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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 1, 195–209

Czechoslovak Mathematical Journal, 60 (135) (2010), 195–209

ON THE BLOW UP CRITERION FOR THE 2-D COMPRESSIBLE NAVIER-STOKES EQUATIONS

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(Received September 12, 2008)

Abstract. Motivated by [10], we prove that the upper bound of the density function ϱ controls the finite time blow up of the classical solutions to the 2-D compressible isentropic Navier-Stokes equations. This result generalizes the corresponding result in [3] concerning the regularities to the weak solutions of the 2-D compressible Navier-Stokes equations in the periodic domain.

Keywords: compressible Navier-Stokes equations, classical solutions, blow up criterion

MSC 2010: 35Q30, 76D03

1. Introduction

In this paper we consider the blow up criterion for the classical solutions to the following 2-D compressible isentropic Navier-Stokes equations in the periodic domain:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho U) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{T}^2, \\ \partial_t(\varrho U) + \operatorname{div}(\varrho U \otimes U) - \mu \Delta U - (\mu+\lambda)\nabla \operatorname{div} U + \nabla p(\varrho) = 0, \end{cases} \tag{1.1}$$

where $\varrho, U = (u,v)$ stand for the density and velocity of the viscous compressible fluid respectively, and μ, λ are the dynamical and volume viscosities such that $\mu > 0$ and $3\lambda + 2\mu \geqslant 0$. For simplicity, in what follows, we always take $\mu = 1$. We complement the above system with the initial data

$$\varrho|_{t=0} = \varrho_0, \qquad U|_{t=0} = U_0. \tag{1.2}$$

Furthermore, we assume that there exist two positive constants m and M such that

$$m \leqslant \varrho_0(x) \leqslant M \quad \text{for} \quad x \in \mathbb{T}^2. \tag{1.3}$$

The Navier-Stokes equations are the basic model describing the evolution of a viscous compressible gas. Before the celebrated works of P. L. Lions, very little was known about the global solutions to the multi-dimensional compressible Navier-Stokes equations. In particular, Lions [7] proved the global existence of weak solutions to (1.1) under the assumptions that

$$p(\varrho) = A\varrho^{\gamma}, \quad \varrho_0 \in L^1(\mathbb{T}^d) \cap L^{\gamma}(\mathbb{T}^d), \quad \varrho_0 \geqslant 0 \quad \text{and} \quad \frac{m_0^2}{\varrho_0} \in L^1(\mathbb{T}^d),$$
$$\gamma \geqslant \frac{3}{2} \quad \text{if } d = 2, \qquad \gamma \geqslant \frac{9}{5} \quad \text{if } d = 3, \qquad \gamma > \frac{d}{2} \quad \text{if } d \geqslant 4,$$

where we agree that $m_0^2/\varrho_0 = 0$ on the set $\{x \in \mathbb{T}^d$ such that $\varrho_0(x) = 0\}$. This result was improved later by Feireisl [4] et al to $\gamma > \frac{1}{2}d$.

On the other hand, as was emphasized in many papers related to the viscous compressible fluid dynamics, vacuum might be a major difficulty when one tries to prove the global classical solutions to (1.1), like in [9], where Xin proved the finite time blow up of classical solutions to the full compressible Navier-Stokes equations when the initial density has compact support. As a matter of fact, starting from initial densities with positive lower bound, local existence of smooth solutions to (1.1) can be proved by classical method (see [5], [8]), while in [3], Desjardins proved that $\sup_{t\in[0,T)} \|\varrho(t)\|_{L^\infty}$ controls the regularities of weak solutions to the 2-D isentropic compressible Navier-Stokes equations.

Motivated by [10], where the authors proved the global existence of classical solutions to (1.1) for $\lambda = \varrho^{\beta}$ with $\beta > 3$, we shall prove in this paper that $\sup_{t\in[0,T)} \|\varrho(t)\|_{L^\infty}$ controls the finite time blow up of classical solutions to (1.1). More precisely, we prove

Theorem 1.1. *Given $\varrho_0, U_0 \in H^3(\mathbb{T}^2)$ with ϱ_0 satisfying* (1.3), *and $p(\varrho) = A\varrho^{\gamma}$ for $A, \gamma > 0$, there exists a positive constant T such that* (1.1)–(1.2) *has a unique solution (ϱ, U) with $\varrho \in C([0,T); H^3(\mathbb{T}^2))$, $U \in C([0,T); H^3(\mathbb{T}^2)) \cap L^2((0,T); H^4(\mathbb{T}^2))$. We denote by T^* the maximal time of the existence of (ϱ, U). Then if $T^* < \infty$, we have*

$$\lim_{T\to T^*} \sup_{t\in[0,T)} \|\varrho(t)\|_{L^\infty} = \infty. \tag{1.4}$$

Let us end this section with the notation we are going to use in this paper.

Notation. In the following, we shall denote by C and C_T uniform positive constants which may be different on different lines. We shall denote by (a, b) the $L^2(\mathbb{T}^2)$ inner product of a and b, and $\|a\|_{L^p}, \|a\|_{H^s}$ the standard $L^p(\mathbb{T}^2)$ and $H^s(\mathbb{T}^2)$ norms of a.

