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FURTHER PROPERTIES OF AZIMI-HAGLER BANACH SPACES

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Abstract. For the Azimi-Hagler spaces more geometric and topological properties are investigated. Any constructed space is denoted by $X_{\alpha,p}$. We show

(i) The subspace $[(e_{n_k})]$ generated by a subsequence (e_{n_k}) of (e_n) is complemented.
(ii) The identity operator from $X_{\alpha,p}$ to $X_{\alpha,q}$ when $p > q$ is unbounded.
(iii) Every bounded linear operator on some subspace of $X_{\alpha,p}$ is compact. It is known that if any $X_{\alpha,p}$ is a dual space, then
(iv) duals of $X_{\alpha,1}$ spaces contain isometric copies of ℓ_∞ and their preduals contain asymptotically isometric copies of c_0.
(v) We investigate the properties of the operators from $X_{\alpha,p}$ spaces to their predual.

Keywords: Banach spaces, compact operator, asymptotic isometric copy of ℓ_1

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1. Introduction and preliminaries

In this paper we continue the study of properties of the classes of Azimi-Hagler Banach spaces which were constructed by Hagler and the first named author. These spaces are denoted by $X_{\alpha,p}$. In [3] classes of spaces containing hereditarily ℓ_1 which fail the Schur property were constructed and studied. In [1] classes of $X_{\alpha,p}$ Banach spaces were constructed which are hereditarily complementably ℓ_p. Here further geometric and topological investigation of the spaces is carried out. In the first result subclasses are constructed where each member has an unconditional basis (u_i) such that $u_i \xrightarrow{w} 0$ but not in norm. Among the other interesting properties, all constructed Azimi-Hagler spaces are dual spaces. We consider properties of the operators from the spaces to their predual. In [11] Popov showed that the classical Pitt theorem on compactness of operators from ℓ_p to ℓ_q for $1 \leqslant q < p < \infty$ it fails in general setting of hereditarily ℓ_p and ℓ_q spaces.

By the Pitt theorem every bounded linear operator from ℓ_p to ℓ_q when $1 \leqslant q < p < \infty$ is compact. The proof of this theorem is based on the fact that any block basis of (e_n) in ℓ_p is equivalent to (e_n) in ℓ_p. But this is not the case for $X_{\alpha,p}$ spaces. In fact there are block basis sequences of (e_n) in $X_{\alpha,p}$ which are not equivalent to (e_n).

Before beginning our detailed analysis, we pass to the construction of $X_{\alpha,p}$ spaces of Azimi and Hagler. Consider a nonnegative sequence (α_i) of reals which satisfies the following conditions:

1. $\alpha_1 = 1$ and $\alpha_{i+1} \leqslant \alpha_i$ for $i = 1, 2, \ldots,$
2. $\lim \alpha_i = 0,$
3. $\sum\limits_{i=1}^{\infty} \alpha_i = \infty.$

A block F is a finite or infinite interval $F \subset \mathbb{N}$ and a sequence of blocks $(F_i)_i$ where the F_i (finite or infinite) is called admissible if $\max F_i < \min F_{i+1}$ $(i \in \mathbb{N})$. We now define a norm which uses the α_i's and admissible sequences of blocks in its definition. For a block F and a finitely non-zero sequence $x = (x_1, x_2, x_3 \ldots)$ of reals we let $\langle x, F\rangle = \sum\limits_{i\in F} x_i$. For $1 \leqslant p < \infty$ we define

$$\|x\| = \max\left[\sum_{i=1}^{n} \alpha_i |\langle x, F_i\rangle|^p\right]^{1/p}$$

where the maximum is taken over all n and an admissible sequence $F_1, F_2, \ldots, F_n$. The Banach space $X_{\alpha,p}$ is the completion of the finitely non-zero sequencees of scalars in this norm. Let $\tilde{X}_{\alpha,p} = [u_j]$ where $u_j = e_{2j} - e_{2j-1}$. In [3] it is shown that $\tilde{X}_{\alpha,p}$ is weakly sequentially complete and $(u_i)_i$ is an unconditional basis such that $u_i \to 0$ weakly but $\|u_i\| = (1 + \alpha_2)^{1/p}$. Let us present the main properties of $X_{\alpha,p}$ spaces [1].

