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THE LAPLACIAN SPECTRAL RADIUS OF GRAPHS

Jianxi Li, Fujian, Wai Chee Shiu, Hong Kong, An Chang, Fujian

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Abstract. The Laplacian spectral radius of a graph is the largest eigenvalue of the associated Laplacian matrix. In this paper, we improve Shi's upper bound for the Laplacian spectral radius of irregular graphs and present some new bounds for the Laplacian spectral radius of some classes of graphs.

Keywords: graph, Laplacian spectral radius, bounds

MSC 2010: 05C50

1. Introduction

Let G be a simple graph with edge set E, vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and corresponding degrees $d_1, d_2, \ldots, d_n$. The maximum and minimum degrees of G are denoted by Δ and δ, respectively. The terms order and size refer to the numbers $n = |V|$ of vertices and $m = |E|$ of edges of G, respectively. For a nonempty subset V_1 of $V(G)$, we use $G[V_1]$ to denote the subgraph of G induced by V_1. The distance of u and v (in G) is the length of the shortest path between u and v, denoted by $d(u, v)$. The diameter of G is the maximum distance over all pairs of vertices. Readers are referred to [1] for undefined terms.

Let $A(G)$ and $D(G) = \operatorname{diag}(d_1, \ldots, d_n)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. Since $A(G)$ is a real symmetric nonnegative matrix, its eigenvalues are all real. We call the largest eigenvalue of $A(G)$ the spectral radius of G, denoted by $\lambda(G)$. When G is connected, the Perron-Frobenius Theorem implies that $\lambda(G)$ is simple and there is a unique positive unit eigenvector (also called the *Perron eigenvector* of $A(G)$). The Laplacian matrix of G is

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$L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Gerschgorin's theorem, it follows that its eigenvalues are nonnegative real numbers. Moreover, since its rows sum to 0, 0 is the smallest eigenvalue of $L(G)$ with the all ones vector as eigenvector. The eigenvalues of $L(G)$ (or the Laplacian eigenvalues of G) are denoted by

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G) = 0.$$

The largest eigenvalue $\mu_1(G)$ is called the *Laplacian spectral radius* of the graph G, denoted by $\mu(G)$. It is known that the multiplicity of 0 as the eigenvalue of $L(G)$ is equal to the number of connected components of G. So a graph G is connected if and only if the second smallest Laplacian eigenvalue $\mu_{n-1}(G)$ (also called the *algebraic connectivity* of G) is strictly greater than 0.

The eigenvalues of the Laplacian matrix are important in graph theory. Because they have relations to numerous graph invariants including connectivity, expanding property, isoperimetric number, maximum cut, independence number, genus, diameter, mean distance, and bandwidth-type parameters of a graph. Especially, the largest and the second smallest eigenvalues of $L(G)$ are probably the most important information contained in the spectrum of a graph (see, for example, [4], [6] and the references therein). Since the sum of the second smallest Laplacian eigenvalue of a graph G and the largest Laplacian eigenvalue of the complement of the graph G is equal to n, it is not surprising at all that the important of one of these eigenvalues implies the importance of the other. In many applications good bounds for the Laplacian spectral radius of G are need (see, for example, [4], [6]).

It is a well-known fact that $\mu(G) \leqslant 2\Delta(G)$ with equality if and only if G is bipartite regular. It is natural to ask how small $2\Delta(G) - \mu(G)$ can be when G is irregular.

Shi [9] gave an upper bound for the Laplacian spectral radius of irregular graphs as follows.

Theorem 1.1 (Shi [9]). *Let G be a connected irregular graph of order n with maximum degree Δ and diameter D. Then $\mu(G) < 2\Delta - 2/(2D+1)n$.*

In this paper, we improve Shi's upper bound for the Laplacian spectral radius of irregular graphs and present some new bounds for the Laplacian spectral radius of some classes of graphs.

