Chapter III.3.2]. Now the proof is similar to that of Theorem 1. We note only that our notation is different from that used in [20]. We have the correspondence $LN_*(1 \to \mathbb{Z}/2^{\pm}) \cong BL_{n+1}(\pm)$ and the map $c\colon LN_n(1 \to \mathbb{Z}/2^{\pm}) \to L_n(\mathbb{Z}/2^{\mp})$ coincides with the map $l_{n+1}\colon BL_{n+1}(\pm) \to L_n(\mathbb{Z}/2^{\mp})$ from [20].

In the case ii) it is sufficient to consider a diagram that is similar to Diagram (2.11) and use the realization of the nontrivial element of the group $L_2(\pi) = \mathbb{Z}/2$ by a normal map of closed manifolds, and the fact that no element of $L_1(\pi^-)$ can be realized [16].

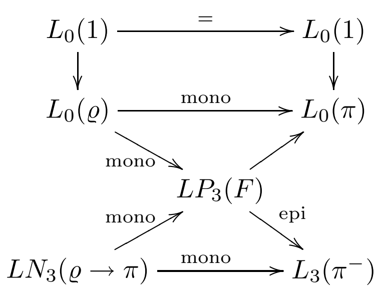
In the case iii) we consider the following part of Diagram (2.7) [21, § 6]:

(2.17)



From Diagram (2.17) it follows that the map p_0 is a monomorphism. In the group $L_3(\pi)$ only the elements of the direct summand S are realized by normal maps of closed manifolds (see [16]). The same argument as in the proof of Theorem 1 and Lemma 1 provides that only the elements from S' are realized by normal maps of closed manifold pairs.

In the case iv), let us consider the commutative diagram



where the two upper vertical maps are monomorphisms induced by the natural inclusions $1 \to L_0(\varrho) \to L_0(\pi)$. In the group $L_0(\pi)$ only the elements of the image $L_0(1) \to L_0(\pi)$, which equals $\mathbb{Z}$, are realized by normal maps of closed manifolds, and only the trivial element of the group $L_3(\pi^-)$ is realized by a normal map of closed manifolds [16]. From this the result follows similarly to Theorem 1. □

Now we construct a new invariant for realizing elements of LP_*-groups by normal maps between pairs of closed manifolds. This invariant is similar to the classical

Browder-Livesay invariant that was constructed in [7] (see also [11], [15], [16], [17], and [19]).

Let

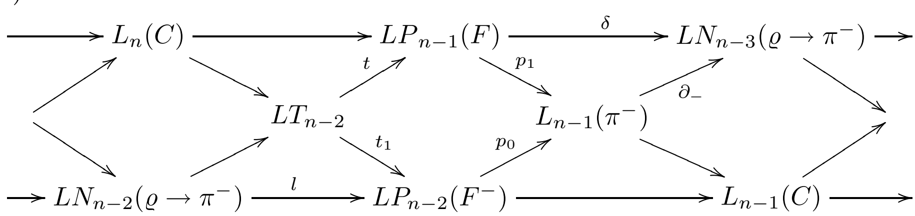
$$Z^{n-2} \subset Y^{n-1} \subset X^n \tag{2.18}$$

be a triple of manifolds such that the pairs (X, Y) and (Y, Z) are Browder-Livesay pairs for the same inclusion $\varrho \to \pi$ of index 2 between oriented groups. We suppose that $n - 2 \geqslant 5$. Let F denote the square in the splitting problem for the Browder-Livesay pair (X, Y). Then the corresponding square for the pair (Y, Z) will be F^-. Such a triple of manifolds is called a *Browder-Livesay triple of manifolds* (see [4], [17], [20], [22], and [23]). This triple defines a stratified space $\mathscr{X}$ in the sense of Browder-Quinn (see [6], [20], [22], and [32]). Thus the Browder-Quinn surgery obstruction groups

$$L^{BQ}_{n-2}(\mathscr{X}) = LT_{n-2}(X, Y, Z) \tag{2.19}$$

are defined (see [22], [23], and [32]). Note that for a filtration $\mathscr{Y}$ given by the manifold pair (X, Y) we have an isomorphism $L^{BQ}_*(\mathscr{Y}) = LP_*(F)$ [25]. For the triple of manifolds considered, the groups LT_* fit in the braid of exact sequences (see [20], [22], and [23])