Theorem 1.1. *Let $X_{\alpha,p}$ denote a specific space of the class. Then*

(1) *$X_{\alpha,p}$ is hereditarily complementably ℓ_p.*
(2) *The sequence (e_i) is a normalized boundedly complete basis for $X_{\alpha,p}$. Thus $X_{\alpha,p}$ is a dual space.*
(3) *The predual of $X_{\alpha,p}$ contains complemented subspaces isomorphic to ℓ_q where $1/p + 1/q = 1$.*
(4) *Each complemented non weakly sequentially complete subspace of $X_{\alpha,p}$ contains a complemented isomorph of $X_{\alpha,p}$. Since $X_{\alpha,p}$ contains ℓ_p hereditarily complementably thus*
(5) *$X_{\alpha,p}$ spaces are not prime.*

Definition and notation are standard. Nevertheless, we list the most important of them. The dual space of a Banach space X is denoted by X^*. Let Y be a subspace of X, then we say that X contains Y hereditarily if every infinite dimensional subspace of X contains an isomorphic copy of Y. A subspace Y is complemented in X

if there is a bounded projection $P\colon X \longrightarrow Y$ such that $P(X) = Y$. Also $[x_n]$ is the closed linear span of (x_n).

The space of all bounded linear operators from X to Y is denoted by $L(X,Y)$ and $B(X)$ is the unit ball of X. Let $T \in L(X,Y)$, then T is called a compact operator (weakly compact operator) if $TB(X)$ is relatively norm compact (relatively weakly compact) in Y. Equivalently, T is compact if for every bounded sequence $(x_n)_n$ in X, the sequence $(Tx_n)_n$ contains a convergent subsequence. We will denote the collection of all compact operators from X to Y by $K(X,Y)$.

Definition 1.2. A Banach space X is called weakly conditionally compact if every bounded sequence in X has a weakly Cauchy subsequence. It is known that all reflexive spaces, as well as any Banach space with a separable dual space, are weakly conditionally compact.

The following theorem is known [9].

Theorem 1.3. *Let X be weakly conditionally compact. $T \in L(X,Y)$ is a compact operator if and only if whenever $(x_n)_n$ converges to zero weakly in X this implies that $(Tx_n)_n$ converges to zero in norm (in Y).*

Definition 1.4. Let X and Y be Banach spaces. Two bases, (x_n) of X and (y_n) of Y, are called equivalent provided a series $\sum\limits_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum\limits_{n=1}^{\infty} a_n y_n$ converges.

Thus the bases are equivalent if the sequence space associated with X by (x_n) is identical to the sequence space associated with Y by (y_n). It follows from the closed graph theorem that (x_n) is equivalent to (y_n) if and only if there is an isomorphism T from X to Y for which $Tx_n = y_n$ for all n.

2. The results

From the definition of the norm of $X_{\alpha,p}$, we can see that the unit vector basis is spreading (equivalent to each of its subsequence) and bi-monotone. That is $\|(P_m - P_n)x\| \leqslant \|x\|$ for each $x = (x_1, x_2, x_3, \dots) \in X_{\alpha,p}$ and $n < m$. Observe that each block F defines a functional which is bounded on $X_{\alpha,p}$. In fact $\langle x, F\rangle = \sum\limits_{i\in F} x_i = \sum\limits_{i\in F} e_i^*(x)$.

