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COMPATIBLE MAPPINGS OF TYPE (β) AND WEAK COMPATIBILITY IN FUZZY METRIC SPACES

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Abstract. The object of this paper is to establish a unique common fixed point theorem for six self-mappings satisfying a generalized contractive condition through compatibility of type (β) and weak compatibility in a fuzzy metric space. It significantly generalizes the result of Singh and Jain [The Journal of Fuzzy Mathematics (2006)] and Sharma [Fuzzy Sets and Systems (2002)]. An example has been constructed in support of our main result. All the results presented in this paper are new.

Keywords: fuzzy metric space, common fixed points, t-norm, compatible maps of type (β), compatible maps of type (α), weak compatible maps

MSC 2010: 54H25, 47H10

1. Introduction

Zadeh [14] initiated the concept of fuzzy sets in 1965. Many authors used this concept in Topology and Analysis and developed the theory of fuzzy sets and its applications. Kramosil and Michalek [7] introduced the concept of fuzzy metric spaces. George and Veeramani [3] modified this concept and defined a Hausdorff topology on fuzzy metric spaces. Grabiec [4] obtained the fuzzy version of the Banach contraction principle, which has been milestone in developing the fixed point theory in fuzzy metric spaces.

Sessa [9] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [5] soon enlarged this concept to compatible maps. The concepts of R-weakly commuting maps and compatible maps in fuzzy metric spaces have been introduced by Vasuki [13] and Mishra et al [8] respectively. Cho [1] introduced the concept of compatible maps of type (α) and compatible maps of type (β). In [2] Cho et al proved a common fixed point theorem for two pairs of (β)-compatible maps assuming continuity of the extreme maps of the two pairs. In [10] Sharma generalized this result to six self-maps by considering the (β)-compatibility of both pairs through continuity of extreme maps of the two (β)-compatible pairs. In [6] Jungck and Rhoades termed a pair of self-maps to be coincidentally commuting or equivalently weak compatible if they commute at their coincidence points. This concept is the most general among all commutativity concepts in this field as every pair of commuting maps or of compatible maps of type (β) is weak compatible but the reverse is not true as shown in Example 2.11

In this paper we establish the existence of a unique common fixed point of six self-maps through weak compatibility and compatibility type (β) satisfying a more general contraction than that adopted in [10] by assuming the continuity of only one map. Moreover, the (α)-compatibility of two pairs of [10] has been reduced to that of only one pair in Corollary 3.3. Also Example 3.9 of this paper shows that unlike [10] the unique common fixed point exists even when both extreme maps AB and ST of (β)-compatible pairs are not continuous while obeying a more general (p, q, a)-contraction, which motivated the authors to establish the main result of this paper.

2. Preliminaries

Definition 2.1. A binary operation $*\colon\ [0,1]\times[0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a*b \leqslant c*d$ whenever $a \leqslant c$ and $b \leqslant d$ for all a, b, c and $d \in [0,1]$.

Examples of the t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2 (Kramosil and Michalek [7])**.** A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) = 0$;
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;
(FM-3) $M(x, y, t) = M(y, x, t)$;
(FM-4) $M(x, y, t) * M(y, z, t) \leqslant M(x, z, t + s)$;
(FM-5) $M(x, y, \cdot)\colon\ [0, \infty) \to [0, 1]$ is left continuous for all $x, y \in X$ and $s, t > 0$.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

E x a m p l e 2.3 (George and Veeramani [3]). Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ for all $x, y \in X$, $M(x, y, t) = t/(t + d(x, y))$ for all $t > 0$ and $M(x, y, 0) = 0$.

Then $(X, M, *)$ is a fuzzy metric space. It is called the fuzzy metric space induced by the metric d.

Lemma 2.4 (Grabiec [4]). *For all $x, y \in X, M(x, y, \cdot)$ is a non-decreasing function.*

Definition 2.5 (Grabiec [4]). Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$. The space is said to be complete if every Cauchy sequence in it converges to a point of it.

R e m a r k 2.6. Since $*$ is continuous, it follows from (FM-4) that the limit of a sequence in a fuzzy metric space is unique, if it exists.