(2.20)



where $C = \pi_1(X \setminus Y)$ and $LT_* = LT_*(X, Y, Z)$. Note that the map p_1 fits in Diagram (2.7) for the manifold pair (X, Y), and the maps p_0 and ∂ fit in Diagram (2.7) for the manifold pair (Y, Z). Recall that the map ∂ in Diagram (2.20) is the Browder-Livesay invariant for the group $\pi^- = \pi(Y)$ [7]. Diagram (2.20) is functorially defined by an inclusion $\varrho \to \pi$ of index 2 between oriented groups [23]. Thus we can write $LT_*(F) = LT_*(X, Y, Z)$. In particular, it is defined for the inclusion $\varrho \to \pi^-$ and the groups $LT_*(F^-)$ are defined, too.

Theorem 3. *Let (X, Y, Z) be a Browder-Livesay triple of manifolds. Let $x \in LP_{n-1}(F)$ be an element with $\delta(x) \neq 0$, where δ is the map from Diagram (2.20). Then the element x cannot be realized by a normal map of closed manifold pairs.*

P r o o f. In Diagram (2.20) the map

$$t\colon\ LT_{n-2}(X,Y,Z)\to LP_{n-1}(F)$$

is the natural forgetful map. Let an element $x\in LP_{n-1}(F)$ be realized by a normal map $f\colon\ (M^n,N^{n-1})\to(X^n,Y^{n-1})$ to a Browder-Livesay manifold pair. We can always suppose, by taking the multiplication by $\mathbb{C}P^2$, that the dimension n is large. Changing f in the class of normal bordism, taking a characteristic codimension 1 submanifold Z of N with respect to the inclusion $\varrho\to\pi$, and taking its transversal preimage $K=f^{-1}(Z)$, give a normal map

$$g\colon\ (M,N,K)\to(X,Y,Z)$$

to the Browder-Livesay triple (X,Y,Z) with an obstruction $z=\sigma(g)\in LT_{n-2}(X,Y,Z)$. By construction and the definition of the map t we obtain $x=t(z)\in LP_{n-1}(F)$. From the exact sequence

$$\ldots\to LT_{n-2}(X,Y,Z)\xrightarrow{t}LP_{n-1}(F)\xrightarrow{\delta}LN_{n-3}(\varrho\to\pi^-)\to\ldots\tag{2.21}$$

fitting in Diagram (2.20) we obtain $\delta x=0$. This implies the result. □

Now we apply Theorem 3 to study the closed manifold problem in the case of the groups $LP_*(F^-)$. Here the results are explicit in all dimensions except the dimension 3.

Theorem 4. *Let F^- be the square in* (2.16) *with* $\operatorname{rank}\varrho=r$.

i) *No element of the group*

$$LP_0(F^-)=(\mathbb{Z})^{2^r}\oplus(\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-2}$$

can be realized as surgery obstructions of normal maps between closed manifold pairs.

ii) *Only the elements of the direct summand*

$$\mathbb{Z}/2=\operatorname{Im}\{\mathbb{Z}/2=L_2(\varrho)\xrightarrow{\text{mono}}LP_1(F^-)\}\subset LP_1(F^-)=(\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-1}$$

are realized as surgery obstructions of normal maps between closed manifold pairs.

iii) *Only the elements of the direct summand*

$$\mathbb{Z}/2=\operatorname{Im}\{\mathbb{Z}/2=LN_2(\varrho\to\pi^-)\xrightarrow{\text{mono}}LP_2(F^-)\}\subset LP_2(F^-)=(\mathbb{Z}/2)^{2^r-r}$$

are realized as surgery obstructions of normal maps between closed manifold pairs.

iv) *The map* $p_1\colon\ LP_3(F^-) \to L_3(\pi)$ *is a monomorphism and its image contains the subgroup* $(Z/2)^{r+1} = S \subset L_3(\pi)$ *defined in the statement of Theorem 2. No element of the group* $LP_3(F^-)$*, except those lying in the direct summand*

$$S = (\mathbb{Z}/2)^{r+1} \subset LP_3(F^-) = (\mathbb{Z}/2)^{2^{r+1}-r-1},$$

can be realized as surgery obstructions of normal maps between closed manifold pairs. The nontrivial element of the direct summand

$$\mathbb{Z}/2 = \operatorname{Im}\{\mathbb{Z}/2 = \mathbb{L}P_3(\Psi^-) \to LP_3(F^-)\} \subset S,$$

where Ψ^- *is the square in the splitting problem for the inclusion* $1 \to \mathbb{Z}/2^-$*, is realized as surgery obstruction of a normal map between closed manifold pairs.*

