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ON PERIODIC SOLUTIONS OF NON-AUTONOMOUS SECOND ORDER HAMILTONIAN SYSTEMS*

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Abstract. The purpose of this paper is to study the existence of periodic solutions for the non-autonomous second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \ \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

Some new existence theorems are obtained by the least action principle.

Keywords: periodic solution, critical point, non-autonomous second-order system, Sobolev's inequality

MSC 2010: 34C25, 37J45, 58E50

1. Introduction

Consider the non-autonomous second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \ \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $F\colon [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

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(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $b \in L^1([0,T];\mathbb{R}^+)$ such that

$$|F(t,x)| \leqslant a(|x|)b(t), \quad |\nabla F(t,x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

The corresponding functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t + \int_0^T F(t,u(t))\,\mathrm{d}t$$

is continuously differentiable and weakly lower semicontinuous on H_T^1, where

$$\begin{aligned} H_T^1 = \{u\colon\ [0,T] \to \mathbb{R}^N \mid\ & u \text{ is absolutely continuous,} \\ & u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T;\mathbb{R}^N)\}\end{aligned}$$

is a Hilbert space with the usual scalar product and norm (see [4]). Moreover, one has

$$(\varphi'(u),v) = \int_0^T [(\dot{u}(t),\dot{v}(t)) + (\nabla F(t,u(t)),v(t))]\,\mathrm{d}t$$

for $u,v \in H_T^1$. It is well known that the solutions of problem (1.1) correspond to the critical points of φ (see [4]).

For $u \in H_T^1$, let $\bar{u} = T^{-1}\int_0^T u(t)\,\mathrm{d}t$ and $\tilde{u}(t) = u(t) - \bar{u}$. Then one has

$$\|\tilde{u}\|_\infty^2 \leqslant \frac{T}{12}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t \quad \text{(Sobolev's inequality)}$$

(see [4], Proposition 1.3).

In many papers (see [1], [3]–[10]) it has been shown by the least action principle that problem (1.1) has at least one solution which minimizes φ on H_T^1 . When $F(t,\cdot)$ is convex for a.e. $t \in [0,T]$, Mawhin-Willem [4] studied the existence of a solution which minimizes φ on H_T^1 for problem (1.1). For non-convex potential cases, using the least action principle, the existence of a solution which minimizes φ on H_T^1 has been investigated by many people (see [3], [6]–[10] and the references therein). Inspired and motivated by the results in [3] and [8]–[10], we consider problem (1.1) with the potential $F(t,x) = F_1(t,x) + F_2(t,x)$. In our Theorem 2.1, it is assumed that

$$F_1(t,x) \geqslant G(x)|f(t)|, \tag{1.2}$$

where $G(x)$ is subconvex and $\nabla F_2(t,x)$ has sublinear growth. In Theorem 2.2, it is assumed that $F_1(t,x)$ satisfies (1.2) and $F_2(t,x)$ has subquadratic growth. In Theorem 2.3, it is assumed that $F_1(t,x)$ satisfies (1.2) and

$$F_2(t,x) \geqslant (h(t),x) + g(t), \tag{1.3}$$

where $h(t) \in L^1(0,T;\mathbb{R}^N)$ and $g(t) \in L^1(0,T;\mathbb{R})$. In Theorem 2.4, it is assumed that $F_1(t,x)$ is subconvex with subquadratic growth and $F_2(t,x)$ satisfies (1.3). In Theorem 2.5, it is assumed that $F_1(t,x) \to +\infty$ uniformly for a.e. $t \in [0,T]$, as $|x| \to \infty$ and $F_2(t,x)$ satisfies (1.3). By using the least action principle, we obtain that system (1.1) has at least one solution. Theorems 2.1–2.4 develop and generalize the corresponding results in [8] and [10] and Theorem 2.5 is a new result.

2. Main results and proofs

We first recall a definition due to Wu-Tang [9].

A function $G\colon \mathbb{R}^N \to \mathbb{R}$ is called (λ,μ)-subconvex if

$$G(\lambda(x+y)) \leqslant \mu(G(x)+G(y))$$

for some $\lambda,\mu > 0$ and all $x,y \in \mathbb{R}^N$. A function is called γ-subadditive if it is $(1,\gamma)$-subconvex. A function is called subadditive if it is 1-subadditive. The convex and subadditive functions are special cases of subconvex functions.

