Czechoslovak Mathematical Journal

Hiroyuki Tachikawa

Loewy coincident algebra and QF-3 associated graded algebra

Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 3, 583–589

Persistent URL: http://dml.cz/dmlcz/140501

Czechoslovak Mathematical Journal, 59 (134) (2009), 583–589

LOEWY COINCIDENT ALGEBRA AND QF-3 ASSOCIATED GRADED ALGEBRA

Hiroyuki Tachikawa, Tsukuba

(Received October 16, 2007)

Abstract. We prove that an associated graded algebra R_G of a finite dimensional algebra R is QF (= selfinjective) if and only if R is QF and Loewy coincident. Here R is said to be Loewy coincident if, for every primitive idempotent e, the upper Loewy series and the lower Loewy series of Re and eR coincide.

QF-3 algebras are an important generalization of QF algebras; note that Auslander algebras form a special class of these algebras. We prove that for a Loewy coincident algebra R, the associated graded algebra R_G is QF-3 if and only if R is QF-3.

Keywords: associated graded algebra, QF algebra, QF-3 algebra, upper Loewy series, lower Loewy series

MSC 2010: 13A30, 16D50, 16L60, 16P70

Introduction

Let K be a field and R a finite dimensional K-algebra; denote its Jacobson radical by J. Given a left R-module X, the chain

$$X \supset JX \supset \ldots \supset J^{\varrho}X$$

of its submodules is called the upper Loewy series of X. On the other hand, the chain of the right annihilators

$$X = r(J^{\varrho+1}\colon X) \supset r(J^{\varrho}\colon X) \supset r(J^{\varrho-1}\colon X) \supset \ldots \supset r(J^1\colon X),$$

where $r(J^i\colon X) = \{x \in X;\ J^i x = 0\}$ for $i = 1, 2, \ldots, \varrho$, is called the lower Loewy series of X.

If the upper Loewy series and the lower Loewy series of X coincide, we shall say that X satisfies the Loewy coincidence condition. For a right R-module, we shall apply the same definitions.

We shall say that R is a left Loewy coincident algebra if every primitive left ideal satisfies the Loewy coincidence condition. That is, R is left Loewy coincident if and only if for every primitive idempotent e of R, $J^{\varrho+1-i}e = r(J^i)e$ for $i = 1, 2, \ldots, \varrho$, where $J^{\varrho}e \neq 0$ but $J^{\varrho+1}e = 0$.

A left and right Loewy coincident algebra is called simply a Loewy coincident algebra.

In [5], the author has proved that the associated graded algebra R_G is quasi-Frobenius if and only if R is Loewy coincident and quasi-Frobenius. It is well known that R is quasi-Frobenius (abbreviated to QF) if and only if R is selfinjective.

Let us point out that R and R_G have very different structures even if R is commutative (cf. Example 2.2 in [5]).

In this paper we will extend our consideration to QF-3 associated graded algebras. For the definition of QF-3 algebras, see Thrall [6] and Tachikawa [4]. Note that Auslander algebras are a special class of QF-3 algebras; recall that an Auslander algebra is the endomorphism algebra of the direct sum of all indecomposable modules over an algebra of finite representation type (Auslander [1]).

In §2, we shall prove that if R is a Loewy coincident algebra, then the associated graded algebra R_G is QF-3 if and only if R is QF-3.

1. Socle condition and Loewy coincidence condition

Let R be an algebra with Jacobson radical J of nilpotency $n+1$. Let us denote the associated graded ring of R by R_G $(= R/J \oplus J/J^2 \oplus \ldots \oplus J^{n-1}/J^n \oplus J^n)$.

We shall say that a positive integer ϱ is the Loewy length of a left R-module X if $J^{\varrho}X \neq 0$ but $J^{\varrho+1}X = 0$.

Then for a left R-module X of the Loewy length ϱ the associated graded left R_G-module X_G is defined as a (formal) direct sum

$$X/JX \oplus JX/J^2X \oplus \ldots \oplus J^{\varrho-1}X/J^{\varrho}X \oplus J^{\varrho}X$$

with the following operation by R_G: $r_G x_G = \sum_{j=0}^{n} \sum_{k=0}^{\varrho} (r_j x_k + J^{j+k+1}X)$, where $r_G = \sum_{j=0}^{n} (r_j + J^{j+1}) \in R_G$ with $r_j \in J^j$ and $x_G = \sum_{k=0}^{\varrho} (x_k + J^{k+1}X) \in X_G$ with $x_k \in J^k X$.