2. An improvement on Shi's bound

Let $Q(G) = D(G) + A(G)$, and let $\varrho(Q(G))$ denote the spectral radius of $Q(G)$ (i.e., the largest eigenvalue of $Q(G)$). For a connected graph G, $Q(G)$ is a non-negative (i.e., all entries are non-negative) and irreducible. By Perron-Frobenius Theorem of irreducible non-negative matrices, $\varrho(Q(G))$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\varrho(Q(G))$. We shall refer to such an eigenvector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ as the *Perron eigenvector* of $Q(G)$, where the positive real number x_i corresponds to the vertex v_i $(i = 1, 2, \ldots, n)$. The following description is well known:

$$
\begin{aligned}
\varrho(Q(G)) &= \langle \boldsymbol{x}, Q(G)\boldsymbol{x} \rangle = \sum_{i=1}^{n} d_i x_i^2 + 2 \sum_{v_i v_j \in E(G)} x_i x_j \\
&= \sum_{i=1}^{n} d_i x_i^2 + \sum_{v_i v_j \in E(G)} (x_i^2 + x_j^2) - \sum_{v_i v_j \in E(G)} (x_i - x_j)^2 \\
&= 2\sum_{i=1}^{n} d_i x_i^2 - \sum_{v_i v_j \in E(G)} (x_i - x_j)^2.
\end{aligned} \tag{2.1}
$$

Lemma 2.1 (Zhang and Luo [11]). *Let G be a graph. Then $\mu(G) \leqslant \varrho(Q(G))$, the equality holds if and only if G is a bipartite graph.*

The spectral radius of an r-regular graph is r with $(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n})$ as its Perron eigenvector. Recently, Stevanović [10], Zhang [12], Liu et al. [5], and Cioabă [2] studied the spectral radius of irregular graphs. The current best result is due to Cioabă as follows: For an irregular graph G of order n with maximum degree Δ and diameter D, then $\lambda(G) < \Delta - 1/nD$.

Similarly, for any r-regular graph G, we have $\varrho(Q(G)) = 2r$ and the Perron eigenvector of $Q(G)$ is $(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n})$. Now, we give an upper bound for $\varrho(Q(G))$ analogous to that of the spectral radius, where G is an irregular graph.

Lemma 2.2. *Let G be a connected irregular graph of order n with maximum degree Δ and diameter D. Then $\varrho(Q(G)) < 2\Delta - 1/nD$.*

Proof. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron eigenvector of $Q(G)$. Let v_k be a vertex of G such that $x_k = \max\limits_i \{x_i\}$. Then $x_k > 1/\sqrt{n}$ since G is irregular.

Suppose that $d_k < \Delta$. From $Q(G)\boldsymbol{x} = \varrho(Q(G))\boldsymbol{x}$, we have

$$\varrho(Q(G))x_k = d_k x_k + \sum_{v_j \in N(v_k)} x_j \leqslant 2(\Delta - 1)x_k.$$

This implies $2\Delta - \varrho(Q(G)) \geqslant 2 > 1/nD$. Hence $\varrho(Q(G)) < 2\Delta - 1/nD$.

In the following, we will assume that $d_k = \Delta$. We consider the following two cases:

Case 1. G contains at least two vertices whose degree is less than Δ.

Without loss of generality, we assume that $d(u), d(v) < \Delta$. Let $P\colon\ u = v_{i0}, v_{i1}, \ldots, v_{ir} = v_k$ be a shortest path of length $r \leqslant D$ from u to v_k. Let Q be a shortest path from v to v_k. Let $t = \min\{j\colon\ v_{ij} \in V(P) \cap V(Q)\}$. Then $t \in \{0, 1, \ldots, r\}$. We consider the following two subcases:

Subcase 1.1. $t = 0$.

Then $d(u, v_k) = r \leqslant D - 1$. From (2.1) and by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
2\Delta - \varrho(Q(G)) &= 2\sum_{i=1}^{n}(\Delta - d_i)x_i^2 + \sum_{v_iv_j \in E(G)}(x_i - x_j)^2 \\
&\geqslant 2x_v^2 + 2x_u^2 + \sum_{v_iv_j \in E(G)}(x_i - x_j)^2 \geqslant 2x_v^2 + 2x_u^2 + \sum_{v_iv_j \in E(P)}(x_i - x_j)^2 \\
&\geqslant 2x_v^2 + 2x_u^2 + \frac{(x_u - x_k)^2}{r} \geqslant 2x_v^2 + \frac{2x_k^2}{2r+1} > \frac{2x_k^2}{2D-1} > \frac{x_k^2}{D} > \frac{1}{nD}.
\end{aligned}$$

Hence $\varrho(Q(G)) < 2\Delta - 1/nD$.

Subcase 1.2. $t \geqslant 1$.

