Czechoslovak Mathematical Journal

Zhibo Chen

On integral sum graphs with a saturated vertex

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 3, 669–674

Persistent URL: `http://dml.cz/dmlcz/140597`

Czechoslovak Mathematical Journal, 60 (135) (2010), 669–674

ON INTEGRAL SUM GRAPHS WITH A SATURATED VERTEX

Zhibo Chen, McKeesport

(Received January 9, 2009)

Abstract. As introduced by F. Harary in 1994, a graph G is said to be an *integral sum graph* if its vertices can be given a labeling f with distinct integers so that for any two distinct vertices u and v of G, uv is an edge of G if and only if $f(u)+f(v)=f(w)$ for some vertex w in G.

We prove that every integral sum graph with a saturated vertex, except the complete graph K_3, has edge-chromatic number equal to its maximum degree. (A vertex of a graph G is said to be *saturated* if it is adjacent to every other vertex of G.) Some direct corollaries are also presented.

Keywords: integral sum graph, saturated vertex, edge-chromatic number

MSC 2010: 05C78, 05C15

1. Introduction

In this note we consider only finite graphs with no loops or multiple edges. In general we follow the standard graph-theoretic notation and terminology (see, for example, [1] or [2]).

In 1994, Harary [8] introduced the notion of an *integral sum graph.* The integral sum graph $G^+(S)$ of a finite subset S of integers is the graph (V,E) where $V=S$ and $uv \in E$ if and only if $u+v \in S$. A graph G is said to be an *integral sum graph* if it is isomorphic to the integral sum graph $G^+(S)$ of a finite subset S of integers. In other words, G is an *integral sum graph* if its vertices can be given a labeling f with distinct integers, so that for any two distinct vertices u and v of G, uv is an edge of G if and only if $f(u)+f(v)=f(w)$ for some vertex w in G. (And such a labeling f is then called an *integral sum labeling* of G.) If there is an integral sum labeling f of G with $f(x)>0$ for all vertices x in G, then G is said to be a *sum graph.* In fact, the concept of a sum graph was introduced by Harary [7] earlier in 1990. It is easily seen that none of nontrivial connected graphs is a sum graph.

Many infinite families of connected graphs, however, are known to be integral sum graphs. For example, Harary [8] found that all paths and stars are integral sum graphs. Sharary[14] showed that the cycles C_n and the wheels W_n are also integral sum graphs for all $n \neq 4$. Ellingham [5] proved a conjecture of Harary that the disjoint union of a single vertex K_1 with any tree is a sum graph. For an arbitrarily given graph G, how can we determine whether or not G is an integral sum graph? This is a basic but difficult problem. It has not been solved even for trees. In 1998 we [3] first posted the conjecture that all trees are integral sum graphs. The same conjecture was also raised independently in 2000 by Liao, Guo and Chang [11]. It is still open up to this date, although several classes of trees (see [3], [11], [9], [13]) have been shown to be integral sum graphs. For a survey of known results on sum graphs and integral sum graphs, the reader is referred to the dynamic survey on graph labeling by J. Gallian [6].

To show a graph G is an integral sum graph, we may try to find an integral sum labeling directly, or we may use some undirect methods such as the methods of identification (see [3], [4], [9] and [13]). On the other hand, however, there is no direct way to prove a graph is not an integral sum graph, and few methods have been discovered. This motivated us to study some graphical properties of integral sum graphs in [4]. In the present note we further study the integral sum graphs with a saturated vertex. (As in [4], a vertex of graph G is said to be *saturated* if it is adjacent to every other vertex of G.) We show that every integral sum graph with a saturated vertex, except the complete graph K_3, is of class 1 (i.e., its edge-chromatic number is equal to its maximum degree.) Some corollaries are also presented.

2. Preliminaries

From now on, we use the notation $G^+\{a_1, a_2, \ldots, a_p\}$ to denote an (integral) sum graph with an (integral) sum graph labeling such that the vertices of G are labeled by the integers $a_1, a_2, \ldots, a_p$. It is clear that $G^+\{a_1, a_2, \ldots, a_p\}$ generated by the integers $\{a_1, a_2, \ldots, a_p\}$ is unique up to isomorphism.

Lemma 2.1 [15]. *Let G be a graph with maximum degree Δ and with edge-chromatic number $\Delta + 1$. Then G contains two distinct vertices x, y and a collection of Δ pairwise edge-disjoint paths each joining x, y.*

Note. A graph G satisfying the assumptions of Lemma 2.1 must have at least two vertices of degree Δ.

