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DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS WITH MIXED DIRICHLET-NEUMANN BOUNDARY CONDITIONS*

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Abstract. The paper is devoted to the analysis of the discontinuous Galerkin finite element method (DGFEM) applied to the space semidiscretization of a nonlinear nonstationary convection-diffusion problem with mixed Dirichlet-Neumann boundary conditions. General nonconforming meshes are used and the NIPG, IIPG and SIPG versions of the discretization of diffusion terms are considered. The main attention is paid to the impact of the Neumann boundary condition prescribed on a part of the boundary on the truncation error in the approximation of the nonlinear convective terms. The estimate of this error allows to analyse the error estimate of the method. The results obtained represent the completion and extension of the analysis from V. Dolejší, M. Feistauer, Numer. Funct. Anal. Optim. *26* (2005), 349–383, where the truncation error in the approximation of the nonlinear convection terms was proved only in the case when the Dirichlet boundary condition on the whole boundary of the computational domain was considered.

Keywords: nonlinear convection-diffusion equation, mixed Dirichlet-Neumann conditions, discontinuous Galerkin finite element method, method of lines, nonconforming meshes, NIPG, SIPG, IIPG versions

MSC 2010: 65M12, 65M15, 65M60

1. Introduction

During last ten years, the *discontinuous Galerkin* (DG) method has become a very popular technique for the solution of partial differential equations. DG method is based on piecewise polynomial discontinuous approximations. For a survey on DG methods, see [4] or [5]. There exist several DG techniques for the discretization of linear elliptic boundary value problems (see [2]). Among them the *interior*

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penalty Galerkin (IPG) methods are very popular. They are based on the so-called primal formulation leading, for example, to the SIPG (symmetric interior penalty Galerkin), NIPG (nonsymmetric interior penalty Galerkin), and IIPG (incomplete interior penalty Galerkin) methods. These techniques were applied to elliptic problems (see, e.g., [15], [20], [25]), parabolic problems (e.g., [1], [19]), and to the convection-diffusion problems with linear convection (e.g., [16], [6], [21], [22], [24]).

In our recent papers [7], [10], [9] we applied the IPG methods to a scalar nonstationary convection-diffusion equation with nonlinear Lipschitz continuous convective terms. This equation represents a model problem for the solution of the system of the compressible Navier-Stokes equations which describes the flow of viscous compressible fluids. In order to deal with nonlinear convective terms, the concept of the so-called *numerical flux* was employed for the approximation of "convective boundary" integrals.

Based on some assumptions on the numerical flux, on the regularity of the exact solution, and the regularity of computational meshes we derived a priori error estimates. In papers [7] and [9], in the formulation of the initial-boundary value problem the mixed Dirichlet-Neumann boundary conditions were considered. However, the reader can recognize that in the basic paper [7], the proof of the truncation error in nonlinear convective terms was carried out in the case when the Dirichlet boundary condition is prescribed on the whole boundary of the computational domain. The technique developed does not allow the straightforward extension of our results to problem, where the Neumann boundary condition is prescribed on a part of the boundary.

In this paper we remove this drawback and analyze the IPG methods for nonlinear convection-diffusion problems with mixed Dirichlet-Neumann boundary conditions. In Section 2 the initial-boundary value problem is formulated. In Section 3 we introduce its discretization by IPG methods. Section 4 contains the numerical analysis of these methods. First, we recall some results from our earlier papers and extend them by replacing a rather limiting assumption on the computational mesh, used e.g. in [7] in the case of nonconforming triangulations with hanging nodes. The main attention is paid to the consistency analysis of the convection form under the assumption that the mixed Dirichlet-Neumann boundary conditions are used. Finally, a priori error estimates are established. Several concluding remarks are given in Section 5.

2. Continuous problem

Let us consider the nonstationary nonlinear convection-diffusion problem: Find $u\colon Q_T = \Omega \times (0,T) \to \mathbb{R}$ such that

$$
\text{(2.1)} \qquad
\begin{aligned}
&\text{a)} \quad \frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} = \varepsilon \Delta u + g \quad \text{in } Q_T,\\
&\text{b)} \quad u|_{\partial\Omega_D \times (0,T)} = u_D,\\
&\text{c)} \quad \varepsilon \frac{\partial u}{\partial n}\Big|_{\partial\Omega_N \times (0,T)} = g_N,\\
&\text{d)} \quad u(x,0) = u^0(x), \quad x \in \Omega.
\end{aligned}
$$

We assume that $\Omega \subset \mathbb{R}^d$, $d = 2,3$, is a bounded polygonal (if $d = 2$) or polyhedral (if $d = 3$) domain with Lipschitz-continuous boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$ and $T > 0$. The diffusion coefficient $\varepsilon > 0$ is a given constant, $g\colon Q_T \to \mathbb{R}$, $u_D\colon \partial\Omega_D \times (0,T) \to \mathbb{R}$, $g_N\colon \partial\Omega_N \times (0,T) \to \mathbb{R}$, and $u^0\colon \Omega \to \mathbb{R}$ are given functions, $f_s \in C^1(\mathbb{R})$, $s = 1,\ldots,d$, are prescribed inviscid fluxes. Without loss of generality we assume that $f_s(0) = 0, s = 1,\ldots,d$. In this paper we consider the case, when the $(d-1)$-dimensional measures of $\partial\Omega_D$ and $\partial\Omega_N$ are positive.

Using the techniques treated for example in [23], it is possible to show that under some assumptions on the data there exists a unique weak solution of problem (2.1). In the theory of error estimates we shall assume a suitable regularity of the solution (specified later in (4.1)).

We shall use the standard notation of function spaces. If $p \in [1,\infty]$, $k \in \{1,2,\ldots\}$ and $G \subset \mathbb{R}^d$, $d = 2,3$, is a bounded domain with Lipschitz-continuous boundary, then we shall use the notation $L^p(G)$ and $L^p(\partial G)$ for the Lebesgue spaces over G and ∂G, respectively. The symbol $H^k(G)$ $(= W^{k,2}(G))$ will denote the Sobolev spaces. By $|\cdot|_{H^k(G)}$ we denote the seminorm in $H^k(G)$. The symbol $L^p(0,T;X)$ will denote the Bochner space of functions in the interval $(0,T)$ with values in a Banach space X. Further, $C([0,T];X)$ will denote the space of continuous mappings of the interval $[0,T]$ into X. (See, e.g., [17].)