In this case, since $J^{\varrho+1}X = 0$, $J^{\varrho}X$ and $\mathrm{Rad}(R_G)^{\varrho} X_G$ can be identified as additive groups. Furthermore, we can identify $J^{\varrho}X$ and $\mathrm{Rad}(R_G)^{\varrho} X_G$ as R_G-modules. In order to indicate this identification, we use the notation ${}_{R_G}J^{\varrho}X$.

We know that the socle $\mathrm{Soc}(X)$ of X can be defined by

$$\mathrm{Soc}(X) = r(J\colon\ X), \quad \text{where } r(J\colon\ X) = \{x \in X\,;\ Jx = 0\}.$$

Similarly, we can define the socle of the left R_G-module X_G by $\mathrm{Soc}(X_G) = r(\mathrm{Rad}(R_G)\colon\ X_G) = \Big\{\sum\limits_{k=0}^{\varrho}(x_k + J^{k+1}X)\,;\ x_k \in J^kX$ and $Jx_k \subseteq J^{k+2}X$ for $0 \leqslant k \leqslant \varrho - 1\Big\}$.

Now, let us consider the submodule of X_G consisting of the elements $\sum\limits_{k=0}^{\varrho}(y_k + J^{k+1}X)$, where $y_k \in \mathrm{Soc}(X) \cap J^kX$. We denote this submodule by $\mathrm{soc}(X_G)$. It is clear that ${}_{R_G}J^{\varrho}X \subset \mathrm{soc}(X_G) \subset \mathrm{Soc}(X_G) \subset X_G$.

Let us point out that if ${}_{R_G}J^{\varrho}X = \mathrm{soc}(X_G)$, then $J^{\varrho}X = \mathrm{Soc}(X)$. Indeed, for $x \in \mathrm{Soc}(X)$ denote by j the positive integer such that $x \in J^jX \setminus J^{j+1}X$. Then $x + J^{j+1}X \in \mathrm{soc}(X_G)$ and it follows from ${}_{R_G}J^{\varrho}X = \mathrm{soc}(X_G)$ that there exists $y \in J^{\varrho}X$ such that $y + J^{\varrho+1} = x + J^{j+1}$. But this implies $j = \varrho$ and $x = y$. Hence $\mathrm{Soc}(X) \subseteq J^{\varrho}X$, which means $J^{\varrho}X = \mathrm{Soc}(X)$.

We say that a left R-module X satisfies the socle condition with respect to X_G if ${}_{R_G}J^{\varrho}X = \mathrm{soc}(X_G) = \mathrm{Soc}(X_G)$. Moreover,we say that a left R-module X satisfies the Loewy coincidence condition if $J^iX = \mathrm{Soc}^{\varrho+1-i}(X)$ $(= r(J^{\varrho+1-i}; X) = \{x \in X\,;\ J^{\varrho+1-i}x = 0\})$ for $i = 1, 2, \ldots, \varrho$. Here of course $\mathrm{Soc}^1(X) = \mathrm{Soc}(X)$. Then we can formulate the following statement.

Lemma 1.1. *Let X be a left R-module. Then the following statements* (i) *and* (ii) *are equivalent:*

(i) *X satisfies the socle condition with respect to X_G.*

(ii) *X satisfies the Loewy coincidence condition.*

Proof. (i) $\Rightarrow$ (ii): Let ϱ be the Loewy length of X and assume that $\mathrm{Soc}^s(X) = J^{\varrho+1-s}X$ for $s \geqslant 1$. For $s = 1$, the assumption is satisfied. For, as mentioned earlier, the condition (i), viz. ${}_{R_G}J^{\varrho}X = \mathrm{soc}(X_G)$ implies $J^{\varrho}X = \mathrm{Soc}(X)$.