Theorem 2.1. *If (e_{i_k}) is a subsequence of (e_k) in $X_{\alpha,p}$, then*
(1) *$[(e_{i_k})]$ is asymptotically isometric to ℓ_p,*
(2) *$[(e_{i_k})]$ is complemented in $X_{\alpha,p}$.*

Proof. Part (1) is an immediate consequence of Theorem 1.1. For the proof of (2) let (F_i) be a sequence of blocks without gaps ($\max F_i + 1 = \min F_{i+1}$) such that if $i_k \in F_k$, then $[(e_{i_k})]$ is complemented by the projection

$$Px = \sum_{i=1}^{\infty} \langle x, F_k \rangle \, e_{i_k}.$$

Since (F_i) has no gaps, any estimate of $\|Px\|$ is also an estimate of $\|x\|$, so $\|P\| = 1$. □

Lemma 2.2. *For each non-increasing sequence of positive numbers (β_i) and*

$$v = (\beta_1, -\beta_1, \beta_2, -\beta_2, \ldots, \beta_n, -\beta_n)$$

in the space $X_{\alpha,p}$ we have

$$\|v\|^p = (\alpha_1 + \alpha_2)\beta_1^p + (\alpha_3 + \alpha_4)\beta_2^p + \ldots + (\alpha_{2n-1} + \alpha_{2n})\beta_n^p.$$

Proof. Let each block F be a singleton with $F_i = \{i\}$. Then $|\langle v, F_{2i-1}\rangle| = |\langle v, F_{2i}\rangle| = \beta_i$. This implies

$$\|v\|^p \geqslant (\alpha_1 + \alpha_2)\beta_1^p + (\alpha_3 + \alpha_4)\beta_2^p + \ldots + (\alpha_{2n-1} + \alpha_{2n})\beta_n^p.$$

We claim that the sequence of (F_i) is the norming sequence for v, otherwise there is a sequence $(F_1, F_2, \ldots, F_k)$ of consecutive blocks such that $k < 2n$ and $\|v\|^p = \sum\limits_{i=1}^{n} \alpha_i |\langle v, F_i\rangle|^p$, since for any block F, $\langle v, F\rangle$ is β_i or 0, the number of blocks such that $\langle v, F\rangle \neq 0$ is equal at most to k, and since $\{\beta_i\}$ is non-increasing and $k < 2n$,

$$\|v\|^p = \sum \alpha_i \beta_i^p \leqslant (\alpha_1 + \alpha_2)\beta_1^p + (\alpha_3 + \alpha_4)\beta_2^p + \ldots + (\alpha_{2n-1} + \alpha_{2n})\beta_n^p.$$

So

$$\|v\|^p = (\alpha_1 + \alpha_2)\beta_1^p + (\alpha_3 + \alpha_4)\beta_2^p + \ldots + (\alpha_{2n-1} + \alpha_{2n})\beta_n^p.$$

□

Corollary 2.3. *In the space $X_{\alpha,p}$ and for any integer n we have*

(2) $$\Big\|\sum_{i=1}^{n} u_i\Big\| = \Big\|\sum_{i=1}^{n}(e_{2i} - e_{2i-1})\Big\| = \Big(\sum_{i=1}^{2n} \alpha_i\Big)^{1/p}.$$

Proof. Put $\beta_i = 1$ in Lemma 3.2 . □

We know that if $p > q$ the identity operator from ℓ_p to ℓ_q is unbounded. Here is a similar result for $X_{\alpha,p}$.

Theorem 2.4. *The identity operator from $X_{\alpha,p}$ to $X_{\alpha,q}$ when $p > q$ is unbounded.*

Proof. Let I be bounded, then for any scalars a_i

$$\Big\|\sum_{i=1}^{n} a_i e_i\Big\|_{X_{\alpha,q}} = \Big\|\sum_{i=1}^{n} I a_i e_i\Big\|_{X_{\alpha,q}} = \Big\|I \sum_{i=1}^{n} a_i e_i\Big\|_{X_{\alpha,q}} \leqslant \|I\| \Big\|\sum_{i=1}^{n} a_i e_i\Big\|_{X_{\alpha,p}}$$

with $a_i = (-1)^i$ and Corollary 3.3 yields

$$\Big(\sum_{i=1}^{n} \alpha_i\Big)^{1/q} = \Big\|\sum_{i=1}^{n}(-1)^i e_i\Big\|_{X_{\alpha,q}} \leqslant \|I\| \Big\|\sum_{i=1}^{n}(-1)^n a_i e_i\Big\|_{X_{\alpha,p}} = \|I\| \Big(\sum_{i=1}^{n} \alpha_i\Big)^{1/p},$$

therefore

$$\Big(\sum_{i=1}^{n} \alpha_i\Big)^{1/q-1/p} \leqslant \|I\|.$$

This is a contradiction, since $\sum\limits_{1}^{\infty} \alpha_i$ diverges. So I is unbounded. □

We use the following lemma from [3].

