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ON THE BEHAVIOR NEAR THE ORIGIN OF DOUBLE SINE SERIES WITH MONOTONE COEFFICIENTS

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Abstract. In this paper we obtain estimates of the sum of double sine series near the origin, with monotone coefficients tending to zero. In particular (if the coefficients $a_{k,l}$ satisfy certain conditions) the following order equality is proved

$$g(x,y) \sim mna_{m,n} + \frac{m}{n}\sum_{l=1}^{n-1} la_{m,l} + \frac{n}{m}\sum_{k=1}^{m-1} ka_{k,n} + \frac{1}{mn}\sum_{l=1}^{n-1}\sum_{k=1}^{m-1} kla_{k,l},$$

where $x \in (\frac{\pi}{m+1}, \frac{\pi}{m}]$, $y \in (\frac{\pi}{n+1}, \frac{\pi}{n}]$, $m,n = 1,2,\ldots$.

Keywords: double sine series, sum of a double sine series with monotone coefficients

MSC 2010: 42A20, 42A16

1. Introduction

Many authors considered the sine series

$$g(x) := \sum_{n=1}^{\infty} a_n \sin nx$$

with monotone coefficients tending to zero.

Young [1] was the first to consider the problem of estimates of $g(x)$ for $x \to 0$ expressed in terms of the coefficients a_n. Then Salem [2], [3], Hartman and Wintner [4], Shogunbekov [5], Aljančić, Bojanić and Tomić [6] considered this problem, as well. Later Telyakovskiĭ in [8] has proved the following facts:

Theorem T1. *Assume that $a_n \downarrow 0$. Then for $x \in (\frac{\pi}{m+1}, \frac{\pi}{m}] \equiv I_m$, $m = 1, 2, \dots$ the following estimate is valid:*

$$g(x) = \sum_{n=1}^{m} n a_n x + O\left(\frac{1}{m^3} \sum_{n=1}^{m} n^3 a_n\right).$$

Theorem T2. *Let $a_n \to 0$ and let the sequence a_k be convex. If $x \in I_m$, where $m \geqslant 11$, then the following estimate holds true:*

$$\frac{a_m}{2} \cot\frac{x}{2} + \frac{1}{2m} \sum_{k=1}^{m-1} k^2 \Delta a_k \leqslant g(x) \leqslant \frac{a_m}{2} \cot\frac{x}{2} + \frac{6}{m} \sum_{k=1}^{m-1} k^2 \Delta a_k,$$

where $\Delta a_k = a_k - a_{k+1}$.

As a special case he has proved an interesting corollary:

Corollary T. *Let the sequence a_k tend to zero and be convex. Then the following order equality is true:*[1]

$$g(x) \sim m a_m + \frac{1}{m} \sum_{k=1}^{m-1} k a_k, \quad x \in I_m, \quad x \to 0.$$

In [9] Popov has proved

Theorem P. *For any nonincreasing sequence of positive numbers a_k tending to zero, the following inequalities hold:*

$$-\frac{1}{2} a_1 \sin\frac{x}{2} \leqslant g(x) < 2 \sin\frac{x}{2} \sum_{k=1}^{m} k a_k, \quad \text{for every} \quad x \in (0, \pi],$$

$$g(x) < \sin x \sum_{k=1}^{m} k a_k, \quad \text{for every} \quad x \in \left(0, \frac{\pi}{2}\right].$$

The problem shown above is considered by the present author and by Braha [7], too.

Our main goal in this work is to prove the above theorems in the two dimensional case.

---

[1] As usuall we write $g(u) \sim h(u), u \to 0$ if there exist absolute positive constants A and B such that $Ah(u) \leqslant g(u) \leqslant Bh(u)$ holds in a neighborhood of the point $u = 0$.

Therefore, let

$$\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} a_{k,l}\sin kx\sin ly \tag{1.1}$$

be a double sine series whose coefficients satisfy the conditions

$$a_{k,l}\to 0 \quad \text{for } k\to\infty \text{ and all } l \text{ fixed, and} \tag{A}$$
$$\text{for } l\to\infty \text{ and all } k \text{ fixed}$$

and

$$\Delta_{1,1}a_{k,l} = a_{k,l} - a_{k+1,l} - a_{k,l+1} + a_{k+1,l+1} \geqslant 0$$

for all k and l.

In the following we denote

$$\begin{aligned}
\Delta_{1,0}a_{k,l} &= a_{k,l} - a_{k+1,l},\\
\Delta_{0,1}a_{k,l} &= a_{k,l+1} - a_{k,l+1},\\
\Delta_{1,0}^{2}a_{k,l} &= a_{k,l} - 2a_{k+1,l} + a_{k+2,l},\\
\Delta_{0,1}^{2}a_{k,l} &= a_{k,l} - 2a_{k,l+1} + a_{k,l+2},\\
\psi_{\nu}(u) &= -\frac{\cos(\nu+1/2)u}{2\sin u/2},\\
\varphi_{\nu}(u) &= \sum_{\mu=1}^{\nu}\psi_{\mu}(u) = -\frac{\sin(\nu+1)u}{4\sin^{2}u/2}.
\end{aligned}$$

We observe that the conditions $\Delta_{1,1}a_{k,l}\geqslant 0$ and (A) yield the conditions $\Delta_{1,0}a_{k,l}\geqslant 0$ and $\Delta_{0,1}a_{k,l}\geqslant 0$ for all k and l (see [12], [13]).

Parallelly with series (1.1) we consider the series

$$\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\Delta_{1,1}a_{k,l}\widetilde{D}_{k}(x)\widetilde{D}_{l}(y), \tag{1.2}$$

where $\widetilde{D}_{r}(u)$ are the conjugate Dirichlet kernels.

Let

$$\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty} c_{\mu,\nu} \tag{1.3}$$

be a double numerical series and

$$S_{m,n} = \sum_{\nu=1}^{n} \sum_{\mu=1}^{m} c_{\mu,\nu}$$

its rectangular partial sums.

If there exists a number S such that for all $\varepsilon > 0$ there exist natural numbers k and l such that

$$|S_{m,n} - S| < \varepsilon$$

for all $n > k$ and $m > l$, then series (1.3) converges in Pringsheim's sense to the number S (see [11], page 27).

It is well-known (see [10]) that if a sequence of numbers $\{a_{k,l}\}$ satisfies conditions (A) and $\Delta_{1,1} a_{k,l} \geqslant 0$ for each k and l, then:

(1) Series (1.1) converges almost everywhere in Pringsheim's sense, in other words there exists a function $g(x, y)$ such that the sum of series (1) is $g(x, y)$.