2. The proof of the theorem

Under the assumptions of Theorem 1.1, in standard method can be applied to prove the local well posedness of (1.1)–(1.2), like the [5], [8], and we omit the details here. Now let (ϱ, U) be the local classical solution of (1.1)–(1.2) given by Theorem 1.1. Then one gets by the standard energy estimate that

(2.1)
$$\begin{aligned}\frac{1}{2}\sup_{0\leqslant s\leqslant t}\|U(s)\|_{L^2}^2+\sup_{0\leqslant s\leqslant t}\int_{\mathbb{T}^2}\varphi(\varrho)\,\mathrm{d}x\,\mathrm{d}y+\int_0^t[\|\nabla U\|_{L^2}^2+(1+\lambda)\|\operatorname{div}U\|_{L^2}^2]\,\mathrm{d}s\\ \leqslant\frac{1}{2}\|U_0\|_{L^2}^2+\int_{\mathbb{T}^2}\varphi(\varrho_0)\,\mathrm{d}x\,\mathrm{d}y\qquad\text{for } 0\leqslant t<T,\end{aligned}$$

with $\varphi(\varrho)\overset{\text{def}}{=}\varrho\int_1^\varrho\tau^{-2}p(\tau)\,\mathrm{d}\tau$.

To get an estimate of the derivatives of U, motivated by [10] we denote

$$\begin{aligned}A\overset{\text{def}}{=}u_y-v_x,\qquad B\overset{\text{def}}{=}(2+\lambda)(u_x+v_y)-p(\varrho),\\ L\overset{\text{def}}{=}\frac{1}{\varrho}(A_y+B_x),\quad\text{and}\quad H\overset{\text{def}}{=}\frac{1}{\varrho}(-A_x+B_y).\end{aligned}\tag{2.2}$$

Then one can rewrite the momentum equations of (1.1) as

$$\begin{cases}u_t+u\partial_x u+v\partial_y u=\dfrac{1}{\varrho}(A_y+B_x),\\ v_t+u\partial_x v+v\partial_y v=\dfrac{1}{\varrho}(-A_x+B_y).\end{cases}\tag{2.3}$$

Lemma 2.1. *Let (ϱ, U) be the classical solution of (1.1)–(1.2) given by Theorem 1.1. We assume that $\sup\limits_{t\in[0,T)}\|\varrho(t)\|_{L^\infty}<\infty$, then*

$$\sup_{0\leqslant t<T}\big(\|A(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2\big)+\int_0^T\big(\|\nabla A(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2\big)\,\mathrm{d}t\leqslant C_T,\tag{2.4}$$

where C_T depends only on $\|U_0\|_1$, $\|\varphi(\varrho_0)\|_{L^1}$ and $\sup\limits_{t\in[0,T)}\|\varrho(t)\|_{L^\infty}$.

P r o o f. First, thanks to (2.2) and (2.3) we obtain

$$A_t+u\partial_x A+v\partial_y A+A\operatorname{div}U=L_y-H_x\tag{2.5}$$

and

$$(\operatorname{div}U)_t+u\partial_x(\operatorname{div}U)+v\partial_y(\operatorname{div}U)+u_x^2+2u_yv_x+v_y^2=L_x+H_y.$$

On the other hand, thanks to the continuity equation of (1.1), one has

$$p(\varrho)_t + u\partial_x p(\varrho) + v\partial_y p(\varrho) = -\varrho p'(\varrho)\operatorname{div} U,$$

from which we deduce that

$$\frac{1}{2+\lambda}[B_t + u\partial_x B + v\partial_y B - \varrho p'(\varrho)\operatorname{div} U] + u_x^2 + 2u_y v_x + v_y^2 = L_x + H_y. \tag{2.6}$$

By virtue of (2.5) and (2.6) we get by the standard energy estimate that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\Big(A^2 + \frac{B^2}{2+\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{T}^2}[(LA_y - HA_x) + (LB_x + HB_y)]\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \frac{1}{2+\lambda}\int_{\mathbb{T}^2}\varrho p'(\varrho)B\operatorname{div} U\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{T}^2}(uA_x + vA_y)A\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{T}^2}\operatorname{div} U A^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \frac{1}{2+\lambda}\int_{\mathbb{T}^2}(uB_x + vB_y)B\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{T}^2}B(u_x^2 + 2u_y v_x + v_y^2)\,\mathrm{d}x\,\mathrm{d}y = 0.
\end{aligned}$$

However, notice that

$$(LA_y - HA_x) + (LB_x + HB_y) = \frac{1}{\varrho}(A_y + B_x)^2 + \frac{1}{\varrho}(-A_x + B_y)^2,$$

and

$$\begin{aligned}
&\Big[\sup_{t\in[0,T)}\|\varrho(t)\|_{L^\infty}\Big]^{-1}\int_{\mathbb{T}^2}(|\nabla A|^2 + |\nabla B|^2)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\leqslant \int_{\mathbb{T}^2}\frac{1}{\varrho}[(A_y + B_x)^2 + (-A_x + B_y)^2]\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

for $0 \leqslant t < T$. Then one gets by using integration by parts

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\Big(A^2 + \frac{B^2}{2+\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \Big[\sup_{t\in[0,T)}\|\varrho(t)\|_{L^\infty}\Big]^{-1}\int_{\mathbb{T}^2}\big[(A_y + B_x)^2 + (-A_x + B_y)^2\big]\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \frac{1}{2}\int_{\mathbb{T}^2}\Big(-A^2 + \frac{B^2}{2+\lambda}\Big)\operatorname{div} U\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2+\lambda}\int_{\mathbb{T}^2}\varrho p'(\varrho)B\operatorname{div} U\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \int_{\mathbb{T}^2}B(u_x^2 + 2u_y v_x + v_y^2)\,\mathrm{d}x\,\mathrm{d}y, \qquad \text{for}\quad 0 \leqslant t < T.
\end{aligned} \tag{2.7}$$