Without loss of generality, we assume that $t = d(u, v_{it}) \geqslant d(v, v_{it})$. Let $P_{u,v_{it}}$ (a shortest path from u to v_{it}), P_{v_{it},v_k} (a shortest path from v_{it} to v_k), and $Q_{v,v_{it}}$ (a shortest path from v to v_{it}) denote the sub-path of P and Q, respectively. The following argument is borrowed from [2], from (2.1) and by Cauchy-Schwarz inequality, we have,

$$\begin{aligned}
&2\Delta - \varrho(Q(G)) \\
&\quad = 2\sum_{i=1}^{n}(\Delta - d_i)x_i^2 + \sum_{v_iv_j \in E(G)}(x_i - x_j)^2 \geqslant 2x_u^2 + 2x_v^2 + \sum_{v_iv_j \in E(G)}(x_i - x_j)^2 \\
&\quad \geqslant \Bigg(2x_u^2 + \sum_{v_iv_j \in E(P_{u,v_{it}})}(x_i - x_j)^2\Bigg) + \Bigg(2x_v^2 + \sum_{v_iv_j \in E(Q_{v,v_{it}})}(x_i - x_j)^2\Bigg) \\
&\qquad + \sum_{v_iv_j \in E(P_{v_{it},v_k})}(x_i - x_j)^2 \\
&\quad \geqslant \left(2x_u^2 + \frac{(x_u - x_{it})^2}{t}\right) + \left(2x_v^2 + \frac{(x_v - x_{it})^2}{d(v, v_{it})}\right) + \frac{(x_k - x_{it})^2}{r - t} \\
&\quad \geqslant \frac{2x_{it}^2}{2t+1} + \frac{2x_{it}^2}{2d(v, v_{it}) + 1} + \frac{(x_k - x_{it})^2}{r - t} \geqslant \frac{4x_{it}^2}{2t+1} + \frac{(x_k - x_{it})^2}{r - t},
\end{aligned}$$

where the right-hand side is a quadratic function of x_{it} that attains its minimum when $x_{it} = (2t+1)x_k/(4r - 2t + 1)$. Recall that $t \geqslant 1$, $r \leqslant D$ and $x_k^2 > 1/n$. This implies that,

$$2\Delta - \varrho(Q(G)) \geqslant \frac{4x_k^2}{4r-2t+1} > \frac{x_k^2}{r} > \frac{1}{nD}.$$

Therefore $\varrho(Q(G)) < 2\Delta - 1/nD$.

Case 2. G contains only one vertex whose degree is less than Δ.

Let $x_l = \min_i\{x_i\}$. Then we have the following properties:

(a) $d_l < \Delta$ (since $2\Delta x_l > \varrho(Q(G))x_l = d_l x_l + \sum_{v_j\in N(v_l)} x_j \geqslant 2d_l x_l$).

(b) $d(v_l, v_k) = D$. Otherwise, similarly to Subcase 1.1, we are done.

(c) There exists at least one vertex $v_j \in N(v_k)$ such that $x_j < 1/\sqrt{n}$. Otherwise, there exists $v_j \in N(v_k)$ such that $x_j > 1/\sqrt{n}$ and $d(v_l, v_j) = D-1$, similarly to Subcase 1.1, we are done.

Recall that $x_k = \max_i\{x_i\}$. We consider the following two subcases:

Subcase 2.1. $x_k/x_l \leqslant D$.

Summing the equalities $\varrho(Q(G))x_i = d_i x_i + \sum_{v_j\in N(v_i)} x_j$ over all $i = 1, 2, \ldots, n$, we have $\varrho(Q(G)) \sum_{i=1}^{n} x_i = 2\sum_{i=1}^{n} d_i x_i$. Thus,

$$2\Delta - \varrho(Q(G)) = \frac{2\sum_{i=1}^{n}(\Delta - d_i)x_i}{\sum_{i=1}^{n} x_i} = \frac{2(\Delta - d_l)x_l}{\sum_{i=1}^{n} x_i} > \frac{2x_l}{nx_k} \geqslant \frac{2}{nD} > \frac{1}{nD}.$$

Hence $\varrho(Q(G)) < 2\Delta - 1/nD$.

Subcase 2.2. $x_k/x_l > D$.