In this paper the fuzzy metric space $(X, M, *)$ is assumed to satisfy the condition
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

Definition 2.7. A pair (A, S) of self-mappings of a fuzzy metric space is said to be compatible type (β) if

$$\lim_{n\to\infty} M(A^2x_n, S^2x_n, t) = 1, \quad \forall t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Definition 2.8. A pair (A, S) of self-mappings of a fuzzy metric space is said to be compatible type (α) if

$$\lim_{n\to\infty} M(ASx_n, S^2x_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} M(SAx_n, A^2x_n, t) = 1, \quad \forall t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Definition 2.9. A pair (A, S) of self-mappings of a fuzzy metric space is said to be weak compatible or coincidentally commuting if A and S commute at their coincidence points, i.e. for $x \in X$ if $Ax = Sx$ then $ASx = SAx$.

R e m a r k 2.10. If self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible type (β) or compatible type (α) then they are weak compatible.

The converse is not true as seen in example below.

E x a m p l e 2.11. Let $(X, M, *)$ be a fuzzy metric space where $X = [0, 2]$, with the t-norm defined by $a * b = \min\{a, b\}$, for $a, b \in [0, 1]$ and $M(x, y, t) = t/(t + d(x, y))$, for $t > 0$ and $M(x, y, 0) = 0$, for $x, y \in X$. Define self maps A and S on X as follows:

$$\begin{aligned} &Ax = 2 \ \text{ if } \ 0 \leqslant x \leqslant 1 \text{ and } Sx = 2 \text{ if } x = 1; \\ &Ax = \tfrac{1}{2}x \text{ if } 1 < x \leqslant 2 \text{ and } Sx = \tfrac{1}{5}(x + 3) \ \text{ otherwise.} \end{aligned}$$

Taking $x_n = 2 - \frac{1}{2n}$ we have $S(1) = A(1) = 2$ and $S(2) = A(2) = 1$. Also $SA(1) = AS(1) = 1$ and $SA(2) = AS(2) = 2$. Thus (A, S) is weak compatible. Again, $Ax_n = 1 - \frac{1}{4n}$ and $Sx_n = 1 - \frac{1}{10n}$. Thus, $Ax_n \to 1$ and $Sx_n \to 1$. Hence $u = 1$. Also

$$\lim_{n\to\infty} M(A^2x_n, S^2x_n, t) = \lim_{n\to\infty} M(2, \tfrac{2}{5} - \tfrac{1}{50n}, t) = t/(t + \tfrac{8}{5}) < 1, \ \forall t > 0.$$

Hence (A, S) is not compatible type (β).

Lemma 2.12 (Cho [1])**.** *Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition* (FM-6)*. If there exists a number $k \in (0, 1)$ such that $M(y_{n+2}, y_{n+1}, kt) \geqslant M(y_{n+1}, y_n, t)$ for all $t > 0$, then $\{y_n\}$ is a Cauchy sequence in X.*

Lemma 2.13 (Mishra et al [8])**.** *If for all $x, y \in X$ and $0 < k < 1$*

$$M(x, y, kt) \geqslant M(x, y, t) \ \text{ for all } t > 0,$$

then $x = y$.

Proposition (Cho [2, Prop. 3.1])**.** *Let $(X, M, *)$ be a fuzzy metric space with $t * t \geqslant t$, $\forall t \in [0, 1]$ and let A and S be compatible maps type (α). If one of A or S is continuous, then A and S are compatible type (β).*

R e m a r k 2.14. It is easy to see that the above result also holds in a fuzzy metric space equipped with the general t-norm $*$.

3. Main results

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geqslant t$ for all $t \in [0, 1]$. Let P, Q, S, T, A and B be self-mappings from X into itself satisfying*

(3.11) *$P(X) \subseteq ST(X)$ and $Q(X) \subseteq AB(X)$;*

(3.12) *there exists a constant* $k \in (0,1)$ *such that*

$$\begin{aligned} M(Px, Qy, kt) + a[M(ABx, STy, kt) * M(Px, Qy, kt)] \\ \geqslant p[M(Px, ABx, kt) * M(Qy, STy, kt)] \\ + q[M(Px, STy, kt) * M(Qy, ABx, kt)] \\ + M(ABx, STy, t) * M(Px, ABx, t) * M(Qy, STy, t) \\ * M(Px, STy, \alpha t) * M(Qy, ABx, (2-\alpha)t), \end{aligned}$$

$\forall\, x, y \in X$, $\forall t > 0$ *and* $\forall \alpha \in (0,2)$ *where* $q \geqslant 0$, $a \leqslant p$;