Note that for any $\varepsilon > 0$, Young's inequality applied gives

$$\begin{aligned}
\Big|\int_{\mathbb{T}^2}\Big(-A^2 + \frac{B^2}{2+\lambda}\Big)\operatorname{div} U\,\mathrm{d}x\,\mathrm{d}y\Big| &\leqslant C(\|A\|_{L^4}^2 + \|B\|_{L^4}^2)(\|u_x\|_{L^2} + \|v_y\|_{L^2})\\
&\leqslant \varepsilon(\|\nabla A\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \frac{C}{\varepsilon}(\|u_x\|_{L^2}^2 + \|v_y\|_{L^2}^2)(\|A\|_{L^2}^2 + \|B\|_{L^2}^2),
\end{aligned}$$

where we have used the trivial interpolation inequality in 2-D,

$$\|A\|_{L^4} \leqslant C\|A\|_{L^2}^{\frac{1}{2}}\|\nabla A\|_{L^2}^{\frac{1}{2}}, \tag{2.8}$$

and it is easy to observe that

$$\left|\int_{\mathbb{T}^2} B(u_x^2+2u_yv_x+v_y^2)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant C\|B\|_{L^4}(\|\nabla u\|_{L^{\frac{8}{3}}}^2+\|\nabla v\|_{L^{\frac{8}{3}}}^2).$$

On the other hand, thanks to (2.2) we have

$$\begin{aligned}
&\|\nabla u\|_{L^{\frac{8}{3}}}^2+\|\nabla v\|_{L^{\frac{8}{3}}}^2 \leqslant C(\|u_x+v_y\|_{L^{\frac{8}{3}}}^2+\|u_y-v_x\|_{L^{\frac{8}{3}}}^2), \qquad \text{and}\\
&\|u_y-v_x\|_{L^{\frac{8}{3}}} = \|A\|_{L^{\frac{8}{3}}} \leqslant C\|A\|_{L^2}^{\frac{3}{4}}\|\nabla A\|_{L^2}^{\frac{1}{4}},\\
&\|u_x+v_y\|_{L^{\frac{8}{3}}} \leqslant \Big\|\frac{B}{2+\lambda}+p(\varrho)\Big\|_{L^{\frac{8}{3}}} \leqslant C_T(1+\|B\|_{L^2}^{\frac{3}{4}}\|\nabla B\|_{L^2}^{\frac{1}{4}}),
\end{aligned}$$

with C_T depending only on $\sup\limits_{t\in[0,T)}\|\varrho(t)\|_{L^\infty}$. As a consequence, we obtain

$$\left|\int_{\mathbb{T}^2} B(u_x^2+2u_yv_x+v_y^2)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \varepsilon(\|\nabla A\|_{L^2}^2+\|\nabla B\|_{L^2}^2)+\frac{C_T}{\varepsilon}(1+\|A\|_{L^2}^4+\|B\|_{L^2}^4)$$

for any $\varepsilon>0$. Then thanks to (2.7), we get by taking $\varepsilon \leqslant \frac{1}{4}\Big[\sup\limits_{t\in[0,T)}\|\varrho(t)\|_{L^\infty}\Big]^{-1}$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{T}^2}\Big(A^2+\frac{B^2}{2+\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}y+\big[\sup_{t\in[0,T)}\|\varrho(t)\|_{L^\infty}\big]^{-1}\int_{\mathbb{T}^2}(|\nabla A|^2+|\nabla B|^2)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\leqslant C_T(1+\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2)\int_{\mathbb{T}^2}\Big(A^2+\frac{B^2}{2+\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

and applying (2.1) and the Gronwall inequality gives (2.4). □

Lemma 2.2. *Under the assumptions of Lemma* 2.1, *one can find a positive constant* m_T *which depends only on* $\sup\limits_{t\in[0,T)}\|\varrho(t)\|_{L^\infty}$, $\|U_0\|_{H^1}$ *and* m, M *in* (1.3) *such that*

$$\varrho(t,x) \geqslant m_T \quad \textit{for} \quad 0\leqslant t<T. \tag{2.9}$$

Proof. We first get by taking divergence to the momentum equation of (1.1) that

$$\Delta\big[(2+\lambda)\operatorname{div}U-p(\varrho)\big] = \partial_t\operatorname{div}(\varrho U)+\operatorname{div}(\operatorname{div}(\varrho U\otimes U)),$$

which implies that

$$\operatorname{div} U = \frac{1}{2+\lambda}\big[\partial_t \Delta^{-1}\operatorname{div}(\varrho U) + \Delta^{-1}\operatorname{div}(\operatorname{div}(\varrho U\otimes U)) + p(\varrho)\big],$$

where $\Delta^{-1} f$ is defined as the unique solution of

$$\Delta N = f \quad \text{on } \mathbb{T}^2, \qquad \int_{\mathbb{T}^2} N \,\mathrm{d}x\,\mathrm{d}y = 0.$$

Then thanks to the continuity equation of (1.1),

$$\partial_t \varrho + U\cdot\nabla\varrho + \varrho \operatorname{div} U = 0,$$

one has

$$\partial_t \log\varrho + u\cdot\nabla\log\varrho + \frac{1}{2+\lambda}\big[\partial_t\Delta^{-1}\operatorname{div}(\varrho U) + \Delta^{-1}\operatorname{div}(\operatorname{div}(\varrho U\otimes U)) + p(\varrho)\big] = 0,$$

which gives

$$\begin{aligned}(2.10)\qquad \partial_t\Big[\log\varrho + \frac{1}{2+\lambda}\operatorname{div}\Delta^{-1}(\varrho U)\Big] + u\cdot\nabla\Big[\log\varrho + \frac{1}{2+\lambda}\operatorname{div}\Delta^{-1}(\varrho U)\Big] \\ = \frac{1}{2+\lambda}\big[u\cdot\nabla\operatorname{div}\Delta^{-1}(\varrho U) - \Delta^{-1}\operatorname{div}\operatorname{div}(\varrho U\otimes U) - p(\varrho)\big].\end{aligned}$$