By property (c), there exists a vertex $v_j \in N(v_k)$ such that $x_j < 1/\sqrt{n}$, we have

$$\varrho(Q(G))x_k = d_k x_k + \sum_{v_j\in N(v_k)} x_j < (2\Delta - 1)x_k + \frac{1}{\sqrt{n}}.$$

This implies

$$2\Delta - \varrho(Q(G)) > 1 - \frac{1}{x_k\sqrt{n}}.$$

If $1 - 1/x_k\sqrt{n} \geqslant 1/nD$, we are done. Next, we consider $1 - 1/x_k\sqrt{n} < 1/nD$, i.e.,

$$x_k < \frac{D\sqrt{n}}{nD-1}. \tag{2.2}$$

Since $(n-1)x_k^2 + x_l^2 \geqslant \sum_{i=1}^{n} x_i^2 = 1$ and (2.2), we have

$$x_l^2 \geqslant 1 - (n-1)x_k^2 \geqslant 1 - \frac{(n-1)nD^2}{(nD-1)^2} = \frac{(D^2-2D)n+1}{(n-1)nD^2}. \tag{2.3}$$

If $D \geqslant 3$, combining (2.2) and (2.3) we have

$$\frac{x_k^2}{x_l^2} < \frac{(n-1)n^2D^4}{((D^2-2D)n+1)(nD-1)^2} < \frac{D^2n}{(D^2-2D)n+1} < D^2.$$

Thus, $x_k/x_l < D$ which is a contradiction with $x_k/x_l > D$. Thus $D = 2$. Consider the matrix $Q^2(G) = [D(G)+A(G)]^2$, since there is at least one path of length 2 from v_l to v_k and $x_k/x_l > 2$, we have

$$\varrho^2(Q(G))x_k \leqslant 3\Delta^2 x_k + (\Delta^2-1)x_k + x_l.$$

This implies

$$\varrho^2(Q(G)) \leqslant 4\Delta^2 - 1 + \frac{x_l}{x_k} < 4\Delta^2 - \frac{1}{2}.$$

Thus, $\varrho(Q(G)) \leqslant \sqrt{4\Delta^2 - \frac{1}{2}} < 2\Delta - 1/4\Delta$.
If $n \geqslant 2\Delta$, we are done.
If $n < 2\Delta$, then there are at least two paths of length 2 from v_l to v_k. We have

$$\varrho^2(Q(G))x_k \leqslant 3\Delta^2 x_k + (\Delta^2-2)x_k + 2x_l.$$

This implies

$$\varrho^2(Q(G)) \leqslant 4\Delta^2 - 2 + \frac{2x_l}{x_k} < 4\Delta^2 - 1.$$

Thus, $\varrho(Q(G)) \leqslant \sqrt{4\Delta^2-1} < 2\Delta - 1/2\Delta < 2\Delta - 1/nD$.
From above discussions, the proof is completed. □

By Lemmas 2.1 and 2.2, the following result is immediate.

Theorem 2.3. *Let G be a connected irregular graph of order n with maximum degree Δ and diameter D. Then $\mu(G) < 2\Delta - 1/nD$.*

Remark 2.1. Since $2/((2D+1)n) < 1/nD$, the upper bound in Theorem 2.3 slightly improves Shi's upper bound.

Remark 2.2. The upper bound in Theorem 2.3 is asymptotically best possible when $\Delta - \delta = 1$. That is shown by the following example. Let $G_1, G_2, \ldots, G_k$ be k disjoint copies of the complete bipartite graph $K_{\Delta,\Delta}$. Remove an edge, say $v_{2i-1}v_{2i}$ from each G_i, and join v_{2i} to v_{2i+1} for each $i = 1, \ldots, k-1$. Let $G_{\Delta,k}$ denote the resulting chain of bipartite graphs (this is illustrated in Fig. 1). Clearly, $G_{\Delta,k}$ is a bipartite graph of order $n = 2k\Delta$ with maximum degree Δ, minimum degree $\delta = \Delta - 1$ and diameter $D = 4k-1$.

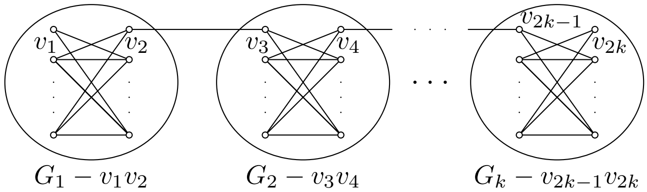

Figure 1. Bipartite graph: $G_{\Delta,k}$.