(2) Series (1.2) converges almost everywhere in Pringsheim's sense to $g(x, y)$.

2. Main results

We begin with

Theorem 2.1. *Assume that $a_{k,l}$ satisfy conditions* (A) *and $\Delta_{1,1} a_{k,l} \geqslant 0$. Then for $x \in I_m$ and $y \in I_n$, $m, n = 1, 2, \ldots$ the following estimate is valid:*

(2.1)

$$\begin{aligned} g(x, y) &= \sum_{l=1}^{n} \sum_{k=1}^{m} k l a_{k,l} x y \\ &\quad + O\Bigl(\frac{1}{mn^3} \sum_{l=1}^{n} \sum_{k=1}^{m} k l^3 a_{k,l} + \frac{1}{m^3 n} \sum_{l=1}^{n} \sum_{k=1}^{m} k^3 l a_{k,l} + \frac{1}{(mn)^3} \sum_{l=1}^{n} \sum_{k=1}^{m} (kl)^3 a_{k,l} \Bigr). \end{aligned}$$

P r o o f. First, according to fact (2), we have

$$g(x, y) = \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \Delta_{1,1} a_{k,l} \widetilde{D}_k(x) \widetilde{D}_l(y).$$

Now we write

$$g(x,y)=\sum_{l=1}^{n}\sum_{k=1}^{m}\Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y) \tag{2.2}$$
$$+\sum_{l=1}^{n}\sum_{k=m+1}^{\infty}\Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y)+\sum_{l=n+1}^{\infty}\sum_{k=1}^{m}\Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y)$$
$$+\sum_{l=n+1}^{\infty}\sum_{k=m+1}^{\infty}\Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y)=\sum_{\mu=1}^{4}I_\mu.$$

Since $\widetilde{D}_r(u)=O(1/u)$ and $\frac{\pi}{m+1}<x\leqslant\frac{\pi}{m}$, $\frac{\pi}{n+1}<y\leqslant\frac{\pi}{n}$, $m,n=1,2,\dots$ we have

$$I_4=O\biggl(\sum_{l=n+1}^{\infty}\sum_{k=m+1}^{\infty}\Delta_{1,1}a_{k,l}\cdot\frac{1}{xy}\biggr) \tag{2.3}$$
$$=O\biggl(mn\sum_{l=n+1}^{\infty}\sum_{k=m+1}^{\infty}\Delta_{1,1}a_{k,l}\biggr)=O(mna_{m,n}).$$

If we use the estimates

$$ma_{m,n}\leqslant\frac{4}{m^3}\sum_{k=1}^{m}k^3a_{k,n},\quad m=1,2,\dots \tag{2.4}$$

and

$$na_{m,n}\leqslant\frac{4}{n^3}\sum_{l=1}^{n}l^3a_{m,l},\quad n=1,2,\dots$$

(because $\Delta_{1,0}a_{k,l}\geqslant 0$ and $\Delta_{0,1}a_{k,l}\geqslant 0$) we obtain

$$mna_{m,n}\leqslant\frac{4}{m^3}\sum_{k=1}^{m}k^3na_{k,n}\leqslant\frac{16}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}. \tag{2.5}$$

From (2.3) and (2.5) we get

$$I_4=O\biggl(\frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}\biggr). \tag{2.6}$$

On the other hand, we have

$$I_2=O\biggl(\sum_{l=1}^{n}\sum_{k=m+1}^{\infty}\Delta_{1,1}a_{k,l}\cdot\frac{1}{x}\cdot\widetilde{D}_l(y)\biggr)=O\biggl(m\sum_{l=1}^{n}\Delta_{0,1}a_{m,l}\widetilde{D}_l(y)\biggr)$$
$$=O\biggl(m\biggl\{\sum_{l=1}^{n}a_{m,l}\sin ly-a_{m,n+1}\widetilde{D}_n(y)\biggr\}\biggr)=O\biggl(m\biggl\{\sum_{l=1}^{n}a_{m,l}\sin ly+na_{m,n}\biggr\}\biggr)$$
$$=O\biggl(\sum_{l=1}^{n}lma_{m,l}y+\sum_{l=1}^{n}l^3ma_{m,l}y^3+mna_{m,n}\biggr),$$

because $\sin y = y + O\left(y^3\right)$ when $y \to 0$. From (2.4), (2.5), $y \in I_n$ and the last estimate we get

$$I_2 = O\left(\frac{1}{m^3 n}\sum_{l=1}^{n}\sum_{k=1}^{m} k^3 l a_{k,l} + \frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3 a_{k,l}\right). \tag{2.7}$$

Using similar arguments as if I_2, $x \in I_m$, we obtain

$$I_3 = O\left(\frac{1}{mn^3}\sum_{l=1}^{n}\sum_{k=1}^{m} k l^3 a_{k,l} + \frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3 a_{k,l}\right). \tag{2.8}$$

Now we will estimate I_1. First, if we denote $\Delta_{1,0} a_{k,l} = b_l$, then

$$\begin{aligned}
I_1 &= \sum_{k=1}^{m} \widetilde{D}_k(x)\left(\sum_{l=1}^{n} \Delta_{1,1} a_{k,l} \widetilde{D}_l(y)\right) = \sum_{k=1}^{m} \widetilde{D}_k(x)\left(\sum_{l=1}^{n} \Delta_{0,1} b_l \widetilde{D}_l(y)\right)\\
&= \sum_{k=1}^{m} \widetilde{D}_k(x)\left(\sum_{l=1}^{n} b_l \sin ly - b_{n+1}\widetilde{D}_n(y)\right)\\
&= \sum_{k=1}^{m} \widetilde{D}_k(x)\left(\sum_{l=1}^{n} b_l \sin ly + O\left(n b_n\right)\right)\\
&= \sum_{l=1}^{n} \sin ly\left(\sum_{k=1}^{m} \Delta_{1,0} a_{k,l} \widetilde{D}_k(x)\right) + O\left(n\sum_{k=1}^{m} \Delta_{1,0} a_{k,n} \widetilde{D}_k(x)\right)\\
&= \sum_{l=1}^{n} \sin ly\left(\sum_{k=1}^{m} a_{k,l} \sin kx - a_{m+1,l}\widetilde{D}_m(x)\right)\\
&\quad + O\left(n\left\{\sum_{k=1}^{m} a_{k,n} \sin kx - a_{m+1,n}\widetilde{D}_m(x)\right\}\right)\\
&= \sum_{l=1}^{n} \sin ly\left(\sum_{k=1}^{m} a_{k,l} \sin kx + O\left(m a_{m,l}\right)\right) + O\left(n\left\{\sum_{k=1}^{m} a_{k,n} \sin kx + O\left(a_{m,n}\right)\right\}\right),
\end{aligned}$$

or

$$\begin{aligned}
I_1 &= \sum_{l=1}^{n}\sum_{k=1}^{m} a_{k,l} \sin kx \sin ly\\
&\quad + O\left(m\sum_{l=1}^{n} a_{m,l} \sin ly + n\sum_{k=1}^{m} a_{k,n} \sin kx + mna_{m,n}\right) = \sum_{i=1}^{4} I_1^{(i)}.
\end{aligned} \tag{2.9}$$