(3.13) *the pair* (P, AB) *is compatible type* (β) *and the pair* (Q, ST) *is weak compatible;*

(3.14) *either* P *or else* AB *is continuous. Then the four self-maps* P, Q, ST *and* AB *have a unique common fixed point* u *in* X. *Further, if*

(3.15) $AB = BA$, $ST = TS$, $PB = BP$, $QT = TQ$ *at a point* u, *then* u *is the unique common fixed point of the six self-maps* P, Q, S, T, A *and* B *in* X.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Construct sequences $\{x_n\}$ and $\{z_n\}$ in X such that

$$(1) \qquad Px_{2n} = STx_{2n+1} = z_{2n+1} \text{ and } Qx_{2n+1} = ABx_{2n+2} = z_{2n+2}, \ \forall n.$$

Step 1. Taking $x = x_{2n}$, $y = x_{2n+1}$ in (3.12) we get

$$\begin{aligned} &M(Px_{2n}, Qx_{2n+1}, kt) + a[M(ABx_{2n}, STx_{2n+1}, kt) * M(Px_{2n}, Qx_{2n+1}, kt)] \\ &\geqslant p[M(Px_{2n}, ABx_{2n}, kt) * M(Qx_{2n+1}, STx_{2n+1}, kt)] \\ &\quad + q[M(Px_{2n}, STx_{2n+1}, kt) * M(Qx_{2n+1}, ABx_{2n}, kt)] \\ &\quad + \{M(ABx_{2n}, STx_{2n+1}, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qx_{2n+1}, STx_{2n+1}, t) \\ &\quad * M(Px_{2n}, STx_{2n+1}, \alpha t) * M(Qx_{2n+1}, ABx_{2n}, (2-\alpha)t)\}. \end{aligned}$$

Using (1), for $\alpha = 1 - \beta$ with $\beta \in (0,1)$ we have

$$\begin{aligned} &M(z_{2n+1}, z_{2n+2}, kt) + a[M(z_{2n}, z_{2n+1}, kt) * M(z_{2n+1}, z_{2n+2}, kt)] \\ &\geqslant p[M(z_{2n+1}, z_{2n}, kt) * M(z_{2n+2}, z_{2n+1}, kt)] \\ &\quad + q[M(z_{2n+1}, z_{2n+1}, kt) * M(z_{2n+2}, z_{2n}, kt)] \\ &\quad + \{M(z_{2n}, z_{2n+1}, t) * M(z_{2n+1}, z_{2n}, t) * M(z_{2n+2}, z_{2n+1}, t) \\ &\quad * M(z_{2n+1}, z_{2n+1}, (1-\beta)t) * M(z_{2n+2}, z_{2n}, (1+\beta)t) \end{aligned}$$

and

$$\begin{aligned}&M(z_{2n+1}, z_{2n+2}, kt) + a[M(z_{2n}, z_{2n+1}, kt) * M(z_{2n+1}, z_{2n+2}, kt)]\\&\geqslant p[M(z_{2n+1}, z_{2n}, kt) * M(z_{2n+2}, z_{2n+1}, kt)] + qM(z_{2n}, z_{2n+2}, kt)\\&\quad + M(z_{2n}, z_{2n+1}, t) * M(z_{2n+2}, z_{2n+1}, t) * 1 * M(z_{2n+2}, z_{2n+1}, \beta t).\end{aligned}$$

Taking the limit $\beta \to 1$ we obtain

$$\begin{aligned}M(z_{2n+1}, z_{2n+2}, kt) \geqslant{}& [p-a][M(z_{2n+1}, z_{2n}, kt) * M(z_{2n+2}, z_{2n+1}, kt)]\\&+ qM(z_{2n}, z_{2n+2}, t) + M(z_{2n}, z_{2n+1}, t) * M(z_{2n+2}, z_{2n+1}, t).\end{aligned}$$

As $p - a > 0$ and $q \geqslant 0$ we get

$$M(z_{2n+1}, z_{2n+2}, kt) \geqslant M(z_{2n}, z_{2n+1}, t) * M(z_{2n+1}, z_{2n+2}, t).$$

Similarly if we take $x = x_{2n+2}, y = x_{2n+1}$ in (3.12) we have

(2) $$M(z_{2n+2}, z_{2n+3}, kt) \geqslant M(z_{2n+1}, z_{2n+2}, t) * M(z_{2n+2}, z_{2n+3}, t),\ \forall t > 0.$$