Suppose now that $\mathrm{Soc}^{s+1}(X) \neq J^{\varrho-s}X$. Since $\mathrm{Soc}^{s+1}(X) \supseteq J^{\varrho-s}X$, there is an element $x \in X$ such that $x \in \mathrm{Soc}^{s+1}(X)$ but $x \notin J^{\varrho-s}X$. Let l be a positive integer such that $x \in J^lX \setminus J^{l+1}X$. Then l is uniquely determined, $l < \varrho - s$ and $Jx \subseteq J^{l+1}X$. In this case we know that $Jx \not\subseteq J^{l+2}X$. Indeed, suppose that $Jx \subseteq J^{l+2}X$. Then $x + J^{l+1}X \in \mathrm{Soc}(X_G) = \mathrm{soc}(X_G) = {}_{R_G}J^{\varrho}X$ by (i). Hence we have $l = \varrho$, which is a contradiction to $l < \varrho - s$.

On the other hand, $x \in \mathrm{Soc}^{s+1}(X)$ implies $J^{s+1}x = J^s(Jx) = 0$ and hence $Jx \subseteq \mathrm{Soc}^s(X) = J^{\varrho+1-s}X$. However, $J^{\varrho+1-s}X \subseteq J^{l+2}X$ because $J^{\varrho+1-s}X \subset J^{l+1}X$

but $J^{\varrho+1-s}X \neq J^{l+1}X$. Hence, we have $Jx \subseteq J^{l+2}X$, which contradicts again $Jx \not\subset J^{l+2}X$.

Consequently, we conclude that $\mathrm{Soc}^{s+1}(X) = J^{\varrho-s}X$. Now by induction on s we can complete the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): Since ϱ is the Loewy length of X, it follows immediately from (ii) that $J^{\varrho}X = \mathrm{Soc}(X)$, which yields ${}_{R_G}J^{\varrho}X = \mathrm{soc}(X_G)$.

Suppose that there is an element $x \in J^tX \setminus J^{t+1}X$ such that $x + J^{t+1}X \in \mathrm{Soc}(X_G)$ and $x \notin \mathrm{Soc}(X)(= J^{\varrho}X)$. Then $t < \varrho$ and $Jx \in J^{t+2}X$. However, (ii) implies that $J^{t+2}X = \mathrm{Soc}^{\varrho+1-(t+2)}(X)$ and hence $J^{\varrho-t-1}Jx = J^{\varrho-t}x = 0$. Thus, $x \in \mathrm{Soc}^{\varrho-t}(X)$.

Again by (ii), we have $\mathrm{Soc}^{\varrho-t}(X) = J^{\varrho+1-(\varrho-t)}X = J^{t+1}X$ and hence $x \in J^{t+1}X$. However,this contradicts $x \in J^tX \setminus J^{t+1}X$. Consequently, $\mathrm{Soc}(X_G) \subseteq \mathrm{soc}(X_G)$. □

In view of Morita equivalence [2], we can assume, without loss of generality, that all algebras are basic. Let e be a primitive idempotent of the ring R. Then $e + J \in R/J$ is a primitive idempotent of R_G that will be briefly denoted by e_G.

We have a ring isomorphism $R/J \simeq R_G/\operatorname{Rad}(R_G)$ and by this isomorphism e corresponds to e_G. Therefore we can identify the simple left R-module Re/Je and the simple left R_G-module $R_Ge_G/\operatorname{Rad}(R_G)e_G$. We note that this identification can be extended to semisimple R-modules and semisimple R_G-modules.

Now, if we apply Lemma 1.1 for $X = Re$ then we obtain immediately the following theorem.

Theorem 1.2. *Let ϱ be the Loewy length of Re. Then $\mathrm{Soc}(R_Ge_G) = \mathrm{Soc}(Re) = N^{\varrho}e$ if and only if Re satisfies the Loewy coincidence condition.*

An algebra R is said to be left (or right) QF-2 if $\mathrm{Soc}(Re)$ (or $\mathrm{Soc}(eR)$) is simple for every primitive idempotent e (cf. [6]).