3. Discretization

3.1. Finite element mesh

Let $\mathcal{T}_h$ be a partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed d-dimensional simplices (triangles for $d = 2$ and tetrahedra for $d = 3$) with pairwise disjoint interiors. We shall call $\mathcal{T}_h$ a triangulation of Ω. We do not require the standard properties of $\mathcal{T}_h$ used in the finite element method. This means that we admit the so-called hanging nodes (and in 3D also hanging edges). In our further considerations we use the following notation. For an element $K \in \mathcal{T}_h$ we set $h_K = \operatorname{diam}(K)$, $h = \max_{K \in \mathcal{T}_h} h_K$. By ϱ_K we denote the radius of the largest d-dimensional ball inscribed into K and by $|K|$ we denote the d-dimensional Lebesgue measure of K.

Let $K, K' \in \mathcal{T}_h$. We say that K and K' are *neighbours*, if the set $\partial K \cap \partial K'$ has positive $(d-1)$-dimensional measure. We say that $\Gamma \subset K$ is a *face* of K, if it is a maximal connected nonempty open subset either of $\partial K \cap \partial K'$, where K' is a neighbour of K, or of $\partial K \cap \partial \Omega$. By $\mathcal{F}_h$ we denote the system of all faces of all elements $K \in \mathcal{T}_h$. Further, we define the set of all inner faces by

$$\mathcal{F}_h^I = \{\Gamma \in \mathcal{F}_h \colon \Gamma \subset \Omega\}, \tag{3.1}$$

and the sets of all Dirichlet and Neumann boundary faces by

$$\mathcal{F}_h^D = \{\Gamma \in \mathcal{F}_h \colon \Gamma \subset \partial\Omega_D\} \quad \text{and} \quad \mathcal{F}_h^N = \{\Gamma \in \mathcal{F}_h \colon \Gamma \subset \partial\Omega_N\}. \tag{3.2}$$

Obviously, $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D \cup \mathcal{F}_h^N$. We denote $\mathcal{F}_h^{ID} = \mathcal{F}_h^I \cup \mathcal{F}_h^D$ and $\mathcal{F}_h^B = \mathcal{F}_h^D \cup \mathcal{F}_h^N$.

For each $\Gamma \in \mathcal{F}_h$ we define a unit normal vector $\boldsymbol{n}_\Gamma$. We assume that for $\Gamma \in \mathcal{F}_h^B$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$. For each face $\Gamma \in \mathcal{F}_h^I$ the orientation of $\boldsymbol{n}_\Gamma$ is arbitrary but fixed. See Fig. 1. Finally, by $d(\Gamma)$ we denote the diameter of $\Gamma \in \mathcal{F}_h$.

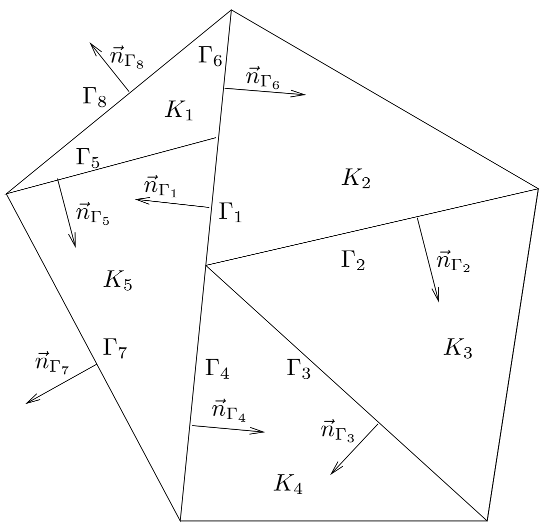

Figure 1. Example of elements K_l, $l = 1, \ldots, 5$, and faces Γ_l, $l = 1, \ldots, 7$, with the corresponding normals $\boldsymbol{n}_{\Gamma_l}$.

3.2. Spaces of discontinuous functions

Over a triangulation $\mathcal{T}_h$ we define the *broken Sobolev spaces*

$$H^k(\Omega,\mathcal{T}_h)=\{v\colon\ v|_K\in H^k(K)\ \forall\, K\in\mathcal{T}_h\} \tag{3.3}$$

equipped with the seminorm

$$|v|_{H^k(\Omega,\mathcal{T}_h)}=\Big(\sum_{K\in\mathcal{T}_h}|v|^2_{H^k(K)}\Big)^{1/2}. \tag{3.4}$$

For each face $\Gamma\in\mathcal{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)},K_\Gamma^{(R)}\in\mathcal{T}_h$ such that $\Gamma\subset\partial K_\Gamma^{(L)}\cap\partial K_\Gamma^{(R)}$. We use the convention that $\boldsymbol{n}_\Gamma$ is the outer normal to the element $K_\Gamma^{(L)}$ and the inner normal to the element $K_\Gamma^{(R)}$, see Fig. 2. For $v\in H^1(\Omega,\mathcal{T}_h)$ and $\Gamma\in\mathcal{F}_h^I$ we introduce the following notation:

$$\begin{aligned} v|_\Gamma^{(L)}&=\text{the trace of } v|_{K_\Gamma^{(L)}}\text{ on }\Gamma,\\ v|_\Gamma^{(R)}&=\text{the trace of } v|_{K_\Gamma^{(R)}}\text{ on }\Gamma,\\ \langle v\rangle_\Gamma&=\frac12\big(v|_\Gamma^{(L)}+v|_\Gamma^{(R)}\big),\\ [v]_\Gamma&=v|_\Gamma^{(L)}-v|_\Gamma^{(R)}. \end{aligned} \tag{3.5}$$

The value $[v]_\Gamma$ depends on the orientation of $\boldsymbol{n}_\Gamma$, but the value $[v]_\Gamma\boldsymbol{n}_\Gamma$ is independent of this orientation.

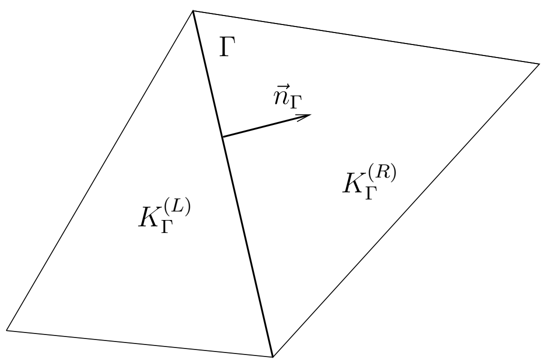

Figure 2. Interior face Γ, elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$, and the orientation of $\boldsymbol{n}_\Gamma$.

Now, let $\Gamma\in\mathcal{F}_h^B$ and let $K_\Gamma^{(L)}\in\mathcal{T}_h$ be such an element that $\Gamma\subset K_\Gamma^{(L)}\cap\partial\Omega$. For $v\in H^1(\Omega,\mathcal{T}_h)$ we set

$$v_\Gamma=v|_\Gamma^{(L)}=v|_\Gamma^{(R)}=\text{ the trace of } v|_{K_\Gamma^{(L)}}\text{ on }\Gamma \tag{3.6}$$

(i.e. we define $v|_\Gamma^{(R)}$ by extrapolation).