Lemma 2.2 [4]**.** *Let G be an integral sum graph. Then*

(i) *G has at most two saturated vertices unless $G = K_3$;*

(ii) *$G \cong G^+\{1, 0, -1, -2, \ldots, -p+2\}$ if G has exactly two saturated vertices and $|V(G)| = p$.*

Lemma 2.3. *Let f be an integral sum labeling of a graph G with more than one vertex. Then $f(u) = 0$ for some vertex u of G if and only if G has a saturated vertex.*

Proof. The necessity is an obvious fact. So we only need to prove the sufficiency. Let v be a saturated vertex of G. If $f(v) = 0$, then there is nothing to prove. So we may distinguish two cases depending on wheather $f(v) > 0$ or $f(v) < 0$.

Case 1. $f(v) > 0$. We prove this case by contradiction. Assume that $f(x) \neq 0$ for any vertex x of G. Let $f(w)$ be the largest label among all vertices other than v. If $f(w) < 0$, then $f(v) > f(v) + f(w) > f(w)$ and so $f(v) + f(w) \neq f(x)$ for any vertex x of G. If $f(w) > 0$, then $f(v) + f(w) > f(x)$ for any vertex x of G. Thus, no matter if $f(w)$ is negative or positive, we always see that v is not adjacent to w. This contracts the condition that v is a saturated vertex. Hence, there must be a vertex u of G such that $f(u) = 0$.

Case 2. $f(v) < 0$. Consider the new labeling g of G defined by $g(x) = (-1)f(x)$ for any $x \in V(G)$. It is an obvious fact that g gives an integral sum labeling of G and $g(v) > 0$. Then from case 1, there must be a vertex u of G such that $g(u) = 0$. It follows that $f(u) = -g(u) = 0$, and so the proof is complete. □

Lemma 2.4 [4]**.** *For any sum graph G, the join $K_1 \vee G$ is an integral sum graph.*

Now we are ready to prove our theorem and its corollaries in the next section.

3. Main results

Theorem 3.1. *Every integral sum graph G with a saturated vertex, except the complete graph K_3, has the edge-chromatic number $\chi'(G)$ equal to the maximum degree $\Delta(G)$.*

Proof. Let $G \neq K_3$ be an integral sum graph with a saturated vertex. Clearly, G is a connected simple graph. If G has less than 4 vertices, then G is a path of length 0, 1 or 2. It is then obvious that the edge-chromatic number $\chi'(G)$ is equal to the maximum degree $\Delta(G)$. So, from now on, we may assume that G has at least 4 vertices.

If G has exactly one saturated vertex, then from the note following Lemma 2.1, one can easily see that $\chi'(G) \neq \Delta(G) + 1$. It follows that $\chi'(G)=\Delta(G)$, since the well-known Vizing's Theorem (see, for example, [1]) asserts that the edge-chromatic number of a simple graph equals either the maximal degree or the maximal degree plus one. Then by Lemma 2.2(i), we only need to consider the remaining case that G has exactly two saturated vertices. By Lemma 2.2(ii), we may further assume that $G = G^+\{1, 0, -1, -2, \ldots, -p+3, -p+2\}$. Clearly, G has p vertices. (Note that $p \geqslant 4$ by assumption.) By Vizing's Theorem, we only need to show that there is a proper edge-coloring of G in $\Delta(G)$ colors. We distinguish two cases according to the parity of p.

Case 1. p is even. Clearly, G is a subgraph of the complete graph K_p with $\Delta(G) = p - 1$. It is known (see, for example, p.96 of [1]) that $\chi'(K_p) = p - 1$. Then we can easily get a proper edge-coloring of G in $\Delta(G) = p - 1$ colors, and so $\chi'(G) = \Delta(G)$.

Case 2. p is odd. The proof goes as follows. Let H be the graph obtained from G by deleting the vertex $-p + 2$ and its incident edges $e_0 = (-p + 2, 0)$ and $e_1 = (-p + 2, 1)$. It is clear that the vertex number of H is $p - 1$ which is an even number greater than or equal to 4. Note that H has saturated vertices. Then, by the same argument as in case 1, we can get $\chi'(H) = \Delta(H) = p - 2$. Clearly, G can be obtained from H by adding the vertex $-p+2$ and the two edges $e_0 = (-p + 2, 0)$ and $e_1 = (-p+2, 1)$ to connect the vertex $-p+2$ with exactly the two vertices 0 and 1 in H. Now a $p - 1$ edge-coloring of G can be given as follows: First give the edges of H a proper coloring in $p-2$ colors and color the edges e_0 and e_1 with a new color. Then, by switching the colors of the two edges e_0 and $(0, -p + 3)$, we immediately obtain a proper edge-coloring of G in $\Delta(G)$ colors. Therefore, $\chi'(G) = \Delta(G)$. □

Recall that a simple graph is said to be of class 1 or of class 2 if its edge-chromatic number is respectively equal to or greater than its maximum degree. Then Theorem 3.1 can be restated as follows:

Any integral sum graph $G \neq K_3$ is of class 1 if G has a saturated vertex.