Theorem 2.1. *Suppose that $F(t,x) = F_1(t,x)+F_2(t,x)$, where F_1 and F_2 satisfy assumption* (A) *and the following conditions:*

(i) *there exist $M > 0$, $f \in L^1(0,T;\mathbb{R})$ and $G\colon \mathbb{R}^N \to \mathbb{R}$ which is continuous and (λ,μ)-subconvex for some $\lambda > \frac{1}{2}$ and $0 < \mu < 2\lambda^2$, such that*

$$F_1(t,x) \geqslant G(x)|f(t)|$$

for all $|x| \geqslant M$ and a.e. $t \in [0,T]$;

(ii) *there exist $p_1, p_2 \in L^1(0,T;\mathbb{R}^+)$ and $\alpha \in [0,1)$ such that*

$$|\nabla F_2(t,x)| \leqslant p_1(t)|x|^\alpha + p_2(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(iii)

$$\frac{1}{|x|^{2\alpha}}\left[\frac{G(\lambda x)}{\mu}\int_0^T |f(t)|\,\mathrm{d}t + \int_0^T F_2(t,x)\,\mathrm{d}t\right] \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1.1) *has at least one solution which minimizes φ on H_T^1.*

P r o o f. Let $\beta = \log_{2\lambda} 2\mu$. Then $\beta < 2$. In a way similar to Wu-Tang [9], by the (λ, μ)-subconvexity and continuity of $G(\cdot)$, one can obtain that there exists a constant $a_0 > 0$ such that

$$|f(t)|G(x) \leqslant a_0(2\mu|x|^\beta + 1)|f(t)|$$

for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^N$. Thus by assumption (A) and condition (i), we have

$$F_1(t, x) \geqslant G(x)|f(t)| + p(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$ and for $p \in L^1(0, T)$ given by

$$p(t) = -\max_{0 \leqslant |x| \leqslant M} a(|x|)b(t) - a_0(2\mu M^\beta + 1)|f(t)|.$$

It follows from (i) and Sobolev's inequality that

$$\begin{aligned} (2.1)\quad & \int_0^T F_1(t, u(t))\,\mathrm{d}t \\ & \geqslant \int_0^T G(u(t))|f(t)|\,\mathrm{d}t + \int_0^T p(t)\,\mathrm{d}t \\ & \geqslant \frac{1}{\mu}\int_0^T G(\lambda\bar{u})|f(t)|\,\mathrm{d}t - \int_0^T G(-\tilde{u}(t))|f(t)|\,\mathrm{d}t + \int_0^T p(t)\,\mathrm{d}t \\ & \geqslant \frac{1}{\mu}\int_0^T G(\lambda\bar{u})|f(t)|\,\mathrm{d}t - a_0(2\mu\|\tilde{u}\|_\infty^\beta + 1)\int_0^T |f(t)|\,\mathrm{d}t + \int_0^T p(t)\,\mathrm{d}t \\ & \geqslant \frac{1}{\mu}\int_0^T G(\lambda\bar{u})|f(t)|\,\mathrm{d}t - C_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{\beta/2} - C_2 + C_3 \end{aligned}$$

for all $u \in H_T^1$ and some constants C_1, C_2, C_3. It follows from assumption (ii) and Sobolev's inequality that

$$\begin{aligned} (2.2)\qquad & \Big|\int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})]\,\mathrm{d}t\Big| \\ & = \Big|\int_0^T\int_0^1 (\nabla F_2(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t))\,\mathrm{d}s\,\mathrm{d}t\Big| \\ & \leqslant \int_0^T\int_0^1 p_1(t)|\bar{u} + s\tilde{u}(t)|^\alpha|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T p_2(t)|\tilde{u}(t)|\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&\leqslant 2(|\bar{u}|^{\alpha} + \|\tilde{u}\|_{\infty}^{\alpha})\|\tilde{u}\|_{\infty} \int_0^T p_1(t)\,\mathrm{d}t + \|\tilde{u}\|_{\infty} \int_0^T p_2(t)\,\mathrm{d}t \\
&\leqslant \frac{3}{T}\|\tilde{u}\|_{\infty}^2 + \frac{T}{3}|\bar{u}|^{2\alpha}\Big(\int_0^T p_1(t)\,\mathrm{d}t\Big)^2 + 2\|\tilde{u}\|_{\infty}^{\alpha+1} \int_0^T p_1(t)\,\mathrm{d}t \\
&\quad + \|\tilde{u}\|_{\infty} \int_0^T p_2(t)\,\mathrm{d}t \\
&\leqslant \frac{1}{4}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + C_4|\bar{u}|^{2\alpha} + C_5\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{(\alpha+1)/2} \\
&\quad + C_6\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2}
\end{aligned}
$$