As $J^{\varrho}e = \mathrm{Soc}(Re)$ if $\mathrm{Soc}(Re)$ is simple, we have immediately

Corollary 1.3. *R_G is left QF-2 if and only if R is left QF-2 and left Loewy coincident.*

P r o o f. If R_G is QF-2, then for every primitive idempotent e_G, $\mathrm{Soc}(R_Ge_G)$ is simple. Hence $\operatorname{Rad}(R_G)^{\varrho}e_G = J^{\varrho}e = \mathrm{Soc}(Re) = \mathrm{Soc}(R_Ge_G)$. Therefore by Lemma 1.1, the Loewy coincidence condition holds for every primitive ideal Re. It follows that R is QF-2.

If R is QF-2 and if the coincidence condition holds for every primitive ideal Re, then by Lemma 1.1 the socle condition for R_Ge_G holds and R_Ge_G has a simple socle. Hence R_G is QF-2. □

By T. Nakayama [3], an algebra R is QF (=quasi-Frobenius) if and only if the following conditions (i) and (ii) are satisfied:

(i) For all primitive idempotents e_i, $i = 1, 2, \ldots, m$, we have $r(J)e_i = l(J)e_i$ and $e_i l(J) = e_i r(J)$, and they are simple left and right R-modules, where $r(J)$ and $l(J)$ denote the right and left annihilators of J, respectively.

(ii) There is a permutation π on $\{1, 2, \ldots, m\}$ such that $r(J)e_i \simeq Re_{\pi(i)}/Je_{\pi(i)}$.

Therefore R is QF if and only if R is a left and right QF-2 algebra with $r(J) = l(J)$ having the above permutation π. Hence, Corollary 1.3 yields immediately the following theorem.

Theorem 1.4. *R_G is QF if and only if R is QF and Loewy coincident.*

(Cf. Theorem 1.7 of [5].)

2. QF-3 Associated graded algebras

In what follows we assume that R is an algebra over a field K and $D(R)$ $(= \mathrm{Hom}_K(R, K))$ is the dual module of R. For a left R-module X, $D(X) = \mathrm{Hom}_K(X, K)$ is the right R-module defined by $(\varphi r)(x) = \varphi(rx)$ for φ from $\mathrm{Hom}_K(X, K)$, $r \in R$ and $x \in X$. Similarly for a right R-module the dual module is defined to be a left R-module.

$D(R)$ is an R-bimodule and it is well known that

$$ {}_R\mathrm{Hom}_K({}_K X_R, {}_K K) \simeq {}_R\mathrm{Hom}_R(X_R, {}_R D(R)_R) $$

for a right R-module X and

$$ \mathrm{Hom}_K({}_R Y_K, K_K)_R \simeq \mathrm{Hom}_R({}_R Y, {}_R D(R)_R)_R $$

for a left R-module Y.

Furthermore, ${}_R D(R)$ (or $D(R)_R$) is an injective cogenerator in R-mod (or mod-R), where R-mod (or mod-R) denotes the category of finitely generated left (or right) R-modules. It is important that $\mathrm{Hom}_R(-, {}_R D(R)_R)$ induces the Morita duality between R-mod and mod-R (cf. [2]).

For a primitive idempotent e, we have $D(Re/Je) \simeq eR/eJ$ and $D(Re) \simeq E(eR/eJ)$, which is the injective hull of simple module eR/eJ. Moreover, by the duality, the upper (or lower) Loewy series of a left R-module ${}_R X$ is transformed to the lower (or upper) Loewy series of the right R-module $D(X)_R$.

From now on we assume that R is a Loewy coincident algebra. Then the injective indecomposable module $D(Re)_R$ (or ${}_R D(eR)$) for any primitive idempotent e satisfies the Loewy coincidence condition.

Let us consider the associated graded algebra R_G of R. Then

$$D(R_G e_G)_{R_G} \simeq E(e_G R_G / e_G \operatorname{rad}(R_G))$$

and by an earlier remark and Lemma 1.1 it satisfies the socle condition for $[{}_G D(Re)]_{R_G}$, i.e.

$$D(Re) J^{\varrho}_{R_G} = \operatorname{soc}({}_G D(Re))_{R_G} = \operatorname{Soc}({}_G D(Re))_{R_G},$$

where ϱ is the Loewy length of $D(Re)$. As $\operatorname{Soc}({}_G D(Re))$ is isomorphic to $\operatorname{Soc}(D(R_G e_G))$, ${}_G D(Re)$ can be imbedded into $D(R_G e_G)$ as a right R_G-module. But the composition lengths of ${}_G D(Re)$ and $D(R_G e_G)$ are the same as the composition length of $D(Re)_R$ (= the composition length of ${}_R Re$). Therefore we get that ${}_G D(Re)_{R_G} \simeq D(R_G e_G)_{R_G}$.