If $[\cdot]_\Gamma$ and $\langle\cdot\rangle_\Gamma$ appear in an integral $\int_\Gamma \ldots \,\mathrm{d}S$, where $\Gamma \in \mathcal{F}_h$, we omit the subscript Γ and write simply $[\cdot]$ and $\langle\cdot\rangle$.

As we have already mentioned in Introduction, the DGFEM is based on the use of discontinuous piecewise polynomial approximations. Let $p \geqslant 1$ be an integer. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$S_{hp} = \{v\colon\ v|_K \in P^p(K),\ \forall\, K \in \mathcal{T}_h\}, \tag{3.7}$$

where $P^p(K)$ denotes the space of all polynomials on K of degree $\leqslant p$.

3.3. Discontinuous Galerkin space semidiscretization

Now we shall introduce a space DG semidiscretization of problem (2.1). To this end, we denote by $(\cdot,\cdot)$ the scalar product in the space $L^2(\Omega)$, i.e.

$$(u,\varphi) = \int_\Omega u\varphi\,\mathrm{d}x, \quad u,\varphi \in L^2(\Omega), \tag{3.8}$$

and define the following forms for functions $u,\varphi \in H^2(\Omega,\mathcal{T}_h)$:

$$\begin{aligned} a_h(u,\varphi) = {} & \sum_{K\in\mathcal{T}_h}\int_K \varepsilon\nabla u\cdot\nabla\varphi\,\mathrm{d}x \\ & - \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \varepsilon\langle\nabla u\rangle\cdot\boldsymbol{n}[\varphi]\,\mathrm{d}S - \theta\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \varepsilon\langle\nabla\varphi\rangle\cdot\boldsymbol{n}[u]\,\mathrm{d}S \\ & - \sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \varepsilon\nabla u\cdot\boldsymbol{n}\varphi\,\mathrm{d}S - \theta\sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \varepsilon\nabla\varphi\cdot\boldsymbol{n}u\,\mathrm{d}S, \end{aligned} \tag{3.9}$$

$$J_h(u,\varphi) = \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma[u][\varphi]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \sigma u\varphi\,\mathrm{d}S, \tag{3.10}$$

$$\begin{aligned} l_h(\varphi)(t) = {} & \int_\Omega g(t)\varphi\,\mathrm{d}x \\ & - \theta\sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \varepsilon\nabla\varphi\cdot\boldsymbol{n}u_D(t)\,\mathrm{d}S + \varepsilon\sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \sigma u_D(t)\varphi\,\mathrm{d}S, \end{aligned} \tag{3.11}$$

$$\begin{aligned} b_h(u,\varphi) = {} & -\sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^d f_s(u)\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \\ & + \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma H\big(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma\big)[\varphi]_\Gamma\,\mathrm{d}S \\ & + \sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma H\big(u|_\Gamma^{(L)}, u|_\Gamma^{(L)}, \boldsymbol{n}_\Gamma\big)\varphi|_\Gamma^{(L)}\,\mathrm{d}S. \end{aligned} \tag{3.12}$$

The forms a_h and J_h represent here the discretization of the diffusion term and the interior and boundary penalty. We choose $\theta = 1$ or $\theta = 0$ or $\theta = -1$ and speak of the SIPG or IIPG or NIPG, respectively, version of the discretization of the diffusion terms. By σ we denote a suitable positive weight, which will be defined in Section 4. The form b_h approximates the convective terms with the aid of a numerical flux $H(u, v, \boldsymbol{n})$. We assume that H has the following properties:

Assumptions (H):

(H1) $H(u, v, \boldsymbol{n})$ is defined in $\mathbb{R}^2 \times B_1$, where $B_1 = \{\boldsymbol{n} \in \mathbb{R}^d \colon |\boldsymbol{n}| = 1\}$, and is *Lipschitz-continuous* with respect to u, v:

$$|H(u, v, \boldsymbol{n}) - H(u^*, v^*, \boldsymbol{n})| \leqslant L_H(|u - u^*| + |v - v^*|), \quad u, v, u^*, v^* \in \mathbb{R},\ \boldsymbol{n} \in B_1. \tag{3.13}$$

(H2) $H(u, v, \boldsymbol{n})$ is *consistent*:

$$H(u, u, \boldsymbol{n}) = \sum_{s=1}^{d} f_s(u) n_s, \quad u \in \mathbb{R},\ \boldsymbol{n} = (n_1, \ldots, n_d) \in B_1. \tag{3.14}$$

(H3) $H(u, v, \boldsymbol{n})$ is *conservative*:

$$H(u, v, \boldsymbol{n}) = -H(v, u, -\boldsymbol{n}), \quad u, v \in \mathbb{R},\ \boldsymbol{n} \in B_1. \tag{3.15}$$

By virtue of assumptions (H1) and (H2), $L_f = 2L_H$ is a Lipschitz-continuity constant of the functions f_s, $s = 1, \ldots, d$.

Similarly to [7] or [10] we can show that a sufficiently regular exact solution u of problem (2.1) satisfies the identity

$$\Big(\frac{\partial u(t)}{\partial t}, \varphi_h\Big) + b_h(u(t), \varphi_h) + a_h(u(t), \varphi_h) + \varepsilon J_h(u(t), \varphi_h) = l_h(\varphi_h)(t) \quad \text{for all } \varphi_h \in S_{hp} \text{ and for a.e. } t \in (0, T). \tag{3.16}$$

On the basis of (3.16) we introduce the *discrete problem*: We say that u_h is a DGFE solution of the convection-diffusion problem (2.1), if

(3.17)

a) $u_h \in C^1([0, T]; S_{hp})$,

b) $$\Big(\frac{\partial u_h(t)}{\partial t}, \varphi_h\Big) + b_h(u_h(t), \varphi_h) + a_h(u_h(t), \varphi_h) + \varepsilon J_h(u_h(t), \varphi_h) = l_h(\varphi_h)(t) \quad \forall\, \varphi_h \in S_{hp},\ \forall\, t \in (0, T),$$

c) $u_h(0) = u_h^0$,

where $u_h^0 \in S_{hp}$ is the $L^2(\Omega)$-projection of the initial condition u^0 onto S_{hp}, i.e. a function defined by

$$(u_h^0 - u^0, \varphi_h) = 0 \quad \forall \varphi_h \in S_{hp}. \tag{3.18}$$

4. Error analysis

4.1. Assumptions

We shall assume that the weak solution u of problem (2.1) is regular, namely

$$u \in L^2(0,T;H^{r+1}(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0,T;H^r(\Omega)), \tag{4.1}$$

where $r \geqslant 1$ is an integer. Then

$$u \in C([0,T];H^r(\Omega)) \tag{4.2}$$

and it is possible to show that u satisfies the identity (3.16).