Let v_1, v_{2k} be the two vertices of degree $\delta = \Delta - 1$. For each unit vector $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)$, we have $\varrho(Q(G_{\Delta,k})) \geqslant \boldsymbol{z}^T Q(G_{\Delta,k})\boldsymbol{z}$. Since $G_{\Delta,k}$ is a bipartite graph, by Lemma 2.1 and (2.1), we have

$$\begin{aligned}2\Delta - \mu(G_{\Delta,k}) &= 2\Delta - \varrho(Q(G_{\Delta,k})) \\ &\leqslant 2\sum_{i=1}^{n}(\Delta - d_i)z_i^2 + \sum_{v_iv_j \in E(G_{\Delta,k})}(z_i - z_j)^2 \\ &= 2(z_1^2 + z_{2k}^2) + \sum_{v_iv_j \in E(G_{\Delta,k})}(z_i - z_j)^2.\end{aligned} \tag{2.4}$$

Let $\boldsymbol{y}$ be the Perron eigenvector corresponding to the greatest eigenvalue $\varrho(Q(P_k)) = 2 + 2\cos \pi/(k+1)$ of $Q(P_k)$. By (2.1) and Lemma 2.1, we have

$$\begin{aligned}&2(y_1^2 + y_k^2) + \sum_{i=1}^{k-1}(y_i - y_{i+1})^2 \\ &\quad = 4 - \varrho(Q(P_k)) = 4 - \mu(P_k) = 2 - 2\cos\frac{\pi}{k+1} < \frac{\pi^2}{(k+1)^2}.\end{aligned} \tag{2.5}$$

For $v \in V(G_{\Delta,k})$, let $z_v = y_i/\sqrt{2\Delta}$ if $v \in V(G_i)$. Then $\boldsymbol{z}^T\boldsymbol{z} = 1$. Substituting $\boldsymbol{z}$ into (2.5), we have

$$\begin{aligned}2\Delta - \mu(G_{\Delta,k}) &= 2\Delta - \varrho(Q(G_{\Delta,k})) \\ &\leqslant 2\frac{y_1^2 + y_k^2}{2\Delta} + \sum_{i=1}^{k-1}\Big(\frac{y_i - y_{i+1}}{\sqrt{2\Delta}}\Big)^2 < \frac{\pi^2}{2\Delta(k+1)^2} < \frac{4\pi^2}{nD}.\end{aligned}$$

Hence, $\mu(G_{\Delta,k}) > 2\Delta - 4\pi^2/nD$.

Let G be an r-regular bipartite graph. For each edge $uv \in E(G)$, let $H = G - uv$. Then H is a connected bipartite graph with maximum degree r. Since $\mu(G) = 2r$, by Theorem 2.3, the following corollary is obvious.

Corollary 2.4. *Let G be a bipartite regular graph of order n with diameter D. If $uv \in E(G)$, then*

$$\mu(G) - \mu(G - uv) > \frac{1}{nD}.$$

3. Bounds for some classes of graphs

In this section, we present some new bounds for the Laplacian spectral radius of some special classes of graphs.

3.1 Triangle-free graphs

Zhang and Luo [11] gave two lower bounds for the Laplacian spectral radius of triangle-free graphs. One is in terms of the numbers of edges and vertices of graphs, and another one is in terms of degrees and average 2-degrees of vertices. In the sequel, we will give a sharp upper bound for the Laplacian spectral radius of triangle-free graphs in terms of the maximum degree and the number of edges of graphs.

Lemma 3.1 (Motzkin and Straus [7]). *Let $\mathcal{F} = \Big\{(x_1, x_2, \ldots, x_n)\colon x_i \geqslant 0, \sum\limits_{i=1}^{n} x_i = 1\Big\}$. Then*

$$1 - \frac{1}{\omega(G)} = \max_{\boldsymbol{x} \in \mathcal{F}} \langle \boldsymbol{x}, A(G)\boldsymbol{x} \rangle,$$

where $\omega(G)$ (clique number of G) is the number of vertices of the largest clique set in G.

Theorem 3.2. *Let G be a triangle-free graph with m edges and maximum degree Δ. Then*

$$\mu(G) \leqslant \Delta + \sqrt{m}. \tag{3.1}$$

P r o o f. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be the Perron eigenvector of $Q(G)$. Then

$$\begin{aligned} \varrho(Q(G)) &= \langle \boldsymbol{x}, Q(G)\boldsymbol{x} \rangle \\ &= \sum_{i=1}^{n} d_i x_i^2 + 2 \sum_{v_i v_j \in E(G)} x_i x_j \leqslant \Delta + 2 \sum_{v_i v_j \in E(G)} x_i x_j. \end{aligned} \tag{3.2}$$

Moreover, by the Cauchy-Schwarz inequality, we have

$$\biggl(\sum_{v_i v_j \in E(G)} 2x_i x_j\biggr)^2 \leqslant 2m\biggl(2 \sum_{v_i v_j \in E(G)} x_i^2 x_j^2\biggr). \tag{3.3}$$

Since $(x_1^2, x_2^2, \ldots, x_n^2) > 0$, $\sum\limits_{i=1}^{n} x_i^2 = 1$ and $\omega(G) = 2$ for any connected triangle-free graph G, by Lemmas 2.1 and 3.1, combining (3.2) and (3.3), we have

$$\mu(G) \leqslant \varrho(Q(G)) \leqslant \Delta + \sqrt{2m\Big(1 - \frac{1}{\omega(G)}\Big)} = \Delta + \sqrt{m}.$$

The proof is completed. □

It is obvious that our upper bound is sharp, since when $G \cong K_{\Delta,\Delta}$, the upper bound in Theorem 3.2 is attained.