In other words, except for K_3, any integral sum graph of class 2 has no saturated vertices.

By Lemma 2.3, we can easily see that the following Theorem 3.1$'$ is equivalent to Theorem 3.1.

Theorem 3.1$'$. *Let S be a set of integers. The integral sum graph $G^+(S)$ is of class 1 if S contains 0 and $S \neq \{0, -n, n\}$ for any integer n.*

Now we apply Theorem 3.1 to two familiar families of graphs. The wheels W_n with vertex number $n \neq 4$ were shown to be integral sum graphs in [14], and the fans $K_1 \vee P_n$ (obtained by joining K_1 with every vertex of P_n) were also shown to be integral sum graphs in [4]. Note that $W_4 = K_4$ is of class 1. Then, the following is a straightforward consequence of Theorem 3.1.

Corollary 3.2. *The wheels W_n and the fans except K_3 are of class* 1.

Corollary 3.3. *For any sum graph G, the join $K_1 \vee G$ is of class* 1.

Proof. It is trivial if $|V(G)| = 1$. When $|V(G)| > 1$, G is not connected, and so $K_1 \vee G$ has exactly one saturated vertex and $G \neq K_3$. Since $K_1 \vee G$ is an integral sum graph from Lemma 2.4, it is of class 1 by Theorem 3.1. □

Finally, we give a corollary concerning graphs which may have no saturated vertices. From a theorem of Mahmoodian [12] (also, see p. 294 of [10]), we know that the Cartesian product of a finite set of graphs is of class 1 if at least one of the factor graphs is not totally disconnected and of class 1. Then we easily obtain the following result.

Corollary 3.4. *The Cartesian product of a finite set of graphs is of class* 1 *if at least one of the factor graphs is not K_3 but is an integral sum graph with a saturated vertex.*

Acknowledgement. The author would like to thank the referee for his (her) helpful suggestions.

References

[1] *J. A. Bondy and U. S. R. Murty*: Graph Theory with Applications. Macmillan, London, 1976.
[2] *G. Chartrand and L. Lesniak*: Graphs and Digraphs, 2nd ed. Wadsworth., Belmont, 1986.
[3] *Z. Chen*: Integral sum graphs from identification. Discrete Math. *181* (1998), 77–90.
[4] *Z. Chen*: On integral sum graphs. Discrete Math. *306* (2006), 19–25 (It first appeared in The Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications, 11 pp. (electronic), Electron. Notes Discrete Math., 11, Elsevier, Amsterdam, 2002.).
[5] *M. N. Ellingham*: Sum graphs from trees. Ars Combin. *35* (1993), 335–349.
[6] *J. Gallian*: A dynamic survey of graph labeling. Electronic J. Combinatorics *15* (2008).
[7] *F. Harary*: Sum graphs and difference graphs. Congr. Numer. *72* (1990), 101–108.
[8] *F. Harary*: Sum graphs over all the integers. Discrete Math. *124* (1994), 99–105.
[9] *W. He, L. Wang, H. Mi, Y. Shen and X. Yu*: Integral sum graphs from a class of trees. Ars Combinatoria *70* (2004), 197–205.
[10] *W. Imrich and S. Klavar*: Product Graphs. John Wiley & Sons, New York, 2000.
[11] *S.-C. Liaw, D. Kuo and G. Chang*: Integral sum numbers of Graphs. Ars Combin. *54* (2000), 259–268.
[12] *E. S. Mahmoodian*: On edge-colorability of Cartesian products of graphs. Canad. Math. Bull. *24* (1981), 107–108.

[13] *A. V. Pyatkin*: Subdivided trees are integral sum graphs. Discrete Math. *308* (2008), 1749–1750.
[14] *A. Sharary*: Integral sum graphs from complete graphs, cycles and wheels. Arab Gulf Sci. Res. *14–1* (1996), 1–14.
[15] *C. Thomassen*: Hajós conjecture for line graphs. J. Combin. Theory Ser. B *97* (2007), 156–157.

Author's address: Zhibo Chen, Department of Mathematics, Penn State University, Greater Allegheny Campus, McKeesport, PA, 15132, U.S.A.