P r o o f. In the case i) we consider the part of Diagram (2.7)

$$\begin{array}{ccccc} L_1(\varrho) & & \xrightarrow{\text{mono}} & & L_1(\pi^-) \\ & \searrow & & \nearrow_{epi} & \\ & & LP_0(F^-) & & \\ & {}^{\text{mono}}\nearrow_{l} & & \searrow^{p_1} & \\ LN_0(\varrho \to \pi^-) & & \xrightarrow{c} & & L_0(\pi) \end{array} \tag{2.22}$$

where the map c is an inclusion of a direct summand. This follows from the commutative diagram

$$\begin{array}{ccc} LN_0(\varrho \to \pi^-) & \xrightarrow{c} & L_0(\pi) \\ \downarrow & & \downarrow \\ L_0(\varrho) & \xrightarrow{i_*} & L_0(\pi) \end{array} \tag{2.23}$$

in which the bottom map is induced by the inclusion $i\colon\ \varrho \to \pi$ and the vertical maps are isomorphisms [29, Corollary 12.9.2]. From Diagrams (2.22) and (2.23) we obtain a decomposition of the group $LP_0(F^-)$ into the direct sum

$$LP_0(F^-) = LN_0(\varrho \to \pi^-) \oplus L_1(\pi^-). \tag{2.24}$$

By [16] the nontrivial elements of $L_1(\pi^-)$ cannot be realized by normal maps of closed manifolds. Hence, by Lemma 1, to complete the proof of part i) of the theorem we have only to prove the nonrealization of the elements of $LP_0(F^-)$ having the form

$$(x, 0) \in LN_0(\varrho \to \pi^-) \oplus L_1(\pi^-), \tag{2.25}$$

that is, the elements lying in the image of the map l. Let us consider the commutative diagram

$$
\begin{array}{ccccc}
 & & LN_2(\varrho \to \pi) & & \\
 & \overset{\delta}{\nearrow} & & \overset{\partial}{\nwarrow} & \\
LP_0(F^-) & & \xrightarrow{\quad p_1 \quad} & & L_0(\pi) \\
 & \underset{l}{\nwarrow} & & \underset{c}{\nearrow} & \\
 & & LN_0(\varrho \to \pi^-) & &
\end{array} \tag{2.26}
$$

in which the upper triangle fits in Diagram (2.20) for the inclusion $\varrho \to \pi^-$ and the bottom triangle fits in Diagram (2.7) for the same inclusion. The map ∂ from (2.26) lies in Diagram (2.7) for the inclusion $\varrho \to \pi^+$. Then the composition

$$LN_0(\varrho \to \pi^-) \xrightarrow{c} L_0(\pi) \xrightarrow{\partial} LN_2(\varrho \to \pi^+) \cong LN_0(\varrho \to \pi^-)$$

coincides with the first differential of the surgery spectral sequence [14], and is given by multiplication by 2. Hence if an element $x \in LN_0(\varrho \to \pi^-)$ has infinite order, then the element $(x, 0)$ in (2.25) cannot be realized by a normal map of closed manifold pairs by Theorem 3. Diagram (2.22) provides that any nontrivial element x lying in the torsion part of the group $LN_0(\varrho \to \pi^-)$ maps into the nontrivial element of the torsion part of the group $L_0(\pi)$. By [16] only the elements of the direct summand $\mathbb{Z} \subset L_0(\pi)$ can be realized by normal maps of closed manifolds. Now we can apply Lemma 1 to complete the proof for case i) of the theorem. In the other cases the proofs are similar to this one and those given for Theorem 1 and Theorem 2. □

Now we give a further application of Theorem 3. Let us consider an index 2 inclusion

$$i\colon\ \varrho = \mathbb{Z}/2^r \to \mathbb{Z}/2^{r+1^-} = \pi^- \tag{2.27}$$

between finite abelian 2-groups, where ϱ has the trivial orientation while the orientation of π^- is nontrivial and $r \geqslant 1$. Denote by Φ the square that corresponds to the inclusion in (2.27).