Thus, for all m we have

$$\begin{aligned}M(z_{m+1}, z_{m+2}, kt) &\geqslant M(z_m, z_{m+1}, t),\\&\geqslant M(z_m, z_{m+1}, t) * M(z_m, z_{m+1}, t/k) * M(z_{m+1}, z_{m+2}, t/k)\\&\geqslant M(z_m, z_{m+1}, t) * M(z_m, z_{m+1}, t/k)\\&\geqslant M(z_m, z_{m+1}, t) * M(z_m, z_{m+1}, t/k^p),\ \forall t > 0,\\&\geqslant M(z_m, z_{m+1}, t),\ \forall t > 0.\end{aligned}$$

Hence by Lemma 2.12, $\{z_n\}$ is a Cauchy sequence in X, which is complete. Therefore $\{z_n\}$ converges to $u \in X$. Also its subsequences satisfy

(3) $$\{Px_{2n}\} \to u \text{ and } \{ABx_{2n}\} \to u.$$

(4) $$\{Qx_{2n+1}\} \to u \text{ and } \{STx_{2n+1}\} \to u.$$

Case 1. P is continuous.

As P is continuous we have

(5) $$PABx_{2n} \to Pu \text{ and } P^2x_{2n} \to Pu.$$

As (P, AB) is compatible type (β) we get

(6) $$(AB)^2x_{2n} \to Pu.$$

S t e p 2. Taking $x = ABx_{2n}$, $y = x_{2n+1}$ and $\alpha = 1$ in (3.12) we get

$$
\begin{aligned}
&M(PABx_{2n}, Qx_{2n+1}, kt) + a[M((AB)^2x_{2n}, STx_{2n+1}, kt) * M(PABx_{2n}, Qx_{2n+1}, kt)]\\
&\geqslant p[M(PABx_{2n}, (AB)^2x_{2n}, kt) * M(Qx_{2n+1}, STx_{2n+1}, kt)]\\
&\quad + q[M(PABx_{2n}, STx_{2n+1}, kt) * M(Qx_{2n+1}, (AB)^2x_{2n}, kt)]\\
&\quad + M((AB)^2x_{2n}, STx_{2n+1}, t) * M(PABx_{2n}, (AB)^2x_{2n}, t) * M(Qx_{2n+1}, STx_{2n+1}, t)\\
&\quad * M(PABx_{2n}, STx_{2n+1}, t) * M(Qx_{2n+1}, (AB)^2x_{2n}, t).
\end{aligned}
$$

Letting $n \to \infty$ and using (4), (5) and (6) we get that

$$
\begin{aligned}
&M(Pu, u, kt) + a[M(Pu, u, kt) * M(Pu, u, kt)]\\
&\geqslant p[M(Pu, Pu, kt) * M(u, u, kt)] + q[M(Pu, u, kt) * M(u, Pu, kt)]\\
&\quad + M(Pu, u, t) * M(Pu, Pu, t) * M(u, u, t) * M(Pu, u, t) * M(u, Pu, t),
\end{aligned}
$$

and consequently

$$M(Pu, u, kt) \geqslant (p + q - a)[M(Pu, u, kt) * M(u, Pu, kt)] + M(u, Pu, t)$$

As $p + q - a \geqslant 0$, we get that

$$M(Pu, u, kt) \geqslant M(Pu, u, t) \ \forall t > 0.$$

Hence by Lemma 2.13 we have

$$Pu = u. \tag{7}$$

S t e p 3. As $P(X) \subseteq ST(X)$, there exists $v \in X$ such that

$$u = Pu = STv. \tag{8}$$

Taking $x = x_{2n}$, $y = v$ and $\alpha = 1$ in (3.12) we get

$$
\begin{aligned}
&M(Px_{2n}, Qv, kt) + a[M(ABx_{2n}, STv, kt) * M(Px_{2n}, Qv, kt)]\\
&\geqslant p[M(Px_{2n}, ABx_{2n}, kt) * M(Qv, STv, kt)]\\
&\quad + q[M(Px_{2n}, STv, kt) * M(Qv, ABx_{2n}, kt)]\\
&\quad + M(ABx_{2n}, STv, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qv, STv, t)\\
&\quad * M(Px_{2n}, STv, t) * M(Qv, ABx_{2n}, t).
\end{aligned}
$$

