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# On Fourth-order Boundary-value Problems

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**Abstract**

We show the existence of solutions to a boundary-value problem for fourth-order differential inclusions in a Banach space, under Lipschitz's contractive conditions, Carathéodory conditions and lower semicontinuity conditions.

**Key words:** Boundary-value problems, set-valued map, fixed point, selection.

**2000 Mathematics Subject Classification:** 34G20, 34B15, 47H10

## 1 Introduction

The aim of this paper is to establish the existence of solutions of the following fourth-order boundary-value problem:

$$\begin{cases} x^{(4)}(t) \in F(t, x(t), \ddot{x}(t)) & \text{a.e. in } [0,1]; \\ x(0) = x(1) = \ddot{x}(0) = \ddot{x}(1) = \theta, \end{cases} \tag{1.1}$$

where $F\colon [0,1] \times E \times E \to 2^E$ is a multi-valued map and $\theta$ is the zero element of $E$.

Boundary-value problems arise from applied mathematical sciences, and they have received a great deal of attention in the literature. Problem (1.1), with $F$ single-valued, models deformations of an elastic beam in equilibrium state, whose two ends are simply supported. For review of results on fourth-order boundary-value problems for differential equations, we refer the reader to the papers by Aftabizadeh [1], Yang [11], Liu [8], and the references therein.

It is known that every control system has an equivalent formulation whose dynamics are described by an inclusion and that many engineering problems can be studied by using differential inclusions. The goal of this paper is to extend the fourth-order boundary-value problems for differential equations to the multi-valued case. We applied three methods different from those used in the above mentioned works. In the first case, we supposed that $F(.,.,.)$ is a closed multifunction, measurable in the first argument and Lipschitz continuous in the second argument. We used the fixed point theorem introduced by Covitz and Nadler for contraction multi-valued maps. In the second case, the multifunction $F(.,.,.)$ is compact and lower semicontinuous. We used Schaefer's fixed point theorem combined with a selection theorem of Bressan and Colombo (see [2]) for lower semicontinuous and nonconvex multi-valued operators with decomposable values. In the third case, we assumed that $F(.,.,.)$ is a compact convex $L^1$-Carathéodory multifunction. We used the fixed point theorem for condensing maps due to Martelli [9].

## 2 Preliminaries and notations

Let $E$ be a real Banach space with the norm $\|\cdot\|$. We denote by $\mathcal{C}([0,1],E)$ the Banach space of continuous functions from $[0,1]$ to $E$ with the norm

$$\|x(.)\|_\infty := \sup\big\{\|x(t)\|; t\in[0,1]\big\}.$$

We say that a subset $A$ of $[0,1]\times E\times E$ is $\mathcal{L}\otimes\mathcal{B}$-measurable if $A$ belongs to the $\sigma$-algebra generated by all sets of the form $I\times D$, where $I$ is Lebesgue measurable in $[0,1]$ and $D$ is measurable in $E\times E$. For $x\in E$ and for nonempty sets $A,B$ of $E$, we denote $d(x,A)=\inf\{d(x,y); y\in A\}$, $e(A,B):=\sup\{d(x,B); x\in A\}$ and $H(A,B):=\max\{e(A,B),e(B,A)\}$. A multifunction is said to be measurable if its graph is measurable. For more detail on measurability theory, we refer the reader to the book of Castaing–Valadier [3].

Now, let $G(t,s)$ be the Green's function of the linear problem

$$-\ddot{z}=0,\ t\in[0,1] \text{ together with } z(0)=z(1)=0,$$

which is explicitly given by

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases}$$

Let $x(.)\in\mathcal{C}([0,1],E)$ and $t\in[0,1