In our further consideration we shall use the notation

$$\mu = \min(p,r). \tag{4.3}$$

We shall consider a *regular system* $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, of triangulations of the domain Ω. This means that there exists a constant $C_T > 0$ such that

$$\frac{h_K}{\varrho_K} \leqslant C_T \quad \forall K \in \mathcal{T}_h \ \forall h \in (0,h_0). \tag{4.4}$$

4.2. Some auxiliary results

In the analysis of the DGFEM we use the following important tools (see, e.g., [7]).

- *Multiplicative trace inequality*: There exists a constant $C_M > 0$ independent of v, h, and K such that

$$\|v\|^2_{L^2(\partial K)} \leqslant C_M(\|v\|_{L^2(K)}|v|_{H^1(K)} + h_K^{-1}\|v\|^2_{L^2(K)}), \\ K \in \mathcal{T}_h, \ v \in H^1(K), \ h \in (0,h_0). \tag{4.5}$$

- *Inverse inequality*: There exists a constant $C_I > 0$ independent of v, h, and K such that

$$|v|_{H^1(K)} \leqslant C_I h_K^{-1}\|v\|_{L^2(K)}, \quad v \in P^p(K), \ K \in \mathcal{T}_h, \ h \in (0,h_0). \tag{4.6}$$

Now, for $v \in L^2(\Omega)$ we denote by $\Pi_{hp} v$ the $L^2(\Omega)$-projection of v to S_{hp}:

$$\Pi_{hp} v \in S_{hp}, \quad (\Pi_{hp} v - v, \varphi_h) = 0 \quad \forall \varphi_h \in S_{hp}. \tag{4.7}$$

It is possible to show (cf., e.g., [14, Lemma 4.1]) that the operator Π_{hp} has the following property: There exists a constant $C_A > 0$ independent of h, K, v such that

$$\begin{aligned} \|\Pi_{hp} v - v\|_{L^2(K)} &\leqslant C_A h_K^{\mu+1} |v|_{H^{\mu+1}(K)}, \\ |\Pi_{hp} v - v|_{H^1(K)} &\leqslant C_A h_K^{\mu} |v|_{H^{\mu+1}(K)}, \\ |\Pi_{hp} v - v|_{H^2(K)} &\leqslant C_A h_K^{\mu-1} |v|_{H^{\mu+1}(K)} \end{aligned} \tag{4.8}$$

for all $v \in H^{\mu+1}(K)$, $K \in \mathcal{T}_h$ and $h \in (0, h_0)$.

Because of our further considerations we introduce the norm in the space $H^1(\Omega, \mathcal{T}_h)$ defined by

$$\|w\|_{\mathrm{DG}} = \Big(\frac{1}{2}(|w|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(w,w))\Big)^{1/2}. \tag{4.9}$$

An important step in the analysis of error estimates is the *coercivity* of the form

$$A_h(u,v) = a_h(u,v) + \varepsilon J_h(u,v), \quad u, v \in H^2(\Omega, \mathcal{T}_h), \tag{4.10}$$

which reads

$$A_h(\varphi_h, \varphi_h) \geqslant \varepsilon \|\varphi\|^2_{\mathrm{DG}}, \quad \varphi \in S_{hp}, \ \ h \in (0, h_0). \tag{4.11}$$

The validity of estimate (4.11) depends on the definition of the weight σ in the form J_h.

(I) Conforming mesh $\mathcal{T}_h$. We assume that the mesh $\mathcal{T}_h$ has the standard properties from the finite element method (cf., e.g. [3]): if $K, K' \in \mathcal{T}_h$, $K \neq K'$, then $K \cap K' = \emptyset$ or $K \cap K'$ is a common vertex or $K \cap K'$ is a common edge (or $K \cap K'$ is a common face in the case $d = 3$) of K and K'. In this case one usually sets

$$\sigma|_\Gamma = \frac{C_W}{d(\Gamma)}, \quad \Gamma \in \mathcal{F}_h. \tag{4.12}$$

It was shown in [7] and [11] that (4.11) holds under the following choice of the constant C_W:

$$\begin{aligned} &C_W > 0 \quad (\text{e.g. } C_W = 1) && \text{for NIPG}, \\ &C_W \geqslant 4C_M(1 + C_I) && \text{for SIPG}, \\ &C_W \geqslant 2C_M(1 + C_I) && \text{for IIPG}, \end{aligned} \tag{4.13}$$

where C_M and C_I are constants from (4.5) and (4.6), respectively.

(II) Nonconforming mesh $\mathcal{T}_h$. In this case $\mathcal{T}_h$ is formed by closed triangles with mutually disjoint interiors with hanging nodes and/or hanging edges (in 3D) in general. It is suitable to define the weight σ by

$$\begin{aligned}\sigma|_\Gamma &= \frac{2C_W}{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}, \quad && \Gamma \in \mathcal{F}_h^I, \\ \sigma|_\Gamma &= \frac{C_W}{h_{K_\Gamma^{(L)}}}, && \Gamma \in \mathcal{F}_h^B,\end{aligned} \tag{4.14}$$

and consider the *local quasi-uniformity* of the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$: there exists a constant C_H such that

$$h_K \leqslant C_H h_{K'}, \quad K, K' \in \mathcal{T}_h \text{ are neighbours, } h \in (0, h_0). \tag{4.15}$$

(Obviously $C_H \geqslant 1$.) Then (4.11) holds provided

$$\begin{aligned}&C_W > 0 \text{ (e.g. } C_W = 1) && \text{for NIPG,} \\ &C_W \geqslant 2C_M(1 + C_I)(1 + C_H) && \text{for SIPG,} \\ &C_W \geqslant C_M(1 + C_I)(1 + C_H) && \text{for IIPG.}\end{aligned} \tag{4.16}$$

See [11].