Let us now consider $I_1^{(1)}$. Applying the relation $\sin u = u + O\left(u^3\right)$, when $u \to 0$ we obtain

$$
\begin{aligned}
(2.10)\qquad I_1^{(1)} &= \sum_{l=1}^{n}\sum_{k=1}^{m} a_{k,l}\{klxy + O[kl^3xy^3 + k^3lx^3y + (klxy)^3]\} \\
&= \sum_{l=1}^{n}\sum_{k=1}^{m} klxya_{k,l} \\
&\quad + O\left[\sum_{l=1}^{n}\sum_{k=1}^{m} kl^3xy^3a_{k,l} + \sum_{l=1}^{n}\sum_{k=1}^{m} k^3lx^3ya_{k,l} + \sum_{l=1}^{n}\sum_{k=1}^{m} (klxy)^3 a_{k,l}\right] \\
&= \sum_{l=1}^{n}\sum_{k=1}^{m} klxya_{k,l} + \Sigma_2 + \Sigma_3 + \Sigma_4.
\end{aligned}
$$

Further,

$$
(2.11)\qquad \Sigma_2 = \sum_{l=1}^{n}\sum_{k=1}^{m} kl^3xy^3a_{k,l} = O\left(\frac{1}{mn^3}\sum_{l=1}^{n}\sum_{k=1}^{m} kl^3a_{k,l}\right).
$$

Similarly, we find

$$
(2.12)\qquad \Sigma_3 = O\left(\frac{1}{m^3n}\sum_{l=1}^{n}\sum_{k=1}^{m} k^3la_{k,l}\right)
$$

and

$$
(2.13)\qquad \Sigma_4 = O\left(\frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m} (kl)^3a_{k,l}\right).
$$

Therefore by virtue of (2.10)–(2.13) we obtain
(2.14)

$$
\begin{aligned}
I_1^{(1)} &= \sum_{l=1}^{n}\sum_{k=1}^{m} klxya_{k,l} \\
&\quad + O\left(\frac{1}{mn^3}\sum_{l=1}^{n}\sum_{k=1}^{m} kl^3a_{k,l} + \frac{1}{m^3n}\sum_{l=1}^{n}\sum_{k=1}^{m} k^3la_{k,l} + \frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
n\sum_{k=1}^{m} a_{k,n}\sin kx &= n\sum_{k=1}^{m} kxa_{k,n} + O\left(\frac{1}{m^3}\sum_{k=1}^{m} k^3na_{k,n}\right) \\
&= O\left(\frac{1}{mn^3}\sum_{l=1}^{n}\sum_{k=1}^{m} kl^3a_{k,l} + \frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}\right)
\end{aligned}
$$

and similarly

$$m\sum_{l=1}^{n}a_{m,l}\sin ly=O\biggl(\frac{1}{m^3n}\sum_{l=1}^{n}\sum_{k=1}^{m}k^3la_{k,l}+\frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}\biggr),$$

we get

$$I_1^{(2)}=O\biggl(\frac{1}{mn^3}\sum_{l=1}^{n}\sum_{k=1}^{m}kl^3a_{k,l}+\frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}\biggr)\tag{2.15}$$

and

$$I_1^{(3)}=O\biggl(\frac{1}{m^3n}\sum_{l=1}^{n}\sum_{k=1}^{m}k^3la_{k,l}+\frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}\biggr).\tag{2.16}$$

But (2.5) implies

$$I_1^{(4)}=O\biggl(\frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}\biggr).\tag{2.17}$$

From (2.14)–(2.17) and (2.9) we obtain

$$\begin{aligned}I_1&=\sum_{l=1}^{n}\sum_{k=1}^{m}klxya_{k,l}\\&\quad+O\biggl(\frac{1}{mn^3}\sum_{l=1}^{n}\sum_{k=1}^{m}kl^3a_{k,l}+\frac{1}{m^3n}\sum_{l=1}^{n}\sum_{k=1}^{m}k^3la_{k,l}+\frac{1}{(mn)^3}\sum_{l=1}^{n}\sum_{k=1}^{m}(kl)^3a_{k,l}\biggr).\end{aligned}\tag{2.18}$$

Finally, (2.2), (2.6), (2.7), (2.8) and (2.18) yield (2.1). The proof of Theorem 2.1 is complete. □

Theorem 2.2. *Let* $a_{k,l}$ *satisfy conditions* (A), $\Delta_{1,1}a_{k,l}\geqslant 0$, $\Delta_{1,0}^2\left(\Delta_{0,1}a_{k,l}\right)\geqslant 0$ *and* $\Delta_{0,1}^2\left(\Delta_{1,0}a_{k,l}\right)\geqslant 0$. *Then for* $x\in I_m$ *and* $y\in I_n$, *where* $m\geqslant 11$, $n\geqslant 11$, *the following estimate is valid:*

$$\begin{aligned}&\frac{a_{m,n}}{4}\cot\frac{x}{2}\cot\frac{y}{2}+\frac{1}{4n}\cot\frac{x}{2}\sum_{l=1}^{n-1}l^2\Delta_{0,1}a_{m,l}\\&\qquad+\frac{1}{4m}\cot\frac{y}{2}\sum_{k=1}^{m-1}k^2\Delta_{1,0}a_{k,n}+\frac{1}{2mn}\sum_{l=1}^{n-1}\sum_{k=1}^{m-1}(kl)^2\Delta_{1,1}a_{k,l}\\&\quad\leqslant g(x,y)\leqslant\frac{a_{m,n}}{4}\cot\frac{x}{2}\cot\frac{y}{2}+\frac{1}{2}\cot\frac{x}{2}\frac{2,4+\pi}{n}\sum_{l=1}^{n-1}l^2\Delta_{0,1}a_{m,l}\\&\qquad+\frac{1}{2}\cot\frac{y}{2}\frac{2,4+\pi}{m}\sum_{k=1}^{m-1}k^2\Delta_{1,0}a_{k,n}+\frac{(\pi+2,4)^2}{mn}\sum_{l=1}^{n-1}\sum_{k=1}^{m-1}(kl)^2\,\Delta_{1,1}a_{k,l}.\end{aligned}\tag{2.19}$$