On the other hand, motivated by [2], we denote $D \overset{\text{def}}{=} \Delta^{-1}\operatorname{div}\operatorname{div}(\varrho U\otimes U) - u\cdot\nabla\operatorname{div}\Delta^{-1}(\varrho U)$ and $G \overset{\text{def}}{=} \operatorname{div}\Delta^{-1}(\varrho U)$. Then

$$\Delta D = \operatorname{div}\big[\varrho U\cdot\nabla U + \operatorname{div} U\nabla G - \nabla U\cdot\nabla G + \nabla G\cdot\nabla U\big],$$

from which and (2.2) one gets by the standard potential theory that

$$\begin{aligned}\|\nabla D\|_{L^3} &\leqslant C\|(\varrho U + |\nabla G|)|\nabla U|\|_{L^3} \\ &\leqslant C\|\varrho\|_{L^\infty}\|U\|_{L^6}\|\nabla U\|_{L^6} \leqslant C_T\|U\|_{L^2}^{\frac13}\|\nabla U\|_{L^2}^{\frac23}[\|A\|_{L^6} + \|B\|_{L^6} + 1].\end{aligned}$$

However, thanks to (2.4) we have

$$\begin{aligned}&\int_0^T (\|A(t)\|_{L^6} + \|B(t)\|_{L^6})\,\mathrm{d}t \\ &\qquad \leqslant C\int_0^T [\|A(t)\|_{L^2}^{\frac13}\|\nabla A(t)\|_{L^2}^{\frac23} + \|B(t)\|_{L^2}^{\frac13}\|\nabla B(t)\|_{L^2}^{\frac23}]\,\mathrm{d}t \leqslant C_T,\end{aligned}$$

which gives

$$\int_0^T \|\nabla D\|_{L^3}\,\mathrm{d}t \leqslant C_T. \tag{2.11}$$

Similarly to the proof of (2.11) we have

$$\|D\|_{L^3} \leqslant C\|(\varrho U + |\nabla G|)|\nabla U|\|_{L^{\frac{6}{5}}} \leqslant C\|\varrho\|_{L^\infty}\|U\|_{L^2}^{\frac{2}{3}}\|\nabla U\|_{L^2}^{\frac{4}{3}} \leqslant C_T,$$

which gives

$$\int_0^T \|D(t)\|_{L^\infty}\,\mathrm{d}t \leqslant C_T. \tag{2.12}$$

Thanks to (1.3), (2.10) and (2.12), one gets by using the characteristic method that there exists a positive constant C_T such that

$$\log\varrho(t,x) + \frac{1}{2+\lambda}G(t,x) \geqslant -C_T \qquad \text{for} \quad 0 \leqslant t < T. \tag{2.13}$$

Now we note that

$$\begin{aligned}
\|G\|_{L^4} &\leqslant C\|\varrho\|_{L^\infty}\|U\|_{L^{\frac{4}{3}}} \leqslant C\|U\|_{L^2},\\
\|\nabla G\|_{L^4} &\leqslant C\|\varrho\|_{L^\infty}\|U\|_{L^4} \leqslant C\|\varrho\|_{L^\infty}\|U\|_{L^2}^{\frac{1}{2}}\|\nabla U\|_{L^2}^{\frac{1}{2}},
\end{aligned}$$

which together with (2.1) and (2.4) implies that

$$\sup_{t\in[0,T)} \|G(t)\|_{L^\infty} \leqslant C_T,$$

from which and (2.13) one obtains (2.9). This completes the proof of Lemma 2.2. $\square$

An estimate of higher derivatives of U is obtained by carrying out calculations similar to but easier than (15) and (16) in [10]:

$$\begin{cases}
\varrho(L_t + U\cdot\nabla L) - \varrho L\operatorname{div}U + u_yA_x + v_yA_y + u_xB_x + v_xB_y + (A\operatorname{div}U)_y\\
\quad -(p'(\varrho)\varrho\operatorname{div}U)_x + (2+\lambda)(u_x^2 + v_y^2 + 2v_xu_y)\\
= (L_y - H_x)_y + (2+\lambda)(L_x + H_y)_x,\\
\varrho(H_t + U\cdot\nabla H) - \varrho H\operatorname{div}U - u_xA_x - v_xA_y + u_yB_x + v_yB_y - (A\operatorname{div}U)_x\\
\quad -(p'(\varrho)\varrho\operatorname{div}U)_y + (2+\lambda)(u_x^2 + v_y^2 + 2v_xu_y)_y\\
= -(L_y - H_x)_x + (2+\lambda)(L_x + H_y)_y.
\end{cases} \tag{2.14}$$

Now let us turn to the estimate of L, H.

Lemma 2.3. *Under the assumptions of Lemma* 2.1, *there exists a positive constant* C_T *which depends on* $\sup\limits_{t\in[0,T)} \|\varrho(t)\|_{L^\infty}$, $\|U_0\|_{H^2}$, $\|\varrho_0\|_{H^1}$ *and* m, M *in* (1.3) *such that*

$$\sup_{0\leqslant t<T}\int_{\mathbb{T}^2}(L^2+H^2)\,\mathrm{d}x\,\mathrm{d}y+\int_0^T\!\!\int_{\mathbb{T}^2}[(L_y-H_x)^2+(L_x+H_y)^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\leqslant C_T. \tag{2.15}$$