On the basis of the above considerations, using the technique from [7] and [13], we can prove that there exist positive constants C_a and C_J such that for any $v \in H^{\mu+1}(\Omega)$, $\varphi_h \in S_{hp}$ and $h \in (0, h_0)$ we have

$$|a_h(\Pi_{hp}v - v, \varphi_h)| \leqslant \varepsilon C_a h^\mu |v|_{H^{\mu+1}(\Omega)} \|\varphi_h\|_{\mathrm{DG}}, \tag{4.17}$$

$$J_h(\Pi_{hp}v - v, \varphi_h) \leqslant J_h(\Pi_{hp}v - v, \Pi_{hp}v - v)^{1/2} J_h(\varphi_h, \varphi_h)^{1/2}, \tag{4.18}$$

$$|J_h(\Pi_{hp}v - v, \, \Pi_{hp}v - v)| \leqslant C_J h^{2\mu} |v|^2_{H^{\mu+1}(\Omega)}. \tag{4.19}$$

In what follows, for the exact solution u of problem (2.1) and the approximate solution u_h defined by (3.17) we set

$$\eta = u - \Pi_{hp}u, \quad \xi = \Pi_{hp}u - u_h. \tag{4.20}$$

Then the error $e_h = u - u_h$ of the method (3.17) can be written as

$$e_h = \eta + \xi. \tag{4.21}$$

Under the regularity assumptions (4.1), in virtue of (4.8) and (4.19) we have

$$\|\eta(t)\|_{L^2(\Omega)} \leqslant Ch^{\mu+1}|u(t)|_{H^{\mu+1}(\Omega)}, \tag{4.22}$$
$$\|\eta(t)\|_{\mathrm{DG}} \leqslant Ch^{\mu}|u(t)|_{H^{\mu+1}(\Omega)}, \quad t \in [0,T], \;\; h \in (0,h_0),$$

$$\Bigl\|\frac{\partial\eta(t)}{\partial t}\Bigr\|_{L^2(\Omega)} \leqslant Ch^{\mu}\Bigl|\frac{\partial u(t)}{\partial t}\Bigr|_{H^{\mu}(\Omega)}, \quad \text{a.e. } t \in (0,T), \;\; h \in (0,h_0). \tag{4.23}$$

4.3. Consistency analysis of the convection form b_h

We shall be concerned with the consistency of the form b_h in the case of a nonempty Neumann part $\partial\Omega_N$ of the boundary $\partial\Omega$. We start with several auxiliary results.

Lemma 1. *There exists a vector-valued function $\boldsymbol{\varphi} \in [W^{1,\infty}(\Omega)]^d$ such that*

$$\boldsymbol{\varphi}\cdot\boldsymbol{n} \geqslant 1 \quad \text{on } \partial\Omega. \tag{4.24}$$

Proof. By [17] or [18], it follows from the Lipschitz-continuity of $\partial\Omega$ that there exist numbers $\alpha, \beta > 0$ and Cartesian coordinate systems

$$X_r = (x_{r,1},\ldots,x_{r,d-1},x_{r,d})^T = (x_r', x_{r,d})^T,$$

Lipschitz-continuous functions

$$a_r\colon \Delta_r = \{x_r' = (x_{r,1},\ldots,x_{r,d-1})^T\colon |x_{r,i}| < \alpha, \; i = 1,\ldots,d-1\} \to \mathbb{R} \tag{4.25}$$

with a Lipschitz constant $L > 0$, and orthogonal transformations $A_r\colon \mathbb{R}^d \to \mathbb{R}^d$, $r = 1,\ldots,m$, such that

$$\forall\, x \in \partial\Omega \;\; \exists\, r \in \{1,\ldots,m\} \;\; \exists\, x_r' \in \Delta_r\colon \; x = A_r^{-1}(x_r', a_r(x_r')), \tag{4.26}$$

and under the notation

$$\hat V_r^+ = \{(x_r', x_{rd})\colon \; a_r(x_r') < x_{rd} < a_r(x_r') + \beta, \; x_r' \in \Delta_r\}, \tag{4.27}$$
$$\hat V_r^- = \{(x_r', x_{rd})\colon \; a_r(x_r') - \beta < x_{rd} < a_r(x_r'), \; x_r' \in \Delta_r\},$$
$$\hat\Lambda_r = \{(x_r', x_{rd})\colon \; x_{rd} = a_r(x_r'), \; x_r' \in \Delta_r\},$$

we have

$$\hat V_r^+ \subset A_r(\Omega), \quad \hat\Lambda_r \subset A_r(\partial\Omega), \quad \hat V_r^- \subset A_r(\mathbb{R}^d \setminus \overline{\Omega}), \quad \partial\Omega \subset \bigcup_{r=1}^m U_r, \tag{4.28}$$

where the sets U_r are defined by the relations

$$\hat{U}_r = \hat{V}_r^+ \cup \hat{\Lambda}_r \cup \hat{V}_r^-, \quad U_r = A_r^{-1}(\hat{U}_r). \tag{4.29}$$

The mappings A_r can be written in the form

$$A_r(x) = \mathbb{Q}_r x + x_r^0, \quad x \in \mathbb{R}^d, \tag{4.30}$$

where $x_r^0 \in \mathbb{R}^d$ and $\mathbb{Q}_r$ are orthogonal $d \times d$ matrices, i.e. $\mathbb{Q}_r \mathbb{Q}_r^T$ is the unit matrix. Then the transformation of d-dimensional vectors $x \in \mathbb{R}^d$ reads

$$y \in \mathbb{R} \to \mathbb{Q}_r y \in \mathbb{R}. \tag{4.31}$$

The sets U_r are open. There exists an open set U_0 such that

$$\overline{U}_0 \subset \Omega, \quad \overline{\Omega} \subset \bigcup_{r=0}^{m} U_r. \tag{4.32}$$

By the theorem on partition of unity ([17]), there exist functions $\varphi_r \in C_0^\infty(U_r)$, $r = 0, \ldots, m$, such that $0 \leqslant \varphi_r \leqslant 1$ and

$$\sum_{r=0}^{m} \varphi_r(x) = 1 \text{ for } x \in \overline{\Omega} \quad \text{and} \quad \sum_{r=1}^{m} \varphi_r(x) = 1 \text{ for } x \in \partial\Omega. \tag{4.33}$$

Since the functions a_r are Lipschitz-continuous in Δ_r, they are differentiable almost everywhere in Δ_r, i.e. there exists the gradient

$$\nabla a_r(x_r') = \Big(\frac{\partial a_r}{\partial x_{r,1}}(x_r'), \ldots, \frac{\partial a_r}{\partial x_{r,d-1}}(x_r')\Big)^T \quad \text{for a.e. } x_r' \in \Delta_r, \tag{4.34}$$

and

$$|\nabla a_r| \leqslant L \quad \text{a.e. in } \Delta_r, \ r = 1, \ldots, m. \tag{4.35}$$

(Here a.e. is meant with respect to the $(d-1)$-dimensional measure.) Then there exists a unit outer normal

$$\boldsymbol{n}_r(x_r', a_r(x_r')) = \frac{1}{\sqrt{1 + |\nabla a_r(x_r')|^2}} (\nabla a_r(x_r'), -1) \tag{4.36}$$

to $\partial \hat{V}_r^+$ for a.e. $X_r = (x_r', a_r(x_r')) \in \hat{\Lambda}_r$ (with respect to the $(d-1)$-dimensional measure defined on $\hat{\Lambda}_r$—cf. [17]) and

$$\boldsymbol{n}(x) = \mathbb{Q}_r^T \boldsymbol{n}_r(A_r(x)), \quad \text{a.e. } x \in \partial\Omega, \ A_r(x) \in \hat{\Lambda}_r, \tag{4.37}$$

is the unit outer normal to $\partial\Omega$.