for all $u \in H_T^1$ and some positive constants C_4, C_5, C_6. It follows from (2.1) and (2.2) that

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T F_1(t,u(t))\,\mathrm{d}t + \int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]\,\mathrm{d}t \\
&\quad + \int_0^T F_2(t,\bar{u})\,\mathrm{d}t \\
&\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \frac{1}{\mu}\int_0^T G(\lambda\bar{u})|f(t)|\,\mathrm{d}t - C_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{\beta/2} - C_2 + C_3 \\
&\quad - \frac{1}{4}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t - C_4|\bar{u}|^{2\alpha} - C_5\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{(\alpha+1)/2} \\
&\quad - C_6\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} + \int_0^T F_2(t,\bar{u})\,\mathrm{d}t \\
&= \frac{1}{4}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t - C_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{\beta/2} - C_2 + C_3 \\
&\quad - C_5\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{(\alpha+1)/2} - C_6\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} \\
&\quad + |\bar{u}|^{2\alpha}\left(\frac{\int_0^T G(\lambda\bar{u})|f(t)|\,\mathrm{d}t}{\mu|\bar{u}|^{2\alpha}} + \frac{\int_0^T F_2(t,\bar{u})\,\mathrm{d}t}{|\bar{u}|^{2\alpha}} - C_4\right)
\end{aligned}
$$

for all $u \in H_T^1$, which implies that

$$
\varphi(u) \to +\infty
$$

as $\|u\| \to \infty$ by (iii), because $\alpha < 1$, $\beta < 2$, and

$$
\|u\| \to \infty \Longleftrightarrow \Big(|\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} \to \infty.
$$

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

Theorem 2.2. *Suppose that $F(t,x) = F_1(t,x) + F_2(t,x)$, where F_1 and F_2 satisfy assumption* (A) *and the following conditions:*

(i) *there exist $M > 0$, $f \in L^1(0,T;\mathbb{R})$ satisfying $\int_0^T |f(t)|\,\mathrm{d}t \neq 0$ and $G\colon \mathbb{R}^N \to \mathbb{R}$ which is continuous and (λ,μ)-subconvex for some $\lambda > \frac{1}{2}$ and $0 < \mu < 2\lambda^2$ such that*

$$F_1(t,x) \geqslant G(x)|f(t)|$$

for all $|x| \geqslant M$ and a.e. $t \in [0,T]$;

(ii) *there exist $\delta \in [0,2)$, $k_1 \in L^1(0,T;\mathbb{R}^+)$, and $k_2 \in L^1(0,T;\mathbb{R})$ such that*

$$|F_2(t,x)| \leqslant k_1(t)|x|^\delta + k_2(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(iii)

$$\frac{G(x)}{|x|^\delta} \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1.1) *has at least one solution which minimizes φ on H_T^1.*

Proof. By condition (ii) and Sobolev's inequality, one has

$$\begin{aligned}
(2.3)\qquad \left|\int_0^T F_2(t,u(t))\,\mathrm{d}t\right| &\leqslant \int_0^T [k_1(t)|u(t)|^\delta + k_2(t)]\,\mathrm{d}t \\
&\leqslant 2^\delta(|\bar{u}|^\delta + \|\tilde{u}\|_\infty^\delta)\int_0^T k_1(t)\,\mathrm{d}t + \int_0^T k_2(t)\,\mathrm{d}t \\
&\leqslant D_1\biggl(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\biggr)^{\delta/2} + D_2|\bar{u}|^\delta + D_3
\end{aligned}$$