Proof. Thanks to (2.14), one gets by using the standard energy estimate that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{T}^2}\varrho(L^2+H^2)\,\mathrm{d}x\,\mathrm{d}y&+\int_{\mathbb{T}^2}\big[(L_y-H_x)^2+(2+\lambda)(L_x+H_y)^2\big]\,\mathrm{d}x\,\mathrm{d}y\\ &-\int_{\mathbb{T}^2}\varrho(L^2+H^2)\operatorname{div}U\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{T}^2}A\operatorname{div}U(H_x-L_y)\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{\mathbb{T}^2}\varrho(L_x+H_y)\operatorname{div}Up'(\varrho)\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{\mathbb{T}^2}L(u_yA_x+v_yA_y+u_xB_x+v_xB_y)\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{\mathbb{T}^2}H(-u_xA_x-v_xA_y+u_yB_x+v_yB_y)\,\mathrm{d}x\,\mathrm{d}y\\ &-(2+\lambda)\int_{\mathbb{T}^2}(L_x+H_y)(u_x^2+2v_xu_y+v_y^2)\,\mathrm{d}x\,\mathrm{d}y=0.\end{aligned} \tag{2.16}$$

Applying the Cauchy-Schwartz inequality, we obtain

$$\Big|\int_{\mathbb{T}^2}\varrho(L^2+H^2)\operatorname{div}U\,\mathrm{d}x\,\mathrm{d}y\Big|\leqslant\|\varrho\|_{L^\infty}\|\nabla U\|_{L^2}(\|L\|_{L^4}^2+\|H\|_{L^4}^2),$$

and using (2.8) we arrive at

$$\begin{aligned}\|L\|_{L^4}^2+\|H\|_{L^4}^2&\leqslant C(\|L\|_{L^2}+\|H\|_{L^2})(\|\nabla L\|_{L^2}+\|\nabla H\|_{L^2})\\ &\leqslant C(\|L\|_{L^2}+\|H\|_{L^2})(\|L_y-H_x\|_{L^2}+\|L_x+H_y\|_{L^2}).\end{aligned}$$

As a consequence, for any $\varepsilon>0$ one has

$$\begin{aligned}&\Big|\int_{\mathbb{T}^2}\varrho(L^2+H^2)\operatorname{div}U\,\mathrm{d}x\,\mathrm{d}y\Big|\\ &\qquad\leqslant\varepsilon(\|L_y-H_x\|_{L^2}^2+\|L_x+H_y\|_{L^2}^2)+\frac{C_T}{\varepsilon}\|\nabla U\|_{L^2}^2(\|L\|_{L^2}^2+\|H\|_{L^2}^2).\end{aligned}$$

A similar argument yields

$$\begin{aligned}\Big|\int_{\mathbb{T}^2}A\operatorname{div}U(H_x-L_y)\,\mathrm{d}x\,\mathrm{d}y\Big|&\leqslant\varepsilon\|H_x-L_y\|_{L^2}^2+\frac{C}{\varepsilon}\|A\|_{L^4}^2\|\operatorname{div}U\|_{L^4}^2\\ &\leqslant\varepsilon\|H_x-L_y\|_{L^2}^2+\frac{C_T}{\varepsilon}(1+\|A\|_{L^2}^2\|\nabla A\|_{L^2}^2+\|B\|_{L^2}^2\|\nabla B\|_{L^2}^2),\end{aligned}$$

where we used (2.2) and (2.8), so that

$$
\begin{aligned}
(2.17) \qquad \|\nabla U\|_{L^4} &\leqslant C(\|\operatorname{div} U\|_{L^4} + \|u_y - v_x\|_{L^4}) \\
&\leqslant C_T(1 + \|A\|_{L^2}^{\frac{1}{2}}\|\nabla A\|_{L^2}^{\frac{1}{2}} + \|B\|_{L^2}^{\frac{1}{2}}\|\nabla B\|_{L^2}^{\frac{1}{2}}).
\end{aligned}
$$

It is easy to observe that

$$
\left|\int_{\mathbb{T}^2} \varrho(L_x + H_y)\operatorname{div} U p'(\varrho)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \varepsilon\|L_x + H_y\|_{L^2}^2 + \frac{C}{\varepsilon}\|\varrho p'(\varrho)\|_{L^\infty}^2\|\nabla U\|_{L^2}^2
$$

and

$$
\left|\int_{\mathbb{T}^2} L(u_yA_x + v_yA_y + u_xB_x + v_xB_y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \|L\|_{L^2}\|\nabla U\|_{L^4}(\|\nabla A\|_{L^4} + \|\nabla B\|_{L^4}),
$$

while thanks to (2.2) and (2.9) one has

$$
\begin{aligned}
\|\nabla A\|_{L^4} + \|\nabla B\|_{L^4} &\leqslant \frac{C}{m_T}(\|L\|_{L^4} + \|H\|_{L^4}) \\
&\leqslant C_T(\|L\|_{L^2} + \|H\|_{L^2})^{\frac{1}{2}}(\|L_x + H_y\|_{L^2} + \|L_y - H_x\|_{L^2})^{\frac{1}{2}}.
\end{aligned}
$$

Then one obtains

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2} L(u_yA_x + v_yA_y + u_xB_x + v_xB_y)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\quad\leqslant \varepsilon(\|L_x + H_y\|_{L^2}^2 + \|L_y - H_x\|_{L^2}^2) \\
&\quad\quad + \frac{C_T}{\varepsilon}(\|L\|_{L^2}^2 + \|H\|_{L^2}^2)(1 + \|A\|_{L^2}^{\frac{1}{2}}\|\nabla A\|_{L^2}^{\frac{1}{2}} + \|B\|_{L^2}^{\frac{1}{2}}\|\nabla B\|_{L^2}^{\frac{1}{2}})^{\frac{4}{3}}.
\end{aligned}
$$

A similar estimate holds for $\int_{\mathbb{T}^2} H(-u_xA_x - v_xA_y + u_yB_x + v_yB_y)\,\mathrm{d}x\,\mathrm{d}y$.