Setting $\boldsymbol{e}_d = (0,\dots,0,-1)^T \in \mathbb{R}^d$, we get by (4.34) and (4.35)

$$\text{(4.38)} \qquad \boldsymbol{n}_r(X_r)\cdot\boldsymbol{e}_d = \frac{1}{\sqrt{1+|\nabla a_r(x_r')|^2}} \geqslant \frac{1}{\sqrt{1+L^2}}, \quad X_r \in \hat{\Lambda}_r,\ r=1,\dots,m.$$

In virtue of the orthogonality of $\mathbb{Q}_r$, for a.e. $x\in\partial\Omega$, with $A_r(x)\in\hat{\Lambda}_r$, we have

$$\begin{aligned}\text{(4.39)} \qquad \boldsymbol{n}(x)\cdot(\mathbb{Q}_r^T\boldsymbol{e}_d) &= (\mathbb{Q}_r^T\boldsymbol{n}_r(A_r(x)))\cdot(\mathbb{Q}_r^T\boldsymbol{e}_d)\\ &= (\mathbb{Q}_r^T\boldsymbol{n}_r(A_r(x)))^T(\mathbb{Q}_r^T\boldsymbol{e}_d)\\ &= (\boldsymbol{n}_r(A_r(x))^T\mathbb{Q}_r)(\mathbb{Q}_r^T\boldsymbol{e}_d)\\ &= \boldsymbol{n}_r(A_r(x))\cdot\boldsymbol{e}_d \geqslant \frac{1}{\sqrt{1+L^2}}, \quad r=1,\dots,m.\end{aligned}$$

Now we define a function $\boldsymbol{\varphi}$ by

$$\text{(4.40)} \qquad \boldsymbol{\varphi}(x) = \sqrt{1+L^2}\sum_{r=1}^m \varphi_r(x)\mathbb{Q}_r^T\boldsymbol{e}_d, \quad x\in\mathbb{R}^d.$$

Obviously, $\boldsymbol{\varphi}\in[C_0^\infty(\mathbb{R}^d)]^d$ and, thus, $\boldsymbol{\varphi}\in[W^{1,\infty}(\Omega)]^d$. Moreover, by (4.33), (4.39), and (4.40),

$$\boldsymbol{\varphi}(x)\cdot\boldsymbol{n}(x) \geqslant \sum_{r=1}^m \varphi_r(x) = 1, \quad x\in\partial\Omega.$$

□

Now we shall prove a "global version" of the multiplicative trace inequality.

Lemma 2. *There exists a constant $C_M' > 0$ such that*

$$\text{(4.41)} \quad \|v\|_{L^2(\partial\Omega)}^2 \leqslant C_M'\Big\{\|v\|_{\mathrm{DG}}\Big(\|v\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}_h} h_K\|v\|_{L^2(\partial K)}^2\Big)^{1/2} + \|v\|_{L^2(\Omega)}^2\Big\}, \quad v\in H^1(\Omega,\mathcal{T}_h),\ h\in(0,h_0).$$

Proof. Let $v\in H^1(\Omega,\mathcal{T}_h)$, $h\in(0,h_0)$, and $K\in\mathcal{T}_h$. Let $\boldsymbol{\varphi}\in[W^{1,\infty}(\Omega)]^d$ be the function from Lemma 1. Starting from functions belonging to $C^\infty(K)$, using Green's theorem, the density of $C^\infty(K)$ in $H^1(K)$, and the theorem on traces, we find that

$$\int_{\partial K} v^2\boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_K (v^2\operatorname{div}\boldsymbol{\varphi} + 2v\boldsymbol{\varphi}\cdot\nabla v)\,\mathrm{d}x.$$

This implies that

$$\text{(4.42)} \quad \int_{\partial\Omega} v^2\boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma [v^2]\boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}S = \sum_{K\in\mathcal{T}_h}\int_K (v^2\operatorname{div}\boldsymbol{\varphi} + 2v\boldsymbol{\varphi}\cdot\nabla v)\,\mathrm{d}x.$$

In view of (4.24) and (4.42),

$$\int_{\partial\Omega} v^2 \,\mathrm{d}S \leqslant \int_{\partial\Omega} v^2 \boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}S$$
$$\leqslant \sum_{K\in\mathcal{T}_h}\int_K |v^2 \operatorname{div}\boldsymbol{\varphi} + 2v\boldsymbol{\varphi}\cdot\nabla v|\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma |[v^2]||\boldsymbol{\varphi}|\,\mathrm{d}S.$$

Taking into account that $\boldsymbol{\varphi}\in[W^{1,\infty}(\Omega)]^d$ and using the Cauchy inequality, we find that

$$\|v\|^2_{L^2(\partial\Omega)} \leqslant \|\boldsymbol{\varphi}\|_{[W^{1,\infty}(\Omega)]^d}\Bigg(\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma |[v^2]|\,\mathrm{d}S + \|v\|^2_{L^2(\Omega)} + 2\sum_{K\in\mathcal{T}_h}\|v\|_{L^2(K)}|v|_{H^1(K)}\Bigg). \tag{4.43}$$

Further, let us consider the case (II) from Section 4.2 of nonconforming meshes. In the case (I) we can proceed similarly. In view of (3.10) and (4.15), we have

$$\begin{aligned}\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma |[v^2]|\,\mathrm{d}S &= 2\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma |[v]\langle v\rangle|\,\mathrm{d}S \\ &\leqslant 2\Bigg(\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma[v]^2\,\mathrm{d}S\Bigg)^{1/2}\Bigg(\sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma^{-1}\langle v\rangle^2\,\mathrm{d}S\Bigg)^{1/2} \\ &\leqslant C_W^{-1/2}(1+C_H)^{1/2}J_h(v,v)^{1/2}\Bigg(\sum_{K\in\mathcal{T}_h} h_K\|v\|^2_{L^2(\partial K)}\Bigg)^{1/2}.\end{aligned} \tag{4.44}$$