Lemma 2.5. *Let (u_i) be a sequence of norm one vectors in $X_{\alpha,p}$ and (G_i) an admissible sequence of blocks such that $\{j\colon\ u_i(j) \neq 0\} \subset G_i$. For each i put $s_i = s(u_i)$. If $\lim s_i = 0$ then a subsequence (v_k) of (u_k) is equivalent to the usual basis of ℓ_p.*

Theorem 2.6. *Let* $T\colon \tilde{X}_{\alpha,p} \longrightarrow \tilde{X}_{\alpha,q}$, $1<q<p$ *be a bounded linear operator and for any normalized block basis let* $y_n = \sum\limits_{i=q_n+1}^{q_{n+1}} a_i u_i$ *where* $u_n = e_{2n-1} - e_{2n}$ *and* $\lim a_i = 0$. *Then* T *is compact.*

Proof. It is enough to show that for every sequence (x_n) in $X_{\alpha,p}$ such that $x_n \overset{w}{\to} 0$ we have $Tx_n \overset{\|\cdot\|}{\to} 0$. Assume that T is not a compact operator, then there is a sequence (x_n) in $X_{\alpha,p}$ such that $x_n \overset{w}{\to} 0$ and $\|Tx_n\| > \varepsilon$ for some $\varepsilon > 0$ and all integers n. By passing to a subsequence and using the Bessaga-Pelczynski selection we can assume that (x_n) is equivalent to the unit vector basis in $\tilde{X}_{\alpha,p}$ and (Tx_n) is equivalent to the vector unit basis in $\tilde{X}_{\alpha,q}$. In fact $x_n \sim y_n$ where

$$y_k = a_{n_{k-1}+1}u_{n_{k-1}+1} + \ldots + a_{n_k}u_{n_k}, \quad k = 1,2,3,\ldots$$

Now let $s_k = \max|\langle y_k, F\rangle|$ where the maximum is taken over all blocks F. Then (s_k) is a subsequence of (a_k). We observe that by Lemma 3.5 and the fact that $s_k \to 0$ the sequences (y_n), and so (x_n), are equivalent to the unit vector basis of ℓ_p. A similar argument shows that (Tx_n) is equivalent to the unit vector basis of ℓ_q. Then there are bounded linear operators S_1 and S_2 such that $x_n = S_1 e_n$ and $S_2 T x_n = e_n$. Now for every scalars a_n we have

$$\begin{aligned}\left(\sum_{n=1}^{m} |a_n|^q\right)^{1/q} &= \left\|\sum_{n=1}^{m} a_n e_n\right\|_{X_{\alpha,q}} = \left\|\sum a_n S_2 T x_n\right\| \\ &\leqslant \|S_2\|\|T\|\left\|\sum a_n x_n\right\| \leqslant \|S_2\|\|T\|\left\|\sum a_n S_1 e_n\right\| \\ &\leqslant \|S_2\|\|T\|\|S_1\|\left\|\sum a_n e_n\right\|_{X_{\alpha,p}} = M\|T\|\left(\sum_{1}^{m} |a_n|^p\right)^{1/p}\end{aligned}$$

where $M = \|S_2\|\|S_1\|$. If $a_i = 1$ for all i then $m^{1/q} \leqslant M\|T\|m^{1/p}$, i.e. $m^{1/q-1/p} \leqslant M\|T\|$. This shows that T is not bounded and this is a contradiction. So T is a compact operator. □

Now we deduce some more results concerning the subspace structure of $X_{\alpha,p}$ spaces.