for all $u \in H_T^1$ and some constants D_1, D_2, and D_3. It follows from (2.1) and (2.3) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T F(t,u(t))\,\mathrm{d}t \\
&\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t - C_1\biggl(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\biggr)^{\beta/2} - C_2 + C_3 \\
&\quad - D_1\biggl(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\biggr)^{\delta/2} \\
&\quad - D_3 + |\bar{u}|^\delta\biggl(\frac{G(\lambda\bar{u})}{\mu|\bar{u}|^\delta}\int_0^T |f(t)|\,\mathrm{d}t - D_2\biggr)
\end{aligned}$$

for all $u \in H_T^1$, which implies that

$$\varphi(u) \to +\infty$$

as $\|u\| \to \infty$ by (iii), because $\delta < 2$, $\beta < 2$, and

$$\|u\| \to \infty \Longleftrightarrow \left(|\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t\right)^{1/2} \to \infty.$$

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

Theorem 2.3. *Suppose that $F(t,x) = F_1(t,x) + F_2(t,x)$, where F_1 and F_2 satisfy assumption* (A) *and the following conditions:*

(i) *there exist $M > 0$, $f \in L^1(0,T;\mathbb{R})$ satisfying $\int_0^T |f(t)| \,\mathrm{d}t \neq 0$, and $G\colon \mathbb{R}^N \to \mathbb{R}$ which is continuous and (λ,μ)-subconvex for some $\lambda > \frac{1}{2}$ and $0 < \mu < 2\lambda^2$ such that*

$$F_1(t,x) \geqslant G(x)|f(t)|$$

for all $|x| \geqslant M$ and a.e. $t \in [0,T]$;

(ii) *there exist $g(t) \in L^1(0,T;\mathbb{R})$ and $h(t) \in L^1(0,T;\mathbb{R}^N)$ such that*

$$F_2(t,x) \geqslant (h(t),x) + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(iii)

$$\frac{G(x)}{|x|} \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1.1) *has at least one solution which minimizes φ on H_T^1.*

P r o o f. By condition (ii) and Sobolev's inequality, one has

$$\begin{aligned}(2.4)\quad \int_0^T F_2(t,u(t)) \,\mathrm{d}t &\geqslant \int_0^T [(h(t), \bar{u} + \tilde{u}(t)) + g(t)] \,\mathrm{d}t \\ &\geqslant -\|\tilde{u}\|_\infty \int_0^T |h(t)| \,\mathrm{d}t - |\bar{u}| \int_0^T |h(t)| \,\mathrm{d}t + \int_0^T g(t) \,\mathrm{d}t \\ &\geqslant -D_4 \left(\int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t\right)^{1/2} - D_5|\bar{u}| + D_6\end{aligned}$$

for all $u \in H_T^1$ and some constants D_4, D_5, and D_6. It follows from (2.1) and (2.4) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T F(t,u(t))\,\mathrm{d}t \\
&\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \frac{1}{\mu}\int_0^T G(\lambda\bar{u})|f(t)|\,\mathrm{d}t - C_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{\beta/2} \\
&\quad - C_2 + C_3 - D_4\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} - D_5|\bar{u}| + D_6 \\
&= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t - C_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{\beta/2} - C_2 + C_3 \\
&\quad - D_4\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} + D_6 + |\bar{u}|\Big(\frac{1}{\mu|\bar{u}|}\int_0^T G(\lambda\bar{u})|f(t)|\,\mathrm{d}t - D_5\Big)
\end{aligned}$$

for all $u \in H_T^1$, which implies that

$$\varphi(u) \to +\infty$$

as $\|u\| \to \infty$ by (iii), because $\beta < 2$ and

$$\|u\| \to \infty \Longleftrightarrow \Big(|\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} \to \infty.$$

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

R e m a r k 2.1. In [8], the case that G is subadditive is considered. Our theorems generalize that result to the case that G is (λ,μ)-subconvex by modifying some conditions. Moreover, the restriction about $F_2(t,x)$ is also modified. Especially, our Theorem 2.1 generalizes the restriction about $|\nabla F_2(t,x)|$ in [8].