P r o o f. We can write

$$g(x,y) = \sum_{l=1}^{n-1}\sum_{k=1}^{m-1} \Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y) + \sum_{l=n}^{\infty}\sum_{k=1}^{m-1} \Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y) + \sum_{l=1}^{n-1}\sum_{k=m}^{\infty} \Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y) + \sum_{l=n}^{\infty}\sum_{k=m}^{\infty} \Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y) = \sum_{\nu=1}^{4} S_\nu(x,y). \tag{2.20}$$

Since

$$\widetilde{D}_r(u) = \sum_{i=1}^{r} \sin iu \leqslant \sum_{i=1}^{r} iu \leqslant r^2 u \leqslant \pi \cdot \frac{r^2}{j}, \quad \frac{\pi}{j+1} < u \leqslant \frac{\pi}{j}, \quad j = 1, 2, \ldots$$

we conclude that

$$S_1(x,y) \leqslant \frac{\pi^2}{mn} \cdot \sum_{l=1}^{n-1}\sum_{k=1}^{m-1} (kl)^2\, \Delta_{1,1}a_{k,l}. \tag{2.21}$$

Next, using Abel's transformation we get

$$\begin{aligned} S_2(x,y) &\leqslant \frac{\pi}{m}\sum_{k=1}^{m-1} k^2 \Big(\sum_{l=n}^{\infty} \Delta_{1,1}a_{k,l}\widetilde{D}_l(y)\Big) \\ &= \frac{\pi}{m}\sum_{k=1}^{m-1} k^2 \Big(\sum_{l=n}^{\infty} \Delta_{1,1}a_{k,l}\Big\{\frac{1}{2}\cot\frac{y}{2} + \varphi_l(y)\Big\}\Big) \\ &= \frac{\pi}{m}\sum_{k=1}^{m-1} k^2 \Big(\frac{\Delta_{1,0}a_{k,n}}{2}\cot\frac{y}{2} + \sum_{l=n}^{\infty} \Delta_{1,1}a_{k,l}\varphi_l(y)\Big). \end{aligned} \tag{2.22}$$

Using Abel's transformation again we arrive at

$$S_2^{(1)}(x,y) := \sum_{l=n}^{\infty} \Delta_{1,1}a_{k,l}\varphi_l(y) = \sum_{l=n}^{\infty} \Delta_{0,1}^2\left(\Delta_{1,0}a_{k,l}\right)\left\{\varphi_l(y) - \varphi_{n-1}(y)\right\}.$$

From the last equality we obtain

$$\begin{aligned} |S_2^{(1)}(x,y)| &\leqslant \frac{\Delta_{1,1}a_{k,n}}{4\sin^2\frac{y}{2}}\,(1+\sin ny) \leqslant \frac{\Delta_{1,1}a_{k,n}}{2\sin^2\frac{y}{2}} \\ &\leqslant \frac{\pi^2\Delta_{1,1}a_{k,n}}{2y^2} \leqslant \frac{(n+1)^2}{2}\cdot \Delta_{1,1}a_{k,n}. \end{aligned}$$

Now for $n \geqslant 11$ we have

$$|S_2^{(1)}(x,y)| \leqslant \Delta_{1,1} a_{k,n} \cdot \frac{2,4}{n} \sum_{l=1}^{n-1} l^2 \leqslant \frac{2,4}{n} \cdot \sum_{l=1}^{n-1} l^2 \Delta_{1,1} a_{k,l}. \tag{2.23}$$

By (2.22) and (2.23) the following estimate holds:

$$S_2(x,y) \leqslant \frac{\pi}{2m} \cot\frac{y}{2} \cdot \sum_{k=1}^{m-1} k^2 \Delta_{1,0} a_{k,n} + \frac{2,4\cdot\pi}{mn} \cdot \sum_{l=1}^{n-1}\sum_{k=1}^{m-1} (kl)^2\, \Delta_{1,1} a_{k,l}. \tag{2.24}$$

In absolutely the same way we can prove that $(m \geqslant 11)$

$$S_3(x,y) \leqslant \frac{\pi}{2n} \cot\frac{x}{2} \cdot \sum_{l=1}^{n-1} l^2 \Delta_{0,1} a_{m,l} + \frac{2,4\cdot\pi}{mn} \cdot \sum_{l=1}^{n-1}\sum_{k=1}^{m-1} (kl)^2\, \Delta_{1,1} a_{k,l}. \tag{2.25}$$

For $S_4(x,y)$ we have

$$\begin{aligned} S_4(x,y) &= \sum_{l=n}^{\infty}\sum_{k=m}^{\infty} \Delta_{1,1} a_{k,l} \Big\{\frac{1}{2}\cot\frac{x}{2} + \varphi_k(x)\Big\}\Big\{\frac{1}{2}\cot\frac{y}{2} + \varphi_l(y)\Big\} \\ &= \frac{a_{m,n}}{4}\cot\frac{x}{2}\cot\frac{y}{2} + \frac{1}{2}\cot\frac{x}{2}\cdot r_1(y) \\ &\quad + \frac{1}{2}\cot\frac{y}{2}\cdot r_2(x) + r_3(x,y), \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} r_1(y) &= \sum_{l=n}^{\infty} \Delta_{0,1} a_{m,l}\varphi_l(y), \\ r_2(x) &= \sum_{k=m}^{\infty} \Delta_{1,0} a_{k,n}\varphi_k(x), \\ r_3(x,y) &= \sum_{l=n}^{\infty}\sum_{k=m}^{\infty} \Delta_{1,1} a_{k,l}\varphi_k(x)\varphi_l(y). \end{aligned} \tag{2.27}$$

Applying Abel's transformation to $r_1(y)$ we obtain

$$r_1(y) = \sum_{l=n}^{\infty} \Delta_{0,1} a_{m,l}\varphi_l(y) = \sum_{l=n}^{\infty} \Delta_{0,1}^2 a_{m,l}\left\{\varphi_l(y) - \varphi_{n-1}(y)\right\}.$$