Definition 2.7. A Banach space X is called a Grothendieck space if every weak*-convergent sequence in X^* is weakly convergent. For example, every reflexive Banach space is a Grothendieck space.

Definition 2.8. A Banach space X is said to be weakly compactly generated whenever there exists a weakly compact subset K of X such that the closed linear span of K is all X ($[K] = X$). Every reflexive and separable Banach space is weakly compactly generated.

Now we state the following theorem from [9].

Theorem 2.9. *Given a Banach space X, the following conditions are equivalent:*

(1) *X is a Grothendieck space;*
(2) *every continuous linear operator $T\colon X \longrightarrow Y$, where Y is separable, is weakly compact;*
(3) *every continuous linear operator $T\colon X \longrightarrow Y$ where Y is weakly compactly generated, is weakly compact;*
(4) *every continuous linear operator $T\colon X \longrightarrow c_0$ is weakly compact;*
(5) *if Y is any Banach space, and for each $n \in \mathbb{N}$, $T_n\colon X \longrightarrow Y$ is weakly compact operator such that (weak) $\lim\limits_n T_n(x) \equiv T_0(x)$ exists for every $x \in X$, then $T_0\colon X \longrightarrow Y$ is weakly compact;*
(6) *the "weak convergence" of $T_n(x)$ in condition (5) can be replaced by "norm convergence".*

Let $Y_{\alpha,p}$ be the predual of $X_{\alpha,p}$. We have the following corollary.

Corollary 2.10. *Every bounded linear operator from $X_{\alpha,p}$ to $X_{\alpha,q}$ and also from $Y_{\alpha,p}$ to $Y_{\alpha,q}$ where $1/p + 1/q = 1$ is weakly compact.*

Definition 2.11. A Banach space X is said to contain an asymptotically isometric copy of c_0 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X such that

$$\sup_n (1-\varepsilon_n)|t_n| \leqslant \Bigl\| \sum_{n=1}^{\infty} t_n x_n \Bigr\| \leqslant \sup_n |t_n|.$$

We say that a Banach space X is asymptotically isometric to c_0 if X has a basis $(x_n)_n$ with the above property.

Definition 2.12. A Banach space X is said to contain an asymptotically isometric copy of l^∞ if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a bounded linear operator $T\colon l^\infty \longrightarrow X$ such that

$$\sup_n (1-\varepsilon_n)|t_n| \leqslant \|T((t_n)_n)\| \leqslant \sup_n |t_n|.$$

Theorem 1.1 and a result of S. Chen and B. L. Lin yield

Theorem 2.13. *For the $X_{\alpha,1}$ spaces*

(1) *the predual of $X_{\alpha,1}$ contains asymptotically isometric copies of c_0;*
(2) *the dual of $X_{\alpha,1}$ contains an asymptotically isometric copy of ℓ_∞.*

$K_{w^*}(X^*, Y)$ denotes the Banach spaces of compact and weak*-weakly continuous linear operators from X^* into Y, endowed with the usual operator norm.

Remark. In [7] Dowling showed that a Banach space containing an asymptotically isometric copy of ℓ_∞ must contain an isometric copy of ℓ_∞.

The following theorems are due to D. Chen [4].

Theorem 2.14. *Let X and Y be two infinite dimensional Banach spaces. If Y contains an asymptotically isometric copy of c_0, then $K_{w^*}(X, Y)$ contains a complemented asymptotically isometric copy of c_0.*

Theorem 2.15. *Let X be an infinite-dimensional normed linear space and Y a Banach space containing an asymptotically isometric copy of c_0. Then $L(X, Y)$ contains an isometric copy of ℓ_∞.*

Suppose that Y is the predual of $X_{\alpha,1}$. Then we have

Theorem 2.16. *$L(X_{\alpha,1}, Y)$ contains an isometric copy of ℓ_∞.*

Theorem 2.17. *$K_{w^*}(X_{\alpha,1}, Y)$ contains complemented asymptotically isometric copies of c_0.*

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