Now, using (4.43), (4.44) and the Cauchy inequality, we get

$$\|v\|^2_{L^2(\partial\Omega)} \leqslant \|\boldsymbol{\varphi}\|_{[W^{1,\infty}(\Omega)]^d}\Bigg\{C_W^{-1/2}(1+C_H)^{1/2}J_h(v,v)^{1/2}\Bigg(\sum_{K\in\mathcal{T}_h} h_K\|v\|^2_{L^2(\partial K)}\Bigg)^{1/2} + \|v\|^2_{L^2(\Omega)} + 2\|v\|_{L^2(\Omega)}|v|_{H^1(\Omega,\mathcal{T}_h)}\Bigg\},$$

which implies (4.41) with $C'_M = \sqrt{2}\max\{C_W^{-1/2}(1+C_H),2\}\|\boldsymbol{\varphi}\|_{[W^{1,\infty}(\Omega)]^d}$. □

Now we shall apply the above results to the derivation of the consistency estimate of the form b_h. This form can be expressed as

$$b_h(w,v) = b_h^{ID}(w,v) + b_h^N(w,v), \tag{4.45}$$

where

$$b_h^{ID}(w,v) = -\sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^{d} f_s(w)\frac{\partial v}{\partial x_s}\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma H\big(w|_\Gamma^{(L)}, w|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma\big)[v]_\Gamma\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma H\big(w|_\Gamma^{(L)}, w|_\Gamma^{(L)}, \boldsymbol{n}_\Gamma\big)v|_\Gamma^{(L)}\,\mathrm{d}S \tag{4.46}$$

and, due to (3.14),

$$b_h^N(w,v) = \sum_{\Gamma\in\mathcal{F}_h^N}\int_\Gamma H\big(w|_\Gamma^{(L)}, w|_\Gamma^{(L)}, \boldsymbol{n}_\Gamma\big)v|_\Gamma^{(L)}\,\mathrm{d}S = \sum_{\Gamma\in\mathcal{F}_h^N}\int_\Gamma \sum_{s=1}^{d} f_s\big(w|_\Gamma^{(L)}\big)n_s v|_\Gamma^{(L)}\,\mathrm{d}S. \tag{4.47}$$

We are interested in the estimation of the expression

$$b_h(u,\xi) - b_h(u_h,\xi) = (b_h^{ID}(u,\xi) - b_h^{ID}(u_h,\xi)) + (b_h^N(u,\xi) - b_h^N(u_h,\xi)). \tag{4.48}$$

Using the same process as in [13], we get

$$|b_h^{ID}(u,\xi) - b_h^{ID}(u_h,\xi)| \leqslant C_D\|\xi\|_{\mathrm{DG}}(h^{\mu+1}|u|_{H^{\mu+1}(\Omega)} + \|\xi\|_{L^2}(\Omega)), \tag{4.49}$$

where C_D is independent of u and h.

It remains to estimate the second term on the right-hand side of (4.48).

Lemma 3. *If the exact solution u satisfies conditions* (4.1), *then*

$$|b_h^N(u,\xi) - b_h^N(u_h,\xi)| \leqslant C_N(h^{2\mu+1}|u|^2_{H^{\mu+1}(\Omega)} + \|\xi\|_{\mathrm{DG}}\|\xi\|_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}), \tag{4.50}$$

where $\mu = \min(p,r)$ and C_N is a constant independent of u, u_h, and h.

Proof. By (4.47), (3.13), (3.14), the Cauchy and Young's inequalities, and the relation $u - u_h = \eta + \xi$,

$$|b_h^N(u,\xi) - b_n^N(u_h,\xi)| \leqslant C_L\|u-u_h\|_{L^2(\partial\Omega_N)}\|\xi\|_{L^2(\partial\Omega_N)} \leqslant C_L\|u-u_h\|_{L^2(\partial\Omega)}\|\xi\|_{L^2(\partial\Omega)} \leqslant C_L\Big(\frac{1}{2}\|\eta\|^2_{L^2(\partial\Omega)} + \frac{3}{2}\|\xi\|^2_{L^2(\partial\Omega)}\Big) \tag{4.51}$$

with $C_L = 2L_H$.

By (4.5) and (4.8),

$$\sum_{K\in\mathcal{T}_h} \|\eta\|^2_{L^2(\partial K)} \leqslant 2C_M C_A^2 h^{2\mu+1}|u|^2_{H^{\mu+1}}(\Omega). \tag{4.52}$$

Since

$$\|\eta\|^2_{L^2(\partial\Omega)} \leqslant \sum_{K\in\mathcal{T}_h} \|\eta\|^2_{L^2(\partial K)},$$

we have

$$\|\eta\|^2_{L^2(\partial\Omega)} \leqslant C^* h^{2\mu+1}|u|^2_{H^{\mu+1}(\Omega)} \tag{4.53}$$

with $C^* = 2C_M C_A^2$.

Now we shall estimate $\|\xi\|^2_{L^2(\partial\Omega)}$ according to Lemma 2. Taking into account that $\xi \in S_{hp}$ and using inequalities (4.5) and (4.6), we find that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} h_K\|\xi\|^2_{L^2(\partial K)} &\leqslant C_M \sum_{K\in\mathcal{T}_h} h_K(\|\xi\|_{L^2(K)}|\xi|_{H^1(K)} + h_K^{-1}\|\xi\|^2_{L^2(K)})\\ &\leqslant C_M(1+C_I)\,\|\xi\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.54}$$

Hence, in view of (4.41) and (4.54), we have

$$\begin{aligned}\|\xi\|^2_{L^2(\partial\Omega)} &\leqslant C'_M\{(C_M^{1/2}(1+C_I)^{1/2}+1)\|\xi\|_{\mathrm{DG}}\|\xi\|_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}\}\\ &\leqslant C^{**}(\|\xi\|_{\mathrm{DG}}\|\xi\|_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}),\end{aligned} \tag{4.55}$$

where $C^{**} = C'_M(C_M^{1/2}(1+C_I)^{1/2}+1)$. Finally, (4.51), (4.53), and (4.55) yield estimate (4.50) with $C_N = \frac{1}{2}C_L\max\{C^*, 3C^{**}\}$, which we wanted to prove. □

4.4. Error estimates

On the basis of the above results we prove now error estimates of the DGFEM applied to the nonstationary nonlinear convection-diffusion problem with mixed Dirichlet-Neumann boundary conditions.