By Turán's theorem, the number of edges for any triangle-free graph is at most $n^2/4$. Hence, we have

Corollary 3.3. *Let G be a triangle-free graph of order n with maximum degree Δ. Then $\mu(G) \leqslant \Delta + n/2$.*

3.2. Bipartite graphs

Consider two sequences of real numbers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_m$ with $m < n$. The second sequence is said to interlace the first one whenever $\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i}$ for $i = 1, 2, \ldots, m$. The interlacing is called tight if there exists an integer $k \in [1, m]$ such that $\lambda_i = \mu_i$ hold for $1 \leqslant i \leqslant k$ and $\lambda_{n-m+i} = \mu_i$ hold for $k + 1 \leqslant i \leqslant m$. Suppose the rows and columns of

$$A_{n\times n} = \begin{pmatrix} A_{11} & \ldots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \ldots & A_{mm} \end{pmatrix}$$

are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The quotient matrix is a matrix $\tilde{B}_{m\times m}$ whose entries are the average row sums of the blocks of $A_{n\times n}$. The partition is called regular if each block $A_{i,j}$ of A has constant row (and column) sum.

Lemma 3.4 (Haemers [3]). *Suppose $\tilde{B}$ is the quotient matrix of a symmetric partitioned matrix A, then the eigenvalues of $\tilde{B}$ interlace the eigenvalues of A. If the interlacing is tight, then the partition is regular.*

For any bipartite graph G of order n with m edges, let $Q(G)$ be partitioned according to a partitioning $X_1 = \{v_1, v_2, \ldots, v_n\}$. Then by Lemma 3.4, we have $\varrho(Q(G)) \geqslant 4m/n$, with equality if and only if the graph is regular bipartite. Moreover, by Lemma 2.1, we have $\mu(G) = \varrho(Q(G)) \geqslant 4m/n$, with equality if and only if

G is bipartite regular. Furthermore, for any $v \in V(G)$, the row-sum of $Q^2(G)$ corresponding to v is $S_v(Q^2(G)) = 2d^2(v) + 2 \sum\limits_{u \in N(v)} d(u)$; apply Lemma 3.4 to $Q^2(G)$ and $\sum\limits_{v \in V} S_v(Q^2(G)) = 4 \sum\limits_{v \in V} d^2(v)$, we have $\varrho(Q(G)) \geqslant \left(4 \sum\limits_{i=1}^{n} d_i^2/n\right)^{1/2}$. Similarly to the proof of Theorem 2 in [8], we can find a lower bound for $\mu(G) - 4m/n$ when G is irregular bipartite.

Theorem 3.5. *Let G be an irregular bipartite graph of order n with m edges. Then*

$$\mu(G) - \frac{4m}{n} \geqslant \frac{1}{m+n}.$$

Theorem 3.6. *Let $G = (X \cup Y, E)$ be a connected bipartite graph, $|X| = a$, $|Y| = b$, $|E| = m$. Then $\mu(G) \geqslant m(1/a + 1/b)$. The equality holds if and only if G is a semiregular bipartite graph.*

Proof. The X, Y give rise to a partition of $L(G)$ with quotient matrix $M = \begin{pmatrix} m/a & -m/a \\ -m/b & m/b \end{pmatrix}$. Then M has two eigenvalues $\alpha = m(1/a + 1/b)$, $\beta = 0$. By Lemma 3.4, $\mu(G) \geqslant \alpha = m(1/a + 1/b)$. If equality holds, which implies that the interlacing is tight, then G is a semiregular bipartite graph.

Shi [9] gave a sharp upper bound on the Laplacian spectral radius of a bipartite graph as follows.