Proposition 2.1. *If the algebra R is Loewy coincident, then ${}_G D(Re)_{R_G}$ and ${}_{R_G} D(eR)_G$ are indecomposable injective for every primitive idempotent e.*

Following Thrall [6] an algebra R is said to be QF-3 if R has a unique minimal faithful left R-module Q. It is well-known that Q is a direct sum of indecomposable projective and injective left ideals (i.e., injective primitive left ideals).

Let us assume that R is QF-3. Then using mutually non-isomorphic primitive idempotents e_i, $1 \leqslant i \leqslant m$ we have a direct sum decomposition ${}_R R \simeq \bigoplus_{i=l+1}^{m} Re_i \oplus Q$, where $Q \simeq \bigoplus_{i=1}^{l} Re_i$ and $l \leqslant m$.

Since, for $i \leqslant l$, Re_i is injective, there exists a primitive right ideal $e_k R$ such that $Re_i \simeq D(e_k R)$. By Proposition 2.1, $R_G e_{i_G}$ ($\simeq D(e_k R)_G$) is an injective ideal of R_G. Hence every indecomposable direct summand of Q_G is a projective and injective left R_G-module.

Since ${}_R Q$ is faithful, $\operatorname{Soc}(Re_j)$, $j > l$, is imbedded into a direct sum of copies of $\operatorname{Soc}(Q)$. On the other hand, since R is Loewy coincident, the socle condition holds for $R_G e_{j_G}$, $j > l$ and every indecomposable direct summand of Q_G, which is an injective left R_G-module, and hence $R_G e_{j_G}$, $j > l$, is imbedded into a direct sum of copies of ${}_{R_G} Q_G$. It follows that ${}_{R_G} Q_G$ is faithful.

Conversely, assume R_G is QF-3. By Proposition 2.1, $R_G e_{i_G}$ is injective if and only if ${}_R Re_i$ is injective and thus, by the socle condition, $\operatorname{Soc}(R_G e_{i_G}) \simeq \operatorname{Soc}(R_G e_{j_G})$ if and only if $\operatorname{Soc}(Re_i) \simeq \operatorname{Soc}(Re_j)$ for $1 \leqslant i, j \leqslant m$. Therefore R is QF-3.

Consequently, the following theorem holds.

Theorem 2.2. *Let the algebra R be Loewy coincident. Then the associated graded algebra R_G of R is QF-3 if and only if R is QF-3.*

We like to point out that all results in this paper hold for Artin algebras in the sense of Auslander [1], i.e. for algebras that are finitely generated over artinian commutative rings.

References

[1] *M. Auslander*: Representation dimension of Artin algebras. Queen Mary College Lecture Notes, 1971.

[2] *K. Morita*: Duality for modules and its applications to the theory of rings with minimum condition. Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A. No. 150 (1958), 1–60.

[3] *T. Nakayama*: On Frobeniusean algebras. II, Ann. Math. *42* (1941), 1–21.

[4] *H. Tachikawa*: Quasi-Frobenius rings and generalizations. LNM 351, 1973.

[5] *H. Tachikawa*: QF rings and QF associated graded rings. Proc. 38th Symposium on Ring Theory and Representation Theory, Japan, pp. 45–51. http://fuji.cec.yamanash.ac.jp/˜ring/oldmeeting/2005/reprint2005/abst-3-2.pdf.

[6] *R. M. Thrall*: Some generalizations of quasi-Frobenius algebras. Trans. Amer. Math. Soc. *64* (1948), 173–183.

Author's address: Hiroyuki Tachikawa, Institute of Mathematics, University of Tsukuba, 1-1-1 Tennowdai, Tsukuba, Ibaraki, 305-8571, Japan, e-mail: `pytha345@mail2.accsnet.ne.jp`.