Therefore

$$\begin{aligned} |r_1(y)| &\leqslant \sum_{l=n}^{\infty} \Delta_{0,1}^2 a_{m,l} \frac{1+\sin ny}{4\sin^2\frac{y}{2}} \leqslant \frac{\Delta_{0,1} a_{m,n}}{2\sin^2\frac{y}{2}} \\ &\leqslant \frac{\pi^2}{2y^2}\cdot \Delta_{0,1} a_{m,n} \leqslant \frac{(n+1)^2}{2}\cdot \Delta_{0,1} a_{m,n}. \end{aligned}$$

For $n \geqslant 11$ we have

$$r_1(y) \leqslant \frac{2,4}{n} \cdot \Delta_{0,1} a_{m,n} \sum_{l=1}^{n-1} l^2 \leqslant \frac{2,4}{n} \sum_{l=1}^{n-1} l^2 \Delta_{0,1} a_{m,l}. \tag{2.28}$$

In the same way, for $m \geqslant 11$ we can find an estimate

$$r_2(x) \leqslant \frac{2,4}{m} \sum_{k=1}^{m-1} k^2 \Delta_{1,0} a_{k,n}. \tag{2.29}$$

Now we estimate the quantity $r_3(x, y)$. Using twice Abel's transformation to (2.27) we get

$$\begin{aligned}
r_3(x,y) &= \sum_{l=n}^{\infty} \varphi_l(y) \Big\{ \sum_{k=m}^{\infty} \Delta_{1,0}^2 \left(\Delta_{0,1} a_{k,l}\right) \left[\varphi_k(x) - \varphi_{m-1}(x)\right] \Big\} \\
&\leqslant \sum_{l=n}^{\infty} \varphi_l(y) \Big\{ \sum_{k=m}^{\infty} \Delta_{1,0}^2 \left(\Delta_{0,1} a_{k,l}\right) \frac{1 + \sin mx}{4 \sin^2 \frac{x}{2}} \Big\} \\
&\leqslant \frac{1}{2 \sin^2 \frac{x}{2}} \sum_{l=n}^{\infty} \Delta_{1,1} a_{m,l} \varphi_l(y) \\
&= \frac{1}{2 \sin^2 \frac{x}{2}} \sum_{l=n}^{\infty} \Delta_{0,1}^2 \left(\Delta_{1,0} a_{m,l}\right) \left[\varphi_l(y) - \varphi_{n-1}(y)\right] \\
&\leqslant \frac{1}{2 \sin^2 \frac{x}{2}} \cdot \frac{1}{2 \sin^2 \frac{y}{2}} \cdot \Delta_{1,1} a_{m,n} \leqslant \frac{(m+1)^2}{2} \cdot \frac{(n+1)^2}{2} \cdot \Delta_{1,1} a_{m,n}.
\end{aligned}$$

Therefore, for $m, n \geqslant 11$ we have

$$\begin{aligned}
r_3(x,y) &\leqslant \Delta_{1,1} a_{m,n} \cdot \frac{(2,4)^2}{mn} \cdot \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} (kl)^2 \\
&\leqslant \frac{(2,4)^2}{mn} \cdot \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} (kl)^2 \Delta_{1,1} a_{k,l}.
\end{aligned} \tag{2.30}$$

From (2.26), (2.28), (2.29) and (2.30) we get

$$\begin{aligned}
S_4 &\leqslant \frac{a_{m,n}}{4} \cdot \cot \frac{x}{2} \cdot \cot \frac{y}{2} + \frac{1}{2} \cot \frac{x}{2} \cdot \frac{2,4}{n} \cdot \sum_{l=1}^{n-1} l^2 \Delta_{0,1} a_{m,l} \\
&\quad + \frac{1}{2} \cot \frac{y}{2} \cdot \frac{2,4}{m} \cdot \sum_{k=1}^{m-1} k^2 \Delta_{1,0} a_{k,n} + \frac{(2,4)^2}{mn} \cdot \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} (kl)^2 \Delta_{1,1} a_{k,l}.
\end{aligned} \tag{2.31}$$

Now, by virtue (2.20), (2.21), (2.24), (2.25) and (2.31) the proof of the upper estimate is complete.

On the other hand, using (2.20) we can write

$$
\begin{aligned}
(2.32)\qquad g(x,y) &= \frac{a_{m,n}}{4}\cot\frac{x}{2}\cot\frac{y}{2} \\
&+ \frac{1}{2}\cot\frac{x}{2}\Big\{\sum_{l=1}^{n-1}\Delta_{0,1}a_{m,l}\widetilde{D}_l(y) + \sum_{l=n}^{\infty}\Delta_{0,1}a_{m,l}\varphi_l(y)\Big\} \\
&+ \frac{1}{2}\cot\frac{y}{2}\Big\{\sum_{k=1}^{m-1}\Delta_{1,0}a_{k,n}\widetilde{D}_l(x) + \sum_{k=m}^{\infty}\Delta_{1,0}a_{k,n}\varphi_k(x)\Big\} \\
&+ \sum_{l=1}^{n-1}\sum_{k=1}^{m-1}\Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y) + \sum_{k=1}^{m-1}\widetilde{D}_k(x)\sum_{l=n}^{\infty}\Delta_{1,1}a_{k,l}\varphi_l(y) \\
&+ \sum_{l=1}^{n-1}\widetilde{D}_l(y)\sum_{k=m}^{\infty}\Delta_{1,1}a_{k,l}\varphi_k(x) + \sum_{l=n}^{\infty}\sum_{k=m}^{\infty}\Delta_{1,1}a_{k,l}\varphi_k(x)\varphi_l(y).
\end{aligned}
$$

Using relation (17) from [8] we conclude that

$$\frac{1}{2}\sum_{l=1}^{n-1}\Delta_{0,1}a_{m,l}\widetilde{D}_l(y) + \sum_{l=n}^{\infty}\Delta_{0,1}a_{m,l}\varphi_l(y) \geqslant 0 \quad \text{for} \quad n \geqslant 11$$

and

$$\frac{1}{2}\sum_{k=1}^{m-1}\Delta_{1,0}a_{k,n}\widetilde{D}_l(x) + \sum_{k=m}^{\infty}\Delta_{1,0}a_{k,n}\varphi_k(x) \geqslant 0 \quad \text{for} \quad m \geqslant 11.$$

Therefore relation (2.32) assumes the form

$$
\begin{aligned}
(2.33)\qquad g(x,y) &\geqslant \frac{a_{m,n}}{4}\cot\frac{x}{2}\cot\frac{y}{2} \\
&+ \frac{1}{4}\cot\frac{x}{2}\sum_{l=1}^{n-1}\Delta_{0,1}a_{m,l}\widetilde{D}_l(y) + \frac{1}{4}\cot\frac{y}{2}\sum_{k=1}^{m-1}\Delta_{1,0}a_{k,n}\widetilde{D}_k(x) \\
&+ S_1(x,y) + s_1(x,y) + s_2(x,y) + r_3(x,y).
\end{aligned}
$$