Theorem 3.7 (Shi [9]). *Let $G = (X \cup Y, E)$ be a bipartite graph of order n with m edges. Then*

$$\mu(G) \leqslant \{\Delta + \delta - 1 + [(\Delta + \delta - 1)^2 + 8(2m - \delta n + \delta)]^{1/2}\}/2, \tag{3.4}$$

where Δ and δ are the maximum and minimum degrees of G, respectively.

Moreover, the equality holds if and only if G is regular.

Let $G = (X \cup Y, E)$ be a bipartite graph with vertex set X and Y, where $X \cap Y = \emptyset$, $|X| = a$, $|Y| = b$, $|E| = m$. Let $\delta_X = \min\limits_{v \in X} d(v)$, $\Delta_X = \max\limits_{v \in X} d(v)$, $\delta_Y = \min\limits_{v \in Y} d(v)$, $\Delta_Y = \max\limits_{v \in Y} d(v)$. In the sequel, we give a new sharp upper bound for the Laplacian spectral radius of a bipartite graph.

Lemma 3.8 (Shi [9]). *Let M be a real symmetric $n \times n$ matrix, and let ϱ be an eigenvalue of M with an eigenvector $\boldsymbol{x}$ all of whose entries are nonnegative. Let P be any polynomial. Then*

$$\min_{1\leqslant i\leqslant n} S_i(P(M)) \leqslant P(\varrho) \leqslant \max_{1\leqslant i\leqslant n} S_i(P(M)).$$

Moreover, if all entries of $\boldsymbol{x}$ are positive then either of the equalities holds if and only if the row sums of $P(M)$ are all equal.

Theorem 3.9. *Let $G = (X \cup Y, E)$ be a connected bipartite graph, $|X| = a$, $|Y| = b$, $|E| = m$. Then*

$$\mu(G) \leqslant \max\{\theta_1, \theta_2\}, \tag{3.5}$$

where

$$\theta_1 = \frac{1}{2}(\Delta_X + \delta_Y + \sqrt{(\Delta_X + \delta_Y)^2 + 8(m - b\delta_Y)}),$$
$$\theta_2 = \frac{1}{2}(\Delta_Y + \delta_X + \sqrt{(\Delta_Y + \delta_X)^2 + 8(m - a\delta_X)}).$$

Moreover, the equality holds if and only if G is a semiregular graph.

Proof. We consider the matrix $Q^2(G) = [D(G) + A(G)]^2$, the row-sums of $Q^2(G)$ corresponding to the each vertex v satisfy

$$\begin{aligned} S_v(Q^2(G)) &= 2d^2(v) + 2\sum_{u\in N(v)} d(u) \\ &= \begin{cases} 2d^2(v) + 2\Big(m - \sum\limits_{u\in Y\setminus N(v)} d(u)\Big), & \text{if } v \in X \\ 2d^2(v) + 2\Big(m - \sum\limits_{u\in X\setminus N(v)} d(u)\Big), & \text{if } v \in Y \end{cases} \\ &\leqslant \begin{cases} 2d(v)\Delta_X + 2[m - (b - d(v))\delta_Y], & \text{if } v \in X \\ 2d(v)\Delta_Y + 2[m - (a - d(v))\delta_X], & \text{if } v \in Y. \end{cases} \end{aligned}$$

Hence for each $v \in V(G)$, we have

$$\begin{cases} S_v(Q^2(G) - (\Delta_X + \delta_Y)Q(G)) \leqslant 2(m - b\delta_Y), & \text{for any } v \in X; \\ S_v(Q^2(G) - (\Delta_Y + \delta_X)Q(G)) \leqslant 2(m - a\delta_X), & \text{for any } v \in Y. \end{cases}$$

This by Lemma 3.8 with $\mu(G) = \varrho(Q(G))$ implies that

$$\mu^2(G) - (\Delta_Y + \delta_X)\mu(G) \leqslant 2(m - b\delta_Y);$$
$$\mu^2(G) - (\Delta_X + \delta_Y)\mu(G) \leqslant 2(m - a\delta_X).$$

Solving these two quadratic inequalities, we obtain,

$$\mu(G) \leqslant \frac{1}{2}(\Delta_X + \delta_Y + \sqrt{(\Delta_X + \delta_Y)^2 + 8(m - b\delta_Y)}),$$
$$\mu(G) \leqslant \frac{1}{2}(\Delta_Y + \delta_X + \sqrt{(\Delta_Y + \delta_X)^2 + 8(m - a\delta_X)}).$$

Let

$$\theta_1 = \frac{1}{2}(\Delta_X + \delta_Y + \sqrt{(\Delta_X + \delta_Y)^2 + 8(m - b\delta_Y)}),$$
$$\theta_2 = \frac{1}{2}(\Delta_Y + \delta_X + \sqrt{(\Delta_Y + \delta_X)^2 + 8(m - a\delta_X)}).$$

We obtain $\mu(G) \leqslant \max\{\theta_1, \theta_2\}$.