Letting $n \to \infty$ and using (3) and (8) we get that

$$\begin{aligned}
&M(u, Qv, kt) + a[M(u, u, kt) * M(u, Qv, kt)]\\
&\geqslant p[M(u, u, kt) * M(Qv, u, kt)] + q[M(u, u, kt) * M(Qv, u, kt)]\\
&\quad * M(u, u, t) * M(u, u, t) * M(Qv, u, t) * M(u, u, t) * M(Qv, u, t)
\end{aligned}$$

and $M(u, Qv, kt) \geqslant (p + q - a)M(Qv, u, kt) + M(Qv, u, t)$. As $p + q - a \geqslant 0$ we get $M(u, Qv, kt) \geqslant M(Qv, u, t)$, $\forall t > 0$. Therefore Lemma 2.13 implies that $Qv = u$. Hence $STv = Qv = u$. As (Q, ST) is weak compatible we have

$$Qu = STu. \tag{9}$$

S t e p 4. Taking $x = x_{2n}$, $y = u$ and $\alpha = 1$ in (3.12) we get

$$\begin{aligned}
&M(Px_{2n}, Qu, kt) + a[M(ABx_{2n}, STu, kt) * M(Px_{2n}, Qu, kt)]\\
&\geqslant p[M(Px_{2n}, ABx_{2n}, kt) * M(Qu, STu, kt)]\\
&\quad + q[M(Px_{2n}, STu, kt) * M(Qu, ABx_{2n}, kt)]\\
&\quad + M(ABx_{2n}, STu, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qu, STu, t) * M(Px_{2n}, STu, t)\\
&\quad * M(Qu, ABx_{2n}, t).
\end{aligned}$$

Letting $n \to \infty$ and using (3) and (9) we get

$$\begin{aligned}
&M(u, Qu, kt) + a[M(u, Qu, kt) * M(u, Qu, kt)]\\
&\geqslant p[M(u, u, kt) * M(Qu, Qu, kt)] + q[M(u, Qu, kt) * M(Qu, u, kt)]\\
&\quad + M(u, Qu, t) * M(u, u, t) * M(Qu, Qu, t) * M(u, Qu, t) * M(Qu, u, t),
\end{aligned}$$

and so as in Step 2 we get that $Qu = u$. Thus $Pu = Qu = STu = u$.

As $Q(X) \subseteq AB(X)$, there exists $w \in X$ such that

$$u = Qu = ABw. \tag{10}$$

S t e p 5. Taking $x = w$, $y = u$ in (3.12) and $\alpha = 1$ we get

$$\begin{aligned}
&M(Pw, Qu, kt) + a[M(ABw, STu, kt) * M(Pw, Qu, kt)]\\
&\geqslant p[M(Pw, ABw, kt) * M(Qu, STu, kt)]\\
&\quad + q[M(Pw, STu, kt) * M(Qu, ABw, kt)]\\
&\quad + M(ABw, STu, t) * M(Pw, ABw, t)\\
&\quad * M(Qu, STu, t) * M(Pw, STu, t) * M(Qu, ABw, t),
\end{aligned}$$

so

$$\begin{aligned}&M(Pw,u,kt)+a[M(u,u,kt)*M(Pw,u,kt)]\\&\geqslant p[M(Pw,u,kt)*M(u,u,kt)]+q[M(Pw,u,kt)*M(u,u,kt)]\\&\quad+M(u,u,t)*M(Pw,u,t)*M(u,u,t)*M(Pw,u,t)*M(u,u,t)\end{aligned}$$

and

$$M(Pw,u,kt)\geqslant(p+q-a)M(Pw,u,kt)+(Pw,u,t).$$

As $p+q-a\geqslant 0$ we get that

$$M(Pw,u,kt)\geqslant M(Pw,u,t),\ \forall t>0.$$

Therefore by Lemma 2.13, we have $Pw=u$. Thus $Pw=ABw=u$. As (P,AB) is compatible type (β) and so it is weak compatible. We get $Pu=ABu$. Therefore $Pu=ABu=Qu=STu=u$.

Case 2. AB is continuous.