However,

$$
\begin{aligned}
(2.34)\qquad S_1(x,y) &\geqslant \Delta_{1,1}a_{m,n}\sum_{l=1}^{n-1}\widetilde{D}_l(y)\sum_{k=1}^{m-1}\widetilde{D}_k(x) \\
&= \frac{\Delta_{1,1}a_{m,n}}{16\sin^2\frac{1}{2}x\sin^2\frac{1}{2}y}(m\sin x - \sin mx)(n\sin y - \sin ny),
\end{aligned}
$$

$$(2.35)\qquad s_1(x,y) \geqslant \sum_{l=n}^{\infty} \Delta_{1,1}a_{m,l}\varphi_l(y)\sum_{k=1}^{m-1}\widetilde{D}_k(x)$$
$$= \frac{1}{4\sin^2\frac{x}{2}}(m\sin x - \sin mx)\sum_{l=n}^{\infty}\Delta_{1,1}a_{m,l}\varphi_l(y)$$
$$\geqslant -\frac{\Delta_{1,1}a_{m,n}}{16\sin^2\frac{x}{2}\sin^2\frac{y}{2}}(m\sin x - \sin mx)(1+\sin ny).$$

In a similar way we can prove that

$$(2.36)\qquad s_2(x,y) \geqslant -\frac{\Delta_{1,1}a_{m,n}}{16\sin^2\frac{x}{2}\sin^2\frac{y}{2}}(1+\sin mx)(n\sin y - \sin ny).$$

Now we are going to estimate $r_3(x,y)$ from below:

$$(2.37)\qquad r_3(x,y) = \sum_{l=n}^{\infty}\varphi_l(y)\sum_{k=m}^{\infty}\Delta_{1,1}a_{k,l}\varphi_k(x)$$
$$\geqslant -\frac{1}{4\sin^2\frac{x}{2}}(1+\sin mx)\sum_{l=n}^{\infty}\Delta_{1,1}a_{m,l}\varphi_l(y)$$
$$\geqslant \frac{\Delta_{1,1}a_{m,n}}{16\sin^2\frac{x}{2}\sin^2\frac{y}{2}}(1+\sin mx)(1+\sin ny)$$
$$\geqslant -\frac{\Delta_{1,1}a_{m,n}}{16\sin^2\frac{x}{2}\sin^2\frac{y}{2}}(1+\sin mx)(1+\sin ny).$$

From relations (2.34)–(2.37) for $m,n \geqslant 11$ we have

$$(2.38)\qquad \frac{1}{2}S_1(x,y) + s_1(x,y) + s_2(x,y) + r_3(x,y)$$
$$\geqslant \frac{\Delta_{1,1}a_{m,n}}{32\sin^2\frac{x}{2}\sin^2\frac{y}{2}}\left[(m\sin x - \sin mx) - 2(1+\sin mx)\right]$$
$$\times\left[(n\sin y - \sin ny) - 2(1+\sin ny)\right]$$
$$+ \frac{\Delta_{1,1}a_{m,n}}{16\sin^2\frac{x}{2}\sin^2\frac{y}{2}}(1+\sin mx)(1+\sin ny) > 0.$$

So by virtue of (2.38) relation (2.33) takes the form

$$(2.39)\qquad g(x,y) \geqslant \frac{a_{m,n}}{4}\cot\frac{x}{2}\cot\frac{y}{2} + \frac{1}{4}\cot\frac{x}{2}\sum_{l=1}^{n-1}\Delta_{0,1}a_{m,l}\widetilde{D}_l(y)$$
$$+ \frac{1}{4}\cot\frac{y}{2}\sum_{k=1}^{m-1}\Delta_{1,0}a_{k,n}\widetilde{D}_k(x) + \frac{1}{2}\sum_{l=1}^{n-1}\sum_{k=1}^{m-1}\Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y).$$

Taking into consideration that $\nu < r$ and $\frac{\pi}{r+1} < u \leqslant \frac{\pi}{r}$ we get

$$\widetilde{D}_\nu(u) \geqslant \frac{2}{\pi}\sum_{i=1}^{\nu} iu = \frac{\nu(\nu+1)}{\pi}u > \frac{\nu^2}{r}.$$

Therefore, using the last estimate and relation (2.39) we find

$$g(x,y) \geqslant \frac{a_{m,n}}{4}\cot\frac{x}{2}\cot\frac{y}{2} + \frac{1}{4n}\cot\frac{x}{2}\sum_{l=1}^{n-1} l^2\Delta_{0,1}a_{m,l} + \frac{1}{4m}\cot\frac{y}{2}\sum_{k=1}^{m-1} k^2\Delta_{1,0}a_{k,n} + \frac{1}{2mn}\sum_{l=1}^{n-1}\sum_{k=1}^{m-1}(kl)^2\Delta_{1,1}a_{k,l}. \tag{2.40}$$

Relations (2.31) and (2.40) prove Theorem 2.2. □

We notice that using relation (2.19) for $x \in I_m$ and $y \in I_n$ we can write

$$g(x,y) \sim mna_{m,n} + \frac{m}{n}\sum_{l=1}^{n-1} l^2\Delta_{0,1}a_{m,l} + \frac{n}{m}\sum_{k=1}^{m-1} k^2\Delta_{1,0}a_{k,n} + \frac{1}{mn}\sum_{l=1}^{n-1}\sum_{k=1}^{m-1}(kl)^2\,\Delta_{1,1}a_{k,l}. \tag{2.41}$$

The last relation can be written in a simpler form as Telyakovskiĭ for the one-dimensional case did.