Theorem 1. *Let assumptions* (H) *and* (4.4) *be satisfied. Let u be the exact solution of problem* (2.1) *satisfying conditions* (4.1) *and let u_h be the numerical solution obtained by method* (3.17), *where the weight σ from the penalty terms and the constant C_W satisfy the conditions discussed in* (I) *and* (II) *of Section* 4.2. *Then there exists a constant $C > 0$ independent of $h \in (0, h_0)$ such that the error $e_h = u - u_h$ satisfies the estimate*

$$\max_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} + \varepsilon\int_0^T \|e_h(\vartheta)\|^2_{\mathrm{DG}}\,\mathrm{d}\vartheta \leqslant Ch^{2\mu}, \quad h\in(0,h_0). \tag{4.56}$$

P r o o f. For simplicity, in the proof we shall denote by c a positive generic constant attaining, in general, different values at different places. It is independent of h, ε but depends on constants appearing in the previous lemmas. We use again the notation $\xi = \Pi_{hp}u - u_h \in S_{hp}$ and $\eta = u - \Pi_{hp}u$. Then $e_h = u - u_h = \xi + \eta$. If we subtract (3.17) b) from (3.16), set $\varphi_h := \xi$, and use (4.11) and the relation

$$\Big(\frac{\partial\xi(t)}{\partial t}, \xi(t)\Big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2_{L^2(\Omega)},$$

we get

$$\begin{aligned}(4.57)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2_{L^2(\Omega)} + \varepsilon\|\xi(t)\|^2_{\mathrm{DG}}\\ &\quad\leqslant b_h(u_h(t),\xi(t)) - b_h(u(t),\xi(t)) - \Big(\frac{\partial\eta(t)}{\partial t},\xi(t)\Big) - a_h(\eta(t),\xi(t))\\ &\quad\quad - \varepsilon J_h(\eta(t),\xi(t)) \quad \text{for a.e. } t\in(0,T).\end{aligned}$$

Now we estimate the individual terms on the right-hand side of (4.57). By (4.48), (4.49), (4.50), and Young's inequality, we have (we omit the argument t)

$$\begin{aligned}(4.58)\quad &|b_h(u_h,\xi) - b_h(u,\xi)|\\ &\qquad\leqslant \frac{3}{2}\alpha\varepsilon\|\xi\|^2_{\mathrm{DG}} + c\Big(1+\frac{1}{\varepsilon}\Big)h^{2\mu+1}|u|^2_{H^{\mu+1}(\Omega)} + c\Big(1+\frac{1}{\varepsilon}\Big)\|\xi\|^2_{L^2(\Omega)},\end{aligned}$$

where $\alpha > 0$ will be chosen later and $c > 0$ is a constant depending on α.

Further, using the relation

$$(4.59)\qquad \frac{\partial\eta}{\partial t} = \frac{\partial u}{\partial t} - \Pi_{hp}\frac{\partial u}{\partial t},$$

conditions (4.1) and (4.8) with $\mu := \mu - 1$, we get

$$\begin{aligned}(4.60)\qquad \Big|\Big(\frac{\partial\eta}{\partial t},\xi\Big)\Big| &\leqslant C_A\, h^{\mu}\Big|\frac{\partial u}{\partial t}\Big|_{H^{\mu}(\Omega)}\|\xi\|_{L^2(\Omega)}\\ &\leqslant \frac{1}{2}C_A\Big(h^{2\mu}\Big|\frac{\partial u}{\partial t}\Big|^2_{H^{\mu}(\Omega)} + \|\xi\|^2_{L^2(\Omega)}\Big).\end{aligned}$$

By (4.17)–(4.19), and Young's inequality, we get

$$(4.61)\qquad |a_h(\eta,\xi)| + \varepsilon|J_h(\eta,\xi)| \leqslant \alpha\varepsilon\|\xi\|^2_{\mathrm{DG}} + ch^{2\mu}|u|^2_{H^{\mu+1}(\Omega)},$$

where $\alpha > 0$ is a constant from (4.58) and $c > 0$ is a constant depending on α, C_a, and C_J.

Choosing now $\alpha = 1/5$, from (4.57)–(4.61) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \varepsilon\|\xi\|^2_{\mathrm{DG}} \leqslant ch^{2\mu}\Big(\Big(1+\frac{1}{\varepsilon}\Big)|u|^2_{H^{\mu+1}(\Omega)} + \Big|\frac{\partial u}{\partial t}\Big|^2_{H^{\mu}(\Omega)}\Big) + c\Big(1+\frac{1}{\varepsilon}\Big)\|\xi\|^2_{L^2(\Omega)}. \tag{4.62}$$

The integration from 0 to $t \in [0,T]$, the relation $\xi(0) = \Pi_{hp}u(0) - u_h(0) = 0$, and the application of Gronwall's lemma yield the estimate

$$\|\xi(t)\|^2_{L^2(\Omega)} + \varepsilon\int_0^t \|\xi(\vartheta)\|^2_{\mathrm{DG}}\,\mathrm{d}\vartheta \leqslant \tilde{C}h^{2\mu}\Big(\|u\|^2_{L^2(0,T;H^{\mu+1}(\Omega))} + \Big\|\frac{\partial u}{\partial t}\Big\|^2_{L^2(0,T;H^{\mu}(\Omega))}\Big), \quad t\in[0,T], \tag{4.63}$$

where $\tilde{C}$ is a positive constant independent of h, u, u_h, but depending on ε.

Finally, the inequalities

$$\|e_h\|^2_{L^2(\Omega)} \leqslant 2(\|\eta\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}),$$
$$\|e_h\|^2_{\mathrm{DG}} \leqslant 2(\|\eta\|^2_{\mathrm{DG}} + \|\xi\|^2_{\mathrm{DG}}),$$

together with estimates (4.63) and (4.22) imply (4.56), which we wanted to prove. □

5. Concluding remarks

In this paper we are concerned with the analysis of an error estimate in the $L^\infty(L^2)$- and $L^2(H^1)$-norms of the discontinuous Galerkin method applied to the space semi-discretization of a nonstationary convection-diffusion problem with linear diffusion and nonlinear convection and mixed Dirichlet-Neumann boundary conditions on nonconforming meshes.

Combining our results with the technique from [12], we can easily extend the estimates obtained to the case of nonlinear convection as well as diffusion. Moreover, it is possible to extend error estimates of the full space-time discretization proved in [8] to the case of the mixed boundary conditions and simplicial meshes.

There are still some open problems:

- analysis of optimal error estimates in the $L^\infty(L^2)$-norm in the case of a weak regularity of the exact solution, which is a consequence of nonconvexity of the domain Ω and/or the use of mixed Dirichlet-Neumann boundary conditions,

- analysis of optimal error estimates for problems with nonlinear convection and diffusion,
- analysis of error estimates of the DGFEM applied to problems with mixed boundary conditions on nonstandard meshes, used in [10] and [8].

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