Since AB is continuous and (P,AB) is compatible type (β) we get

$$\{ABPx_{2n}\}\to ABu \text{ and } \{(AB)^2x_{2n}\}\to ABu \tag{11}$$

$$\{P^2x_{2n}\}\to ABu. \tag{12}$$

Step 6. Taking $x=Px_{2n}$, $y=x_{2n+1}$ and $\alpha=1$ in (3.12) we get

$$\begin{aligned}&M(P^2x_{2n},Qx_{2n+1},kt)+a[M(ABPx_{2n},STx_{2n+1},kt)*M(P^2x_{2n},Qx_{2n+1},kt)]\\&\geqslant p[M(P^2x_{2n},ABPx_{2n},kt)*M(Qx_{2n+1},STx_{2n+1},kt)]\\&\quad+q[M(P^2x_{2n},STx_{2n+1},kt)*M(Qx_{2n+1},ABPx_{2n},kt)]\\&\quad+M(ABPx_{2n},STx_{2n+1},t)*M(P^2x_{2n},ABPx_{2n},t)\\&\quad*M(Qx_{2n+1},STx_{2n+1},t)*M(P^2x_{2n},STx_{2n+1},t)*M(Qx_{2n+1},ABPx_{2n+1},t).\end{aligned}$$

Letting $n\to\infty$ and using (4), (11) and (12) we get that

$$\begin{aligned}&M(ABu,u,kt)+a[M(ABu,u,kt)*M(ABu,u,kt)]\\&\geqslant p[M(ABu,ABu,kt)*M(u,u,kt)]\\&\quad+q[M(ABu,u,kt)*M(u,ABu,kt)]\\&\quad+M(ABu,u,t)*M(ABu,ABu,t)*M(u,u,t)*M(ABu,u,t)*M(u,ABu,t),\end{aligned}$$

and so, as in Step 2, it follows that

$$ABu=u. \tag{13}$$

S t e p 7. Taking $x = u$, $y = x_{2n+1}$ and $\alpha = 1$ in (3.12) we get

$$\begin{aligned}
&M(Pu, Qx_{2n+1}, kt) + aM(ABu, STx_{2n+1}, kt) * M(Pu, Qx_{2n+1}, kt)\\
&\geqslant p[M(Pu, ABu, kt) * M(Qu, STx_{2n+1}, kt)]\\
&\quad + q[M(Pu, STx_{2n+1}, kt) * M(Qx_{2n+1}, ABu, kt)]\\
&\quad + M(ABu, STx_{2n+1}, t) * M(Pu, ABu, t) * M(Qx_{2n+1}, STx_{2n+1}, t)\\
&\quad * M(Pu, STx_{2n+1}, t) * M(Qx_{2n+1}, ABu, t).
\end{aligned}$$

Letting $n \to \infty$ and using (4) and (13) we get

$$\begin{aligned}
&M(Pu, u, kt) + a[M(u, u, kt) * M(Pu, u, kt)]\\
&\geqslant p[M(Pu, u, kt) * M(u, u, kt)] + q[M(Pu, u, kt) * M(u, u, kt)]\\
&\quad + M(u, u, t) * M(Pu, u, t) * M(u, u, t) * M(Pu, u, t) * M(u, u, t).
\end{aligned}$$

As in Step 5, we get $Pu = u$. Hence

$$Pu = ABu = u. \tag{14}$$

Hence by Step 3 and Step 4 of case 1, it follows that $Pu = STu = Qu = u$. Thus in both the cases we have $Pu = Qu = ABu = STu = u$.

S t e p 8. Taking $x = u$, $y = Tu$ and $\alpha = 1$ in (3.12) we get

$$\begin{aligned}
&M(Pu, QTu, kt) + a[M(ABu, ST(Tu), kt) * M(Pu, QTu, kt)]\\
&\geqslant p[M(Pu, ABu, kt) * M(QTu, ST(Tu), kt)]\\
&\quad + q[M(Pu, ST(Tu), kt) * M(QTu, ABu, kt)]\\
&\quad + M(ABu, ST(Tu), t) * M(Pu, ABu, t) * M(QTu, ST(Tu), t)\\
&\quad * M(Pu, ST(Tu), t) * M(QTu, ABu, t).
\end{aligned}$$

As $QT = TQ$ and $ST = TS$ we have $ST(Tu) = Tu$ and $Q(Tu) = Tu$. Thus

$$\begin{aligned}
&M(u, Tu, kt) + a[M(u, Tu, kt) * M(u, Tu, kt)]\\
&\geqslant p[M(u, u, kt) * M(Tu, Tu, kt)]\\
&\quad + q[M(u, Tu, kt) * M(Tu, u, kt)]\\
&\quad + M(u, Tu, t) * M(u, u, t) * M(Tu, Tu, t) * M(u, Tu, t) * M(Tu, u, t).
\end{aligned}$$

Then as in Step 2, it follows that $Tu = u$. Now $STu = u$ and $Tu = u$ gives $Su = u$.