Since

$$\begin{aligned}\frac{1}{n}\sum_{l=1}^{n-1} l^2\Delta_{0,1}a_{m,l} &= \frac{1}{n}\{1^2a_{m,1} + (2^2-1^2)a_{m,2} + (3^2-2^2)a_{m,3} + \dots \\ &\quad + \left[(n-1)^2-(n-2)^2\right]a_{m,n-1} - (n-1)^2a_{m,n}\} \\ &= \frac{1}{n}\left\{\sum_{l=1}^{n-1}(2l-1)a_{m,l} - (n-1)^2a_{m,n}\right\},\end{aligned}$$

we have

$$\frac{1}{n}\sum_{l=1}^{n-1}(2l-1)a_{m,l} - na_{m,n} \leqslant \frac{1}{n}\sum_{l=1}^{n-1} l^2\Delta_{0,1}a_{m,l} \leqslant \frac{1}{n}\sum_{l=1}^{n-1}(2l-1)a_{m,l}. \tag{2.42}$$

Similarly we can prove that

$$\frac{1}{m}\sum_{k=1}^{m-1}(2k-1)a_{k,n} - ma_{m,n} \leqslant \frac{1}{m}\sum_{k=1}^{m-1} k^2\Delta_{1,0}a_{k,n} \leqslant \frac{1}{m}\sum_{k=1}^{m-1}(2k-1)a_{k,n}. \tag{2.43}$$

On the other hand, putting $\Lambda_{m,n} = (mn)^{-1} \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} (kl)^2 \Delta_{1,1} a_{k,l}$ we have

$$(2.44) \quad \Lambda_{m,n} \leqslant \frac{1}{mn} \sum_{l=1}^{n-1} l^2 \sum_{k=1}^{m-1} (2k-1)\,\Delta_{0,1} a_{k,l} \leqslant \frac{1}{mn} \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} (2k-1)\,(2l-1)\,a_{k,l}$$

and

$$(2.45) \qquad \begin{aligned} \Lambda_{m,n} &\geqslant \frac{1}{n} \sum_{l=1}^{n-1} l^2 \left\{ \frac{1}{m} \sum_{k=1}^{m-1} (2k-1)\,\Delta_{0,1} a_{k,l} - m\Delta_{0,1} a_{m,l} \right\} \\ &\geqslant \frac{1}{m} \sum_{k=1}^{m-1} (2k-1) \left\{ \frac{1}{n} \sum_{l=1}^{n-1} (2l-1)\,a_{k,l} - n a_{k,n} \right\} - \frac{m}{n} \sum_{l=1}^{n-1} l^2 \Delta_{0,1} a_{m,l} \\ &= \frac{1}{mn} \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} (2k-1)\,(2l-1)\,a_{k,l} \\ &\quad - \frac{n}{m} \sum_{k=1}^{m-1} (2k-1)\,a_{k,n} - \frac{m}{n} \sum_{l=1}^{n-1} l^2 \Delta_{0,1} a_{m,l} \\ &\geqslant \frac{1}{mn} \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} (2k-1)\,(2l-1)\,a_{k,l} \\ &\quad - \frac{n}{m} \sum_{k=1}^{m-1} (2k-1)\,a_{k,n} - \frac{m}{n} \sum_{l=1}^{n-1} (2l-1)\,a_{m,l}. \end{aligned}$$

Relations (2.42)–(2.45) and (2.41) immediately imply the following corollary:

Corollary 2.3. *Let $a_{k,l}$ satisfy conditions* (A), $\Delta_{1,1} a_{k,l} \geqslant 0$, $\Delta_{1,0}^2 (\Delta_{0,1} a_{k,l}) \geqslant 0$ *and* $\Delta_{0,1}^2 (\Delta_{1,0} a_{k,l}) \geqslant 0$. *Then for $x \in I_m$, $x \to 0$, $y \in I_n$, $y \to 0$ the following order equality is true:*

$$g(x,y) \sim mn a_{m,n} + \frac{m}{n} \sum_{l=1}^{n-1} l a_{m,l} + \frac{n}{m} \sum_{k=1}^{m-1} k a_{k,n} + \frac{1}{mn} \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} kl a_{k,l}.$$

Theorem 2.4. *Assume that $a_{k,l}$ satisfy conditions* (A) *and* $\Delta_{1,1} a_{k,l} \geqslant 0$. *Then the following estimates are valid:*

$$(2.46) \quad \frac{a_{1,1}}{4} \sin\frac{x}{2} \sin\frac{y}{2} \leqslant g(x,y) < 4 \sin\frac{x}{2} \sin\frac{y}{2} \sum_{l=1}^{n} \sum_{k=1}^{m} kl a_{k,l}, \text{ for every } x, y \in (0, \pi]$$

and

$$(2.47)\qquad g(x,y) < \sin x \sin y \sum_{l=1}^{n}\sum_{k=1}^{m} k l a_{k,l}, \quad \textit{for every } x,y\in(0,\pi/2].$$

P r o o f. First, for $\widetilde{D}_\nu(u)$ the estimate

$$\begin{aligned}\widetilde{D}_\nu(u) &= \frac{\cos\frac{1}{2}u-\cos(n+\frac{1}{2})u}{2\sin\frac{1}{2}u}\\ &\geqslant \frac{\cos\frac{1}{2}-1}{2\sin\frac{1}{2}u} = -\frac{\sin^2\frac{1}{4}u}{\sin\frac{1}{2}u} = -\frac{1}{2}\tan\frac{u}{4} \geqslant -\frac{1}{2}\sin\frac{u}{2},\end{aligned}$$

for every $u\in(0,\pi]$, holds.

Therefore, by this and fact (2) we obtain

$$\begin{aligned}g(x,y) &= \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\Delta_{1,1}a_{k,l}\widetilde{D}_k(x)\widetilde{D}_l(y)\\ &\geqslant \frac{1}{4}\sin\frac{x}{2}\sin\frac{y}{2}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\Delta_{1,1}a_{k,l} = \frac{a_{1,1}}{4}\sin\frac{x}{2}\sin\frac{y}{2}.\end{aligned}$$

To prove the upper estimate we denote $a_{k,l}=b_k$ and $a_{m+1,l}=d_l$. As in [2] for $0<t\leqslant\pi$ and $c_k\downarrow$, we get

$$(2.48)\qquad r_\mu(t)=\sum_{k=\mu+1}^{\infty}c_k\sin kt = \sum_{k=\mu+1}^{\infty}\Delta_{1,0}c_k\widetilde{D}_k(t)-c_{\mu+1}\widetilde{D}_\mu(t)\leqslant\frac{1}{2}c_{\mu+1}\sin\frac{t}{2}.$$

Now by virtue of (2.48) we have

$$(2.49)\qquad \begin{aligned}r_{m,n}(x,y) &= \sum_{l=n+1}^{\infty}\sin ly\Bigg\{\sum_{k=m+1}^{\infty}b_k\sin kx\Bigg\}\\ &= \sum_{l=n+1}^{\infty}\sin ly\Bigg\{\sum_{k=m+1}^{\infty}\Delta_{1,0}b_k\widetilde{D}_k(x)-b_{m+1}\widetilde{D}_m(x)\Bigg\}\\ &\leqslant \frac{1}{2}\sin\frac{x}{2}\sum_{l=n+1}^{\infty}a_{m+1,l}\sin ly = \frac{1}{2}\sin\frac{x}{2}\sum_{l=n+1}^{\infty}d_l\sin ly\\ &= \frac{1}{2}\sin\frac{x}{2}\Bigg\{\sum_{l=n+1}^{\infty}\Delta_{0,1}d_l\widetilde{D}_l(y)-d_{n+1}\widetilde{D}_n(y)\Bigg\}\\ &\leqslant \frac{1}{4}a_{m+1,n+1}\sin\frac{x}{2}\sin\frac{y}{2}.\end{aligned}$$

We now pass to the proof of the right hand sides of inequalities (2.46) and (2.47). Suppose that $x \in (\pi/2, \pi]$, $y \in (\pi/2, \pi]$, then $m = 1, n = 1$. By (2.48) and (2.49) we obtain

$$
\begin{aligned}
g(x,y) &= \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} a_{k,l} \sin kx \sin ly \\
&= a_{1,1} \sin x \sin y + \sin x \sum_{l=2}^{\infty} a_{1,l} \sin ly \\
&\quad + \sin y \sum_{k=2}^{\infty} a_{k,1} \sin kx + \sum_{l=2}^{\infty}\sum_{k=2}^{\infty} a_{k,l} \sin kx \sin ly \\
&\leqslant a_{1,1} \sin x \sin y + \frac{a_{1,2}}{2} \sin x \sin \frac{y}{2} \\
&\quad + \frac{a_{2,1}}{2} \sin \frac{x}{2} \sin y + \frac{a_{2,2}}{4} \sin \frac{x}{2} \sin \frac{y}{2} \\
&= 4 \sin \frac{x}{2} \sin \frac{y}{2} \Big\{ a_{1,1} \cos \frac{x}{2} \cos \frac{y}{2} + \frac{a_{1,2}}{4} \cos \frac{x}{2} + \frac{a_{2,1}}{4} \cos \frac{y}{2} + \frac{a_{2,2}}{16} \Big\}.
\end{aligned}
$$

Since $a_{1,2} \leqslant a_{1,1}$, $a_{2,1} \leqslant a_{1,1}$, $a_{2,2} \leqslant a_{1,1}$ and $\cos \frac{t}{2} < 0,71$ for $t \in (\pi/2, \pi]$, we have

$$
g(x,y) < 3,6864 \cdot a_{1,1} \cdot \sin \frac{x}{2} \sin \frac{y}{2} < 4 \cdot a_{1,1} \cdot \sin \frac{x}{2} \sin \frac{y}{2}.
$$

The right-hand inequality in (2.46) for $x \in (\pi/2, \pi]$ and $y \in (\pi/2, \pi]$ is proved.

Now suppose that $x \in (0, \pi/2]$ and $y \in (0, \pi/2]$. Then $m \geqslant 2$, $n \geqslant 2$ and $g(x,y)$ can be writen in the form

$$
\begin{aligned}
g(x,y) = &\sum_{l=1}^{n-1}\sum_{k=1}^{m-1} a_{k,l} \sin kx \sin ly + \sin ny \sum_{k=1}^{m-1} a_{k,n} \sin kx \\
&+ \sum_{l=n+1}^{\infty}\sum_{k=1}^{m-1} a_{k,l} \sin kx \sin ly + \sin mx \sum_{l=1}^{n-1} a_{m,l} \sin ly \\
&+ \sum_{l=1}^{n-1}\sum_{k=m+1}^{\infty} a_{k,l} \sin kx \sin ly + a_{m,n} \sin mx \sin ny \\
&+ \sin ny \sum_{k=m+1}^{\infty} a_{k,n} \sin kx + \sin mx \sum_{l=n+1}^{\infty} a_{m,l} \sin ly \\
&+ \sum_{l=n+1}^{\infty}\sum_{k=m+1}^{\infty} a_{k,l} \sin kx \sin ly.
\end{aligned}
$$

To find the upper estimate of $g(x,y)$, we use the obvious inequalities

$$
\sin \nu u \leqslant \nu \sin u, \quad 1 \leqslant \nu \leqslant p-1, \quad 0 < u \leqslant \pi/p, \quad p \geqslant 2
$$

and

$$\sin \nu u \leqslant \sin u, \quad \pi/(p+1) < u \leqslant \pi/p.$$

Therefore

$$\begin{aligned} g(x,y) \leqslant\ & \sin x \sin y \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} kl a_{k,l} + \sin x \sin ny \sum_{k=1}^{m-1} k a_{k,n} \\ & + \sin x \sum_{l=n+1}^{\infty} \sum_{k=1}^{m-1} k a_{k,l} \sin ly + \sin mx \sin y \sum_{l=1}^{n-1} l a_{m,l} \\ & + \sin y \sum_{l=1}^{n-1} \sum_{k=m+1}^{\infty} l a_{k,l} \sin kx + a_{m,n} \sin mx \sin ny \\ & + \sin ny \sum_{k=m+1}^{\infty} a_{k,n} \sin kx + \sin mx \sum_{l=n+1}^{\infty} a_{m,l} \sin ly \\ & + \sum_{l=n+1}^{\infty} \sum_{k=m+1}^{\infty} a_{k,l} \sin kx \sin ly. \end{aligned}$$

From estimates (2.48), (2.49) and $\sin u/2 \leqslant \sin u$ the last estimate takes on the form

$$\begin{aligned} g(x,y) &\leqslant \sin x \sin y \bigg\{ \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} kl a_{k,l} + \sum_{k=1}^{m-1} k a_{k,n} + \frac{1}{2} \sum_{k=1}^{m-1} k a_{k,n+1} + \sum_{l=1}^{n-1} l a_{m,l} \\ & \quad + \frac{1}{2} \sum_{l=1}^{n-1} l a_{m+1,l} + a_{m,n} + \frac{1}{2} a_{m+1,n} + \frac{1}{2} a_{m,n+1} + \frac{1}{4} a_{m+1,n+1} \bigg\} \\ &\leqslant \sin x \sin y \bigg\{ \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} kl a_{k,l} + 1,5 \sum_{k=1}^{m-1} k a_{k,n} + 1,5 \sum_{l=1}^{n-1} l a_{m,l} + 2,25 a_{m,n} \bigg\} \\ &\leqslant \sin x \sin y \sum_{l=1}^{n-1} \sum_{k=1}^{m-1} kl a_{k,l}. \end{aligned}$$

Estimate (2.47) for $x, y \in (0, \pi/2]$ is proved. Clearly the right hand side of estimate (2.46) is a consequence of estimate (2.47). □

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