Finally, again thanks to (2.17) one has

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2} (L_x + H_y)(u_x^2 + 2v_xu_y + v_y^2)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\quad\leqslant \varepsilon\|L_x + H_y\|_{L^2}^2 + \frac{C_T}{\varepsilon}(1 + \|A\|_{L^2}^2\|\nabla A\|_{L^2}^2 + \|B\|_{L^2}^2\|\nabla B\|_{L^2}^2).
\end{aligned}
$$

Combining the above estimates, we get by taking ε small enough

$$
\begin{aligned}
(2.18) \quad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2} \varrho(L^2 + H^2)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{T}^2}\big[(L_y - H_x)^2 + (2+\lambda)(L_x + H_y)^2\big]\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\leqslant \frac{C_T}{\varepsilon}(1 + \|L\|_{L^2}^2 + \|H\|_{L^2}^2) \\
&\quad\quad \times \big[1 + \|\nabla U\|_{L^2}^2 + \|A\|_{L^2}^2\|\nabla A\|_{L^2}^2 + \|B\|_{L^2}^2\|\nabla B\|_{L^2}^2)\big].
\end{aligned}
$$

Then thanks to (2.1), (2.4) and (2.9) one obtains (2.15) by using the Gronwall inequality. This completes the proof of the lemma.

□

With the above estimate for the first derivatives of U, we now turn to an estimate of the first derivatives of ϱ.

Proposition 2.1. *Under the assumptions of Theorem* 1.1, *there is a positive constant* C_T, *which depends on* $\sup_{t\in[0,T)} \|\varrho(t)\|_{L^\infty}$, $\|\varrho_0\|_{H^2}$, $\|U_0\|_{H^2}$ *and* m, M *in* (1.3) *such that*

$$\sup_{0\leqslant t\leqslant T} \|\nabla\varrho(t)\|_{L^q} \leqslant C_T \qquad \text{for any } q < \infty. \tag{2.19}$$

P r o o f. First, thanks to (2.15) we deduce that both L and H are bounded in $L^p([0,T];L^q(\mathbb{T}^2))$ with p, q satisfying $1/q = \frac{1}{2} - 1/p$. As a consequence, we get using (2.2) and $\sup_{t\in[0,T)} \|\varrho(t)\|_{L^\infty} < \infty$ that

$$\nabla A,\ \nabla B \qquad \text{are bounded in} \quad L^p([0,T];L^q(\mathbb{T}^2)), \tag{2.20}$$

in particular, $B \in L^p([0,T];L^\infty(\mathbb{T}^2))$ for any $p < \infty$.

Now let X be the flow of U given by

$$\partial_t X(t,s,x) = U(t, X(t,s,x)), \qquad X(t,s,x)|_{t=s} = x,$$

from which, (2.2) and (2.20) we deduce that

$$\begin{aligned} \det\Big(\frac{\partial X(t,s,x)}{\partial x}\Big) &= \exp\Big(\int_0^t \operatorname{div} U(t', X(t',s,x))\,\mathrm{d}t'\Big) \\ &= \exp\bigg(\frac{1}{2+\lambda}\int_0^t (B + p(\varrho))(t', X(t',s,x))\,\mathrm{d}t'\bigg) \leqslant C_T. \end{aligned} \tag{2.21}$$

Thanks to the continuity equation of (1.1) and (2.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho(t, X(t,s,x)) = -\frac{\varrho}{2+\lambda}(B + A\varrho^\gamma)(t, X(t,s,x)),$$

which gives

$$\begin{aligned} \varrho^{-\gamma}(t,x) &= \varrho_0^{-\gamma}(X(0,t,x))\mathrm{e}^{\gamma(2+\lambda)^{-1}\int_0^t B(s,X(s,t,x))\,\mathrm{d}s} \\ &\quad + \frac{A\gamma}{2+\lambda}\int_0^t \mathrm{e}^{\gamma(2+\lambda)^{-1}\int_\tau^t B(s,X(s,t,x))\,\mathrm{d}s}\,\mathrm{d}\tau \overset{\text{def}}{=} M_1(t,x) + M_2(t,x), \end{aligned}$$

and

$$\begin{aligned} &\varrho(t,x) = (M_1(t,x) + M_2(t,x))^{-1/\gamma} \qquad \text{and} \\ &\nabla\varrho(t,x) = -\frac{1}{\gamma}(M_1(t,x) + M_2(t,x))^{-1/\gamma-1}(\nabla M_1(t,x) + \nabla M_2(t,x)). \end{aligned} \tag{2.22}$$

Note that

$$\nabla M_1(t,x) = \nabla \varrho_0^{-\gamma}(X(0,t,x)) \cdot \nabla X(0,t,x) \mathrm{e}^{\gamma(2+\lambda)^{-1}\int_0^t B(s,X(s,t,x))\,\mathrm{d}s}$$
$$+ \frac{\gamma}{2+\lambda}\varrho_0^{-\gamma}(X(0,t,x))\mathrm{e}^{\gamma(2+\lambda)^{-1}\int_0^t B(s,X(s,t,x))\,\mathrm{d}s}\int_0^t \nabla B(s,X(s,t,x)\cdot\nabla X(s,t,x)\,\mathrm{d}s$$

and

$$\sup_{(s,t)\in[0,T]^2} |\nabla X(s,t,x)| \leqslant \exp\Big(\int_0^T |\nabla U(s,x)|\,\mathrm{d}s\Big).$$

Then thanks to (2.2) and (2.15), we obtain that $\nabla U \in L^2([0,T]; BMO(\mathbb{T}^2))$, and

$$\|\nabla M_1(t,\cdot)\|_{L^q} \leqslant C_T\Big(\int_{\mathbb{T}^2} \exp\Big(q'\int_0^T |\nabla U(s,x)|\,\mathrm{d}s\Big)\,\mathrm{d}x\Big)^{1/q'} \tag{2.23}$$
$$\leqslant C_T\Big(\frac{1}{T}\int_0^T\int_{\mathbb{T}^2}\exp(q'|\nabla U(s,x)|)\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/q'} \leqslant C_T.$$