Theorem 2.4. *Suppose that $F(t,x) = F_1(t,x) + F_2(t,x)$, where F_1 and F_2 satisfy assumption* (A) *and the following conditions:*

(i) *$F_1(t,x)$ is (λ,μ)-subconvex for a.e. $t \in [0,T]$ and there exist $\delta \in [0,2)$, $\theta \in L^1(0,T;\mathbb{R}^+)$ and $\omega \in L^1(0,T;\mathbb{R})$ such that*

$$F_1(t,x) \leqslant \theta(t)|x|^\delta + \omega(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(ii) *there exist $q(t) \in L^1(0,T;\mathbb{R})$ and $h(t) \in L^1(0,T;\mathbb{R}^N)$ such that*

$$F_2(t,x) \geqslant (h(t),x) + q(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(iii)

$$\frac{1}{|x|}\int_0^T F_1(t,x)\,\mathrm{d}t \to +\infty \quad \text{as } |x|\to\infty.$$

Then problem (1.1) *has at least one solution which minimizes* φ *on* H_T^1.

P r o o f. By the (λ,μ)-subconvexity of $F_1(t,\cdot)$, one has

$$\begin{aligned}
\int_0^T F_1(t,u(t))\,\mathrm{d}t &\geqslant \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,\mathrm{d}t - \int_0^T F_1(t,-\tilde{u}(t))\,\mathrm{d}t \\
&\geqslant \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,\mathrm{d}t - \int_0^T [\theta(t)|\tilde{u}(t)|^\delta + \omega(t)]\,\mathrm{d}t \\
&\geqslant \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,\mathrm{d}t - \|\tilde{u}\|_\infty^\delta\int_0^T \theta(t)\,\mathrm{d}t - \int_0^T \omega(t)\,\mathrm{d}t \\
&\geqslant \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,\mathrm{d}t - E_1\Bigl(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Bigr)^{\delta/2} - E_2
\end{aligned}\tag{2.5}$$

for all $u\in H_T^1$ and some constants E_1, E_2. By condition (ii), one has

$$\begin{aligned}
\int_0^T F_2(t,u(t))\,\mathrm{d}t &\geqslant \int_0^T [(h(t),\bar{u}+\tilde{u}(t))+q(t)]\,\mathrm{d}t \\
&= \int_0^T (h(t),\bar{u})\,\mathrm{d}t + \int_0^T (h(t),\tilde{u}(t))\,\mathrm{d}t + \int_0^T q(t)\,\mathrm{d}t \\
&\geqslant -|\bar{u}|\int_0^T |h(t)|\,\mathrm{d}t - \|\tilde{u}\|_\infty\int_0^T |h(t)|\,\mathrm{d}t + \int_0^T q(t)\,\mathrm{d}t \\
&\geqslant -E_3\Bigl(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Bigr)^{1/2} - E_4|\bar{u}| + E_5
\end{aligned}\tag{2.6}$$

for all $u\in H_T^1$ and some constants E_3, E_4, E_5. It follows from (2.5) and (2.6) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T F(t,u(t))\,\mathrm{d}t \\
&\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t - E_1\Bigl(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Bigr)^{\delta/2} - E_2 - E_3\Bigl(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Bigr)^{1/2} \\
&\quad + E_5 + |\bar{u}|\left(\frac{\int_0^T F_1(t,\lambda\bar{u})\,\mathrm{d}t}{\mu|\bar{u}|} - E_4\right)
\end{aligned}$$

for all $u\in H_T^1$, which implies that

$$\varphi(u)\to+\infty$$

as $\|u\| \to \infty$ by (iii), because $\delta < 2$ and

$$\|u\| \to \infty \Longleftrightarrow \left(|\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t\right)^{1/2} \to \infty.$$

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

Corollary 2.1. *Suppose that $F(t,x) = F_1(t,x) + F_2(t,x)$, where F_1 and F_2 satisfy assumption* (A) *and the following conditions:*

(i) *$F_1(t,x)$ is (λ,μ)-subconvex for a.e. $t \in [0,T]$, where $\lambda > \frac{1}{2}$ and $\mu < 2\lambda^2$;*

(ii) *there exist $q(t) \in L^1(0,T;\mathbb{R})$ and $h(t) \in L^1(0,T;\mathbb{R}^N)$ such that*

$$F_2(t,x) \geqslant (h(t),x) + q(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(iii)

$$\frac{1}{|x|} \int_0^T F_1(t,x) \,\mathrm{d}t \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1.1) *has at least one solution which minimizes φ on H_T^1.*