Proposition 3. *Under the above assumptions, we have the isomorphisms*

$$LP_n(\Phi) = \mathbb{Z}^{2^{n-1}},\ \mathbb{Z}/2,\ \mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}/2,\ 0 \tag{2.28}$$

for $n = 0, 1, 2, 3 \pmod 4$, *respectively.*

Proof. All the groups $L_*(\varrho)$, $L_*(\pi^\pm)$, and $LN_*(\varrho \to \pi^-)$ are known (see [7], [21], and [31]). Now it suffices to apply a diagram chasing in Diagram (2.7). □

Theorem 5.

i) *No element of the group*

$$LP_0(\Phi) = (\mathbb{Z})^{2^{r-1}}$$

can be realized as surgery obstruction of normal maps between closed manifold pairs.

ii) *All elements of the group*

$$\mathbb{Z}/2 = LP_1(\Phi)$$

are realized as surgery obstructions of normal maps between closed manifold pairs.

iii) *The natural map*

$$\mathbb{Z}^{2^{n-1}} = LN_2(\varrho \to \pi^-) \xrightarrow{l} LP_2(\Phi) = \mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}/2$$

is a monomorphism. The elements of the group $LP_2(\Phi)$ lying in the image of the map l cannot be realized as surgery obstructions of normal maps between closed manifold pairs.

P r o o f. In case ii) the map $p_0\colon LP_1(\Phi) \to L_2(\pi^-)$ is an isomorphism. Now it suffices to apply Lemma 2 similarly to the proof of Theorem 1. In cases i) and iii) it is necessary to use the same argument as in case i) of Theorem 4. We point out only that the composition

$$LN_{2k}(\varrho \to \pi^-) \to L_{2k}(\pi) \to LN_{2k+2}(\varrho \to \pi^+) \xrightarrow{\cong} LN_{2k}(\varrho \to \pi^-),$$

where the first map fits in Diagram (2.7) for the inclusion $\varrho \to \pi^-$ and the second fits in Diagram (2.7) for the inclusion $\varrho \to \pi$, coincides with the first differential in the surgery spectral sequence [14], and it is the multiplication by 2. So the theorem is completely proved. □

3. An application to LT_*-groups

The classification of the manifolds with a filtration as well as that of the stratified manifolds requires to know what elements of the Browder-Quinn obstruction groups are realized by normal maps of closed stratified manifolds. In this section we give an application of the methods developed above to the manifold surgery problem for triples of closed manifolds. Let us consider the Browder-Livesay triple of closed manifolds as in (2.18). The maps t and t_1 from Diagram (2.20) together with the corresponding maps from (1.3) provide the natural forgetful maps

$$\begin{aligned} LT_{n-2}(X,Y,Z) &\xrightarrow{\tau_0} L_n(\pi_1(X)),\\ LT_{n-2}(X,Y,Z) &\xrightarrow{\tau_1} L_{n-1}(\pi_1(Y)),\\ LT_{n-2}(X,Y,Z) &\xrightarrow{\tau_2} L_{n-2}(\pi_1(Z)). \end{aligned} \tag{3.1}$$

Lemma 3. *Let $x \in LT_{n-2}(X,Y,Z)$ be an element for which either one of the elements $\tau_0(x)$, $\tau_1(x)$, and $\tau_2(x)$ cannot be realized by a normal map of closed manifolds, or one of the elements $t(x)$ and $t_1(x)$ cannot be realized by a normal map of manifold pairs. Then the element x cannot be realized by a normal map to a Browder-Livesay triple of manifolds.*

Proof. This result is similar to that of Lemma 1, and follows from the geometrical definitions of the maps in (2.20) and (3.1). □

Note that an invariant, similar to the Browder-Livesay invariant and to the invariant δ from Section 2, can be also defined for LT_*-groups. Let

$$t_1\colon\ LT_{n-2}(F) \to LP_{n-2}(F^-) \tag{3.2}$$

be the map from Diagram (2.20) for the square F, and let

$$\delta'\colon\ LP_{n-2}(F^-) \to LN_n(\varrho \to \pi) \tag{3.3}$$

be the map from Diagram (2.20) for the square F^-. Let

$$\Delta = \delta' t_1\colon\ LT_{n-2}(F) \to LN_n(\varrho \to \pi)$$

be the composition of the maps from (3.2) and (3.3).