S t e p 9. Taking $x = Bu$, $y = u$ and $\alpha = 1$ in (3.12) and using $P(Bu) = Bu$ and $AB(Bu) = Bu$ we get

$$\begin{aligned}
&M(Bu, u, kt) + a[M(Bu, u, kt) * M(Bu, u, kt)] \\
&\geqslant p[M(Bu, Bu, kt) * M(u, u, kt)] + q[M(Bu, u, kt) * M(u, Bu, kt)] \\
&\quad + M(Bu, u, t) * M(Bu, Bu, t) * M(u, u, t) * M(Bu, u, t) * M(u, Bu, t).
\end{aligned}$$

Then as in Step 2, it follows that $Bu = u$. Now $ABu = u$ and $Bu = u$ gives $Au = u$. Combining all the above results we get $Pu = Qu = Su = Tu = Au = Bu = u$.

S t e p 9. Let z be another common fixed point of P, Q, S, T, A and B, i.e., $Pz = Qz = Sz = Tz = Az = Bz = z$. Taking $x = u$, $y = z$ and $\alpha = 1$ in (3.12) we get

$$\begin{aligned}
&M(Pu, Qz, kt) + a[M(ABu, STz, kt) * M(Pu, Qz, kt)] \\
&\geqslant p[M(Pu, ABu, kt) * M(Qz, STz, kt)] \\
&\quad + q[M(Pu, STz, kt) * M(Qz, ABu, kt)] \\
&\quad + M(ABu, STz, t) * M(Pu, ABu, t) \\
&\quad * M(Qz, STz, t) * M(Pu, STz, t) * M(Qz, ABu, t),
\end{aligned}$$

i.e.

$$\begin{aligned}
&M(u, z, kt) + a[M(u, z, kt) * M(u, z, kt)] \\
&\geqslant p[M(u, u, kt) * M(z, z, kt)] \\
&\quad + q[M(u, z, kt) * M(z, u, kt)] \\
&\quad + M(u, z, t) * M(u, u, t) * M(z, z, t) * M(u, z, t) * M(z, u, t).
\end{aligned}$$

As in Step 2, it follows that $u = z$ and thus u is the unique common fixed point of the self-maps P, Q, S, T, A and B. $\square$

In [12] Singh and Jain have established the following result:

Theorem 3.2 [12]. *Let A, B, S, T, P and Q be self-maps on a complete fuzzy metric space $(X, M, *)$ with $t * t \geqslant t$ for all $t \in [0, 1]$ satisfying*

▷ *$P(X) \subseteq ST(X)$ and $Q(X) \subseteq AB(X)$;*

▷ *there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned}
M(Px, Qy, kt) \geqslant\ & M(ABx, Px, t) * M(STy, Qy, t) \\
& * M(STy, Px, \beta t) * M(ABx, Qy, (2 - \beta)t) * M(ABx, STy, t)
\end{aligned}$$

for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$;

▷ *the pair (P, AB) is compatible type (β) and the pair (Q, ST) is weak compatible;*
▷ *either P or AB is continuous;*
▷ $AB = BA$, $ST = TS$, $PB = BP$, $QT = TQ$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Remark 3.2. The above result of [12] follows from Theorem 3.1 by taking $p = q = a = 0$ in (3.12). Obviously the contractive condition of our theorem is more general than that adopted in [12].

In view of Proposition 3.1 of [2], we have the following result:

Corollary 3.3. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geqslant t$. Let P, Q, S, T, A and B be self-maps from X satisfying* (3.11), (3.12), (3.14), (3.15) *and*
▷ *the pair (P, AB) is compatible type (α) and the pair (Q, ST) is weak compatible.*
Then P, Q, S, T, A and B have a unique common fixed point in X.
The result follows from Proposition 3.1 *of* [2].