In the case the equality holds in (3.5), all inequalities in the above argument must be equalities. In particular, $d(v) = \Delta_X$, for each $v \in X$; $d(v) = \Delta_Y$, for each $v \in Y$. And, $d(u) = \delta_Y$, $u \in Y \setminus N(v)$, for each $v \in X$; $d(u) = \delta_X$, $u \in X \setminus N(v)$, for each $v \in Y$. Combining that, we have equality holds if and only if $\Delta_X = \delta_X = d(v)$, for each $v \in X$ and $\Delta_Y = \delta_Y = d(v)$, for each $v \in Y$. Therefore, G is a semiregular graph. The proof is completed. $\square$

4. Conclusion

In the conclusion of this paper, we give one example to illustrate our main results.

Example. Let G_1, G_2 and G_3 be graphs of order 5, 7 and 8 respectively, as follows:

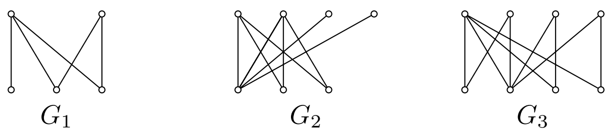

Figure 2. Bipartite graphs: G_1, G_2 and G_3.

The Laplacian spectral radius of graphs G_1, G_2 and G_3 and their upper bounds are as follows, respectively.

	$\mu(G)$	bound in (3.6)	bound in (3.9)	bound in (3.10)
G_1	4.48	5.24	5.27	4.83
G_2	5.68	6.83	6.90	6.27
G_3	6	7.16	6.77	6.61

Here we see that the bounds in (3.6), (3.10) are better than the bound in (3.9) in some cases, but, in general cases, they are incomparable.

References

[1] *J. Bondy, U. Murty*: Graph Theory with Applications. MacMillan, New York, 1976.

[2] *S. Cioabă*: The spectral radius and the maximum degree of irregular graphs. Electron. J. Combin. *14,* Research paper *R38* (2007).

[3] *W. Haemers*: Interlacing eigenvalues and graphs. Linear Algebra Appl. *226–228* (1995), 593–616.

[4] *R. Merris*: Laplacian matrix of graphs: a survey. Linear Algebra Appl. *197–198* (1994), 143–176.

[5] *B. Liu, J. Shen, X. Wang*: On the largest eigenvalue of non-regular graphs. J. Combin. Theory Ser. B. *97* (2007), 1010–1018.

[6] *B. Mohar*: Some applications of Laplace eigenvalues of graphs. Notes taken by Martin Juvan. Graph Symmetry: Algebraic Methods and Applications. NATO ASI Ser., Ser. C, Math. Phys. Sci. Vol. 497 (G. Hahn et al., eds.). Kluwer Academic Publishers, Dordrecht, 1997, pp. 225–275.

[7] *T. Motzkin, E. Straus*: Maxima for graphs and a new proof of a theorem of Turán. Canad. J. Math. *17* (1965), 533–540.

[8] *V. Nikiforov*: Bounds on graph eigenvalues II. Linear Algebra Appl. *427* (2007), 183–189.

[9] *L. Shi*: Bounds on the (Laplacian) spectral radius of graphs. Linear Algebra. Appl. *422* (2007), 755–770.

[10] *D. Stevanović*: The largest eigenvalue of nonregular graphs. J. Combin. Theory Ser. B. *91* (2004), 143–146.

[11] *X.-D. Zhang, R. Luo*: The spectral radius of triangle-free graphs. Australas. J. Comb. *26* (2002), 33–39.

[12] *X.-D. Zhang*: Eigenvector and eigenvalues of non-regular graphs. Linear Algebra Appl. *409* (2005), 79–86.

Authors' addresses: J. Li, Department of Mathematics and Information Science, Zhangzhou Normal University, Fujian, P. R. China, e-mail: `fzjxli@tom.com`; W. C. Shiu, Department of Mathematics, Hong Kong Baptist University, Hong Kong, P. R. China, e-mail: `wcshiu@hkbu.edu.hk`; A. Chang, Software College/Center of Discrete Mathematics, Fuzhou University, Fujian, 350002, P. R. China, e-mail: `anchang@fzu.edu.cn`.