A similar estimate holds for ∇M_2 as well, which together with (2.20) and (2.22) completes the proof of the proposition. □

Now we are in position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let (ϱ, U) be the local classical solution of (1.1)–(1.2) given by Theorem 1.1. Then if $T^* < \infty$, we have

$$\lim_{T\to T^*}\sup_{t\in[0,T)}\big[\|\varrho(t)\|_{H^3} + \|U(t)\|_{H^3}\big] = \infty.$$

Therefore to complete the proof, we need to prove that if $\sup_{t\in[0,T)} \|\varrho(t)\|_{L^\infty} < \infty$, then

$$\sup_{t\in[0,T)}\big[\|\varrho(t)\|_{H^3} + \|U(t)\|_{H^3}\big] < \infty. \tag{2.24}$$

In fact, thanks to (2.5) and (2.6), we get by the standard energy estimate that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|A\|_{H^2}^2 + \frac{1}{2+\lambda}\|B\|_{H^2}^2\Big) \\
&\quad+ \sum_{|\alpha|\leqslant 2}\int_{\mathbb{T}^2}\big[\partial^\alpha H(\partial^\alpha B_y - \partial^\alpha A_x) + \partial^\alpha L(\partial^\alpha A_y + \partial^\alpha B_x)\big]\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{|\alpha|\leqslant 2}\Big[-(\partial^\alpha(U\cdot\nabla A), \partial^\alpha A) + \frac{1}{2+\lambda}(\partial^\alpha(U\cdot\nabla B),\partial^\alpha B) \\
&\quad - \frac{1}{2+\lambda}(\partial^\alpha[(B+p(\varrho))A], \partial^\alpha A) \\
&\quad + \Big(\partial^\alpha\Big[\frac{1}{2+\lambda}\varrho p'(\varrho)\operatorname{div} U - (u_x^2 + 2u_y v_x + v_y^2)\Big], \partial^\alpha B\Big)\Big].
\end{aligned} \tag{2.25}$$

Taking into account (2.2) we get

$$\begin{aligned}
&\sum_{|\alpha|\leqslant 2}\int_{\mathbb{T}^2}\big[\partial^\alpha H(\partial^\alpha B_y-\partial^\alpha A_x)+\partial^\alpha L(\partial^\alpha A_y+\partial^\alpha B_x)\big]\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{\mathbb{T}^2}\frac{1}{\varrho}\sum_{|\alpha|\leqslant 2}\big[|\partial^\alpha B_y-\partial^\alpha A_x|^2+|\partial^\alpha A_y+\partial^\alpha B_x|^2\big]\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\sum_{|\alpha|\leqslant 2}\int_{\mathbb{T}^2}\Big\{\Big[\partial^\alpha;\frac{1}{\varrho}\Big](-A_x+B_y)\times(\partial^\alpha B_y-\partial^\alpha A_x)\\
&\quad+\Big[\partial^\alpha;\frac{1}{\varrho}\Big](A_y+B_x)\times(\partial^\alpha A_y+\partial^\alpha B_x)\Big\}\,\mathrm{d}x\,\mathrm{d}y\overset{\text{def}}{=}R_1+R_2.
\end{aligned}$$

However, it is easy to observe that for any $\varepsilon>0$, we have

$$|R_2|\leqslant\varepsilon(\|B_y-A_x\|_{H^2}^2+\|A_y+B_x\|_{H^2}^2)+\frac{C_T}{\varepsilon}(\|\nabla A\|_{H^1}^2+\|\nabla B\|_{H^1}^2)\|\varrho\|_{H^3}^2,$$

from which we deduce that

$$\begin{aligned}
&\sum_{|\alpha|\leqslant 2}\int_{\mathbb{T}^2}\big[\partial^\alpha H(\partial^\alpha B_y-\partial^\alpha A_x)+\partial^\alpha L(\partial^\alpha A_y+\partial^\alpha B_x)\big]\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\geqslant(c_T-\varepsilon)(\|B_y-A_x\|_{H^2}^2+\|A_y+B_x\|_{H^2}^2)-\frac{C_T}{\varepsilon}(\|\nabla A\|_{H^1}^2+\|\nabla B\|_{H^1}^2)\|\varrho\|_{H^3}^2.
\end{aligned}$$

Applying a Moser type inequality gives

$$\begin{aligned}
&\Big|\sum_{|\alpha|\leqslant 2}(\partial^\alpha(U\cdot\nabla A),\partial^\alpha A)\Big|\\
&\qquad=\Big|\sum_{|\alpha|\leqslant 2}\big[(U\cdot\nabla\partial^\alpha A,\partial^\alpha A)+(\partial^\alpha(U\cdot\nabla A)-U\cdot\nabla\partial^\alpha A,\partial^\alpha A)\big]\Big|\\
&\qquad\leqslant C\big[\|\nabla U\|_{L^\infty}\|A\|_{H^2}+\|\nabla A\|_{H^1}\|U\|_{H^s}\big]\|A\|_{H^2},
\end{aligned}$$

and thanks to (2.2), we have

$$\|U\|_{H^s}\leqslant\|U\|_{L^2}+C[\|\varrho\|_{H^{s-1}}+\|A\|_{H^{s-1}}+\|B\|_{H^{s-1}}],\tag{2.26}$$

which gives

$$\Big|\sum_{|\alpha|\leqslant 2}(\partial^\alpha(U\cdot\nabla A),\partial^\alpha A)\Big|\leqslant C(\|\nabla A\|_{H^1}+\|\nabla U\|_{L^\infty})(1+\|\varrho\|_{H^2}^2+\|A\|_{H^2}^2+\|B\|_{H^2}^2).$$

A similar estimate holds for $\sum\limits_{|\alpha|\leqslant 2}(\partial^\alpha(U\cdot\nabla B),\partial^\alpha B)$.