Proposition 4. *If $\Delta(x) \neq 0$ for an element $x \in LT_{n-2}(F)$, then x cannot be realized by a normal map of triples of closed manifolds.*

Proof. The map t_1 in (3.2) is the natural forgetful map. Hence, if an element $x \in LT_{n-2}(F)$ is realized by a normal map of triples of closed manifolds, then $t_1(x)$ is realized by the induced normal map of the pairs of closed manifolds. By Theorem 3 the map δ' is the forbidden invariant for such a realization. This yields the result. □

Now we give an example of explicit computation. Let us consider the triple

$$\mathbb{R}P^{2k-1} \subset \mathbb{R}P^{2k} \subset \mathbb{R}P^{2k+1} \tag{3.4}$$

of real projective spaces with $2k-1 \geqslant 5$. In this case we have the square

$$F^{\pm} = \left(\begin{array}{ccc} 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}/2^{\mp} & \longrightarrow & \mathbb{Z}/2^{\pm} \end{array} \right).$$

Let us denote by $\mathscr{X}$ the filtration in (3.4), and by $LT_* = L_*^{BQ}(\mathscr{X})$ the Browder-Quinn surgery obstruction groups for the triple $\mathscr{X}$ (see [5], [22], [23], and [32]). These groups fit in the braid of exact sequences (2.20). In this case, $LT_{2k-1} = \mathbb{Z}/2$ and $LT_{2k} = \mathbb{Z} \oplus \mathbb{Z}/2$ (see [17] and [23]).

Theorem 6.

i) *All elements of the groups*

$$L_{2k-1}^{BQ}(\mathscr{X}) = \mathbb{Z}/2 \quad (k = 0, 1)$$

are realized as obstructions to surgery on triples of closed manifolds.

ii) *Only the elements of the direct summand*

$$\mathbb{Z}/2 = \mathrm{Im}\{L_2(1) \to L_0^{BQ}(\mathscr{X})\} \subset L_0^{BQ}(\mathscr{X}) = \mathbb{Z} \oplus \mathbb{Z}/2$$

can be realized by normal maps between triples of closed manifolds. The elements of the direct summand

$$\mathbb{Z} = \mathrm{Im}\{L_0(1) \to L_2^{BQ}(\mathscr{X})\} \subset L_2^{BQ}(\mathscr{X}) = \mathbb{Z} \oplus \mathbb{Z}/2$$

can be realized by normal maps between triples of closed manifolds.

Proof. A diagram chasing in Diagram (2.20) provides that the maps

$$LT_{2k-1} \to LP_{2k}(F^+)$$

are isomorphisms. Now the odd-dimensional case follows from Theorem 2 and Lemma 3 by using the same argument as in Section 2. In even dimensions, Diagram (2.20) yields a decomposition

$$L^{BQ}_{2k}(\mathscr{X}) = LP_{2k+1}(F^+) \oplus LN_{2k}(1 \to \mathbb{Z}/2^-).$$

Now the result follows from Theorems 3 and 4 by using the same argument as before. □

Remark. It is not difficult to extend the argument of this paper to the case of filtered manifolds. For a Browder-Livesay filtration $\mathscr{X}$

$$X_k \subset X_{k-1} \subset \ldots \subset X_2 \subset X_1 \subset X_0 = X$$

of a closed connected topological n-dimensional manifold X by locally flat closed submanifolds, the Browder-Quinn surgery obstruction groups are defined (see [6], [10], [22], and [32]). We have natural forgetful maps

$$\tau_i \colon L^{BQ}_k(\mathscr{X}) \to L_i(\pi_1(X_i)) \quad (i = 0, 1, \ldots, k).$$

Thus it is easy to obtain a result that is similar to Lemma 1 and Lemma 2. A result that is similar to Theorem 3 and Proposition 4 also takes place. It follows easily from the relations between the surgery spectral sequence and the Browder-Quinn surgery obstruction groups (see [14], [10], and [22]). This forbidden invariant is given by the composition

$$L^{BQ}_k(\mathscr{X}) \xrightarrow{\tau_k} L_k(\pi_1(X_k)) \to LN_{k-2}(\varrho \to \pi^*),$$

where the second map is the Browder-Livesay invariant on the group $L_k(\pi_1(X_k))$.

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Authors' addresses: A. Cavicchioli, Dipartimento di Matematica, Università di Modena e Reggio Emilia, Via Campi 213/B, 41100 Modena, Italia, e-mail: cavicchioli.alberto@unimo.it; Y. V. Muranov, Physical Department, Vitebsk State University, Moskovskii pr. 33, 210026 Vitebsk, Belarus, e-mail: ymuranov@mail.ru; F. Spaggiari, Dipartimento di Matematica, Università di Modena e Reggio Emilia, Via Campi 213/B, 41100 Modena, Italia, e-mail: spaggiari.fulvia@unimo.it.