P r o o f. Let $\beta = \log_{2\lambda} 2\mu$. Then $\beta < 2$. In a way similar to Wu-Tang [9], by the (λ,μ)-subconvexity of $F_1(t,\cdot)$ and assumption (A) one can prove that

$$F_1(t,x) \leqslant c_0(2\mu|x|^\beta + 1)b(t)$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$, where $\beta < 2$, $c_0 = \max\limits_{0 \leqslant s \leqslant 1} a(s)$. Thus by Theorem 2.4, the proof is completed. □

R e m a r k 2.2. In [10], the case with $\int_0^T h(t)\,\mathrm{d}t = 0$ is considered. Our Theorem 2.4 and Corollary 2.1 prove the conclusion holds as $\int_0^T h(t)\,\mathrm{d}t = 0$ is omitted by modifying some conditions.

Lemma A (see [7]). *Assume that F satisfies assumption* (A) *and*

$$F(t,x) \to +\infty \quad \text{as } |x| \to \infty$$

uniformly for a.e. $t \in [0,T]$. Then there exist $\eta(t) \in L^1(0,T;\mathbb{R})$ and a subadditive function $G\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$G(x) + \eta(t) \leqslant F(t,x)$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$ *and*

$$G(x) \to +\infty \quad \textit{as } |x| \to \infty$$

and

$$0 \leqslant G(x) \leqslant |x| + 1$$

for all $x \in \mathbb{R}^N$.

Theorem 2.5. *Suppose that* $F(t,x) = F_1(t,x) + F_2(t,x)$, *where* F_1 *and* F_2 *satisfy assumption* (A) *and the following conditions:*

(i)
$$F_1(t,x) \to +\infty \quad \textit{as } |x| \to \infty$$
uniformly for a.e. $t \in [0,T]$;

(ii) *there exist* $v(t) \in L^1(0,T;\mathbb{R})$ *and* $h(t) \in L^1(0,T;\mathbb{R}^N)$ *with* $\int_0^T h(t)\,\mathrm{d}t = 0$ *such that*
$$F_2(t,x) \geqslant (h(t),x) + v(t)$$
for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$.

Then problem (1.1) *has at least one solution which minimizes* φ *on* H_T^1.

P r o o f. By condition (ii) and Sobolev's inequality one has

$$\begin{aligned}
\int_0^T F_2(t,u(t))\,\mathrm{d}t &\geqslant \int_0^T [(h(t),\bar{u}+\tilde{u}(t)) + v(t)]\,\mathrm{d}t \\
&\geqslant -\|\tilde{u}\|_\infty \int_0^T |h(t)|\,\mathrm{d}t + \int_0^T v(t)\,\mathrm{d}t \\
&\geqslant -H_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} + H_2
\end{aligned} \tag{2.7}$$

for all $u \in H_T^1$ and some constants H_1 and H_2. By Lemma A, (2.7), and Sobolev's inequality one has

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T F_1(t,u(t))\,\mathrm{d}t + \int_0^T F_2(t,u(t))\,\mathrm{d}t \\
&\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T G(u(t))\,\mathrm{d}t + \int_0^T \eta(t)\,\mathrm{d}t - H_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} + H_2 \\
&\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T G(\bar{u})\,\mathrm{d}t - \int_0^T G(-\tilde{u}(t))\,\mathrm{d}t \\
&\quad + \int_0^T \eta(t)\,\mathrm{d}t - H_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} + H_2 \\
&\geqslant \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + TG(\bar{u}) - T(\|\tilde{u}\|_\infty + 1) + H_3 - H_1\Big(\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} + H_2
\end{aligned}$$

for all $u \in H_T^1$ and some constant H_3. From the coercivity of G we obtain

$$\varphi(u) \to +\infty \quad \text{as } \|u\| \to \infty,$$

because

$$\|u\| \to \infty \Longleftrightarrow \left(|\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t\right)^{1/2} \to \infty.$$

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

Remark 2.3. In [2], A. Fonda and J.-P. Gossez obtained an abstract theorem in which it is necessary to seek a functional $\hat{b}$. However, we find that in general it is difficult to find the functional $\hat{b}$ satisfying the conditions of the theorem. It is therefore not very suitable for practical use. In our conclusions, for the second order Hamiltonian systems, we start from the property of F itself to seek suitable restrictive conditions so that the necessity of finding $\hat{b}$ is avoided. This is easier.

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