Applying a Moser type inequality again gives

$$\Big|\sum_{|\alpha|\leqslant 2}(\partial^\alpha[(B+p(\varrho))A],\partial^\alpha A)\Big|\leqslant C_T(1+\|A\|_{L^\infty}+\|B\|_{L^\infty})(1+\|\varrho\|_{H^2}^2+\|A\|_{H^2}^2+\|B\|_{H^2}^2)$$

and

$$\begin{aligned}\Big|\sum_{|\alpha|\leqslant 2}\Big(\partial^\alpha\Big[\frac{1}{2+\lambda}\varrho p'(\varrho)\operatorname{div}U-(u_x^2+2u_yv_x+v_y^2)\Big],\partial^\alpha B\Big)\Big|&\\ \leqslant C_T(1+\|\nabla U\|_{L^\infty})(1+\|\varrho\|_{H^2}^2+\|A\|_{H^2}^2+\|B\|_{H^2}^2).&\end{aligned}$$

Then thanks to (2.25) we obtain by taking ε small enough in the above inequalities that

$$\begin{aligned}\text{(2.27)}\qquad &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|A\|_{H^2}^2+\frac{1}{2+\lambda}\|B\|_{H^2}^2\Big)+c_T(\|B_y-A_x\|_{H^2}^2+\|A_y+B_x\|_{H^2}^2)\\ &\leqslant C_T(1+\|\nabla U\|_{L^\infty}+\|A\|_{H^2}^2+\|B\|_{H^2}^2)(1+\|\varrho\|_{H^3}^2+\|A\|_{H^2}^2+\|B\|_{H^2}^2).\end{aligned}$$

On the other hand, rewriting the continuity equation of (1.1) as

$$\partial_t\varrho+U\cdot\nabla\varrho+\frac{A}{2+\lambda}\varrho^{1+\gamma}+\frac{1}{2+\lambda}B\varrho=0,$$

we get by the standard energy estimate that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|_{H^3}^2\leqslant C\big[(1+\|B\|_{L^\infty}+\|\nabla U\|_{L^\infty})\|\varrho\|_{H^3}+\|U\|_{H^3}\|\nabla\varrho\|_{L^\infty}+\|\varrho\|_{L^\infty}\|B\|_{H^3}\big]\|\varrho\|_{H^3}.$$

However, trivial interpolation inequality gives

$$\|U\|_{H^3}\leqslant C\|U\|_{H^1}^{\frac13}\|U\|_{H^4}^{\frac23},\qquad \|\nabla\varrho\|_{L^\infty}\leqslant\|\nabla\varrho\|_{L^2}^{\frac23}\|\varrho\|_{H^3}^{\frac13},$$

which together with (2.26) implies that

$$\|U\|_{H^3}\|\nabla\varrho\|_{L^\infty}\|\varrho\|_{H^3}\leqslant\varepsilon(\|A\|_{H^3}^2+\|B\|_{H^3}^2)+\frac{C}{\varepsilon}(\|U\|_{H^1}+\|\nabla\varrho\|_{L^2})(1+\|\varrho\|_{H^3}^2).$$

Therefore, we get

$$\begin{aligned}\text{(2.28)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|_{H^3}^3\leqslant\varepsilon(\|A\|_{H^3}^2+\|B\|_{H^3}^2)+\frac{C_T}{\varepsilon}(\|U\|_{H^1}+\|B\|_{L^\infty}&\\ +\|\nabla\varrho\|_{L^2}+\|\nabla U\|_{L^\infty})(1+\|\varrho\|_{H^3}^2).&\end{aligned}$$

Finally, thanks to [6] and (2.26) we have

$$
\begin{aligned}
(2.29)\qquad & \|\nabla U\|_{L^\infty} \\
& \leqslant C(1+\|\nabla U\|_{BMO}\times \log^+\|U\|_{H^3}) \\
& \leqslant C_T\bigl[1+(\|L\|_{H^1}+\|H\|_{H^1})\log^+(1+\|\varrho\|_{H^2}+\|A\|_{H^2}+\|B\|_{H^2})\bigr].
\end{aligned}
$$

On the other hand, we deduce from (2.2), (2.15) and (2.19) that $\varrho L, \varrho H \in L^2((0,T);H^1(\mathbb{T}^2))$, which together with (2.2) yields that both ∇A and ∇B are bounded in $L^2((0,T);H^1(\mathbb{T}^2))$. Then combining (2.27)–(2.29), we get

$$
\begin{aligned}
\sup_{0\leqslant t<T}\bigl(\|A(t)\|_{H^2}^2+\|B(t)\|_{H^2}^2+\|\varrho(t)\|_{H^3}^2\bigr)+\int_0^T\bigl(\|B_y-A_x\|_{H^2}^2+\|A_y+B_x\|_{H^2}^2\bigr)\,\mathrm{d}t \\
\leqslant C(m,M,\|\varrho_0\|_{H^3},\|U_0\|_{H^3},T),
\end{aligned}
$$

which together with (2.1) implies that

$$
\sup_{0\leqslant t<T}\bigl(\|\varrho(t)\|_{H^3}^2+\|U(t)\|_{H^3}^2\bigr)+\int_0^T\|\nabla U\|_{H^3}^2\,\mathrm{d}t\leqslant C(m,M,\|\varrho_0\|_{H^3},\|U_0\|_{H^3},T).
$$

This implies (2.24), and we have completed the proof of Theorem 1.1. □

Acknowledgments. The authors would like to thank Professors Ping Zhang and Zhifei Zhang for profitable discussions concerning this topic. We also thank the referee for valuable suggestions.

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