In [10] Sharma has established the following result:

Theorem [10, Th. 3.1]**.** *Let A, B, S, T, P and Q be self-maps on a complete fuzzy metric space $(X, M, *)$ with $t * t \geqslant t$, for all $t \in [0, 1]$ satisfying*
▷ $P(X) \subseteq ST(X)$ *and* $Q(X) \subseteq AB(X)$;
▷ $AB = BA$, $ST = TS$, $PB = BP$, $QT = TQ$, $QS = SQ$;
▷ *A, B, S and T are continuous;*
▷ *the pairs (P, AB) and (Q, ST) are compatible type (α);*
▷ *there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned} M(Px, Qy, kt) \geqslant\ & M(ABx, STy, t) * M(Px, ABx, t) * M(Qy, STy, t) \\ & * M(Px, STy, \alpha t) * M(Qy, ABx, (2-\alpha)t) \end{aligned}$$

for $x, y \in X$, $t > 0$ and $a \in (0, 2)$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Remark 3.4. A generalization and improvement of the result of [10] follows from Corollary 3.3 by taking $p = q = a = 0$ in (3.12). Precisely, the assumed (α)-compatibility of both pairs of the main result of [10] has been reduced here to the (α)-compatibility of only one pair and the weak compatibility of the other pair. Moreover, the assumed continuity of the four maps A, B, S and T in [10] has been reduced to continuity of only one map of the (α)-compatible pair. At the same time, the contractive condition of our corollary is more general than that adopted in [10].

Corollary 3.5. *Let $(X, M, *)$ be a complete fuzzy metric space and let P, Q, S, T, A and B be self-maps from X satisfying* (3.11), (3.12), (3.15) *and*

▷ *pairs (P, AB) and (Q, ST) are compatible of type (α);*

▷ *one of the maps from P, Q, AB and ST is continuous.*

Then P, Q, S, T, A and B have a unique common fixed point in X.

P r o o f. The result follows from Remark 2.10 and Corollary 3.3. □

R e m a r k 3.6. Another improvement of the result of [10] follows from Corollary 3.5. Precisely, the assumed continuity of A, B, S and T has been reduced to the continuity of only one map.

Taking $B = T = I$ in Theorem 3.1, we conclude

Corollary 3.7. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geqslant t$ and let P, Q, S and A be self-maps from X satisfying*

▷ *$P(X) \subseteq S(X)$ and $Q(X) \subseteq A(X)$;*

▷ *there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned}
&M(Px, Qy, kt) + aM(Ax, Sy, kt) * M(Px, Qy, kt)\\
&\geqslant p[M(Px, Ax, kt) * M(Qy, Sy, kt)]\\
&\quad + q[M(Px, Sy, kt) * M(Qy, Ax, kt)]\\
&\quad + M(Ax, Sy, t) * M(Px, Ax, t) * M(Qy, Sy, t)\\
&\quad * M(Px, Sy, \alpha t) * M(Qy, ABx, (2-\alpha)t)
\end{aligned}$$

for all $x, y \in X$, for all $t > 0$ and for all $\alpha \in (0, 2)$ where $q \geqslant 0, a \leqslant p$;

▷ *the pair (P, A) is compatible type (β) and the pair (Q, S) is weak compatible;*

▷ *either P or A is continuous.*

Then P, Q, S and A have a unique common fixed point.

R e m a r k 3.8. Taking $p = a$ and $q = 0$ we have an alternate result of Cho et al [2] under weaker conditions.

E x a m p l e 3.9 (of Theorem 3.1). Let $(X, M, *)$ be a fuzzy metric space where $X = [0, 1]$ and the t-norm is defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$ and $M(x, y, t) = t/(t + d(x, y))$ for all $x, y \in X$, $t > 0$. Then the induced fuzzy metric space is complete. Define self-maps P, Q, S, T, A and B by $Px = Qx = Sx = \frac{1}{2}$ if $x \in [0, 1]$, $Bx = Tx = x$ if $x \in [0, 1]$, $Ax = x$ if $x \in [\frac{1}{8}, \frac{1}{4}]$ and $Ax = \frac{1}{2}$ otherwise.

Then $P(X) \subseteq ST(X)$ and $Q(X) \subseteq AB(X)$. Further, (P, AB) is compatible type (β) and (Q, ST) is weak compatible and P is continuous. Taking $p + q = a$, all the conditions of Theorem 3.1 are satisfied and $\frac{1}{2}$ is the unique common fixed point of the six self-maps P, Q, S, T, A and B.

It is to be observed in the above Example that the pair (P, AB) is compatible of type (α), and though AB is discontinuous, yet the unique common fixed point exists. Further, the contraction in Corollary 3.3 and in this example is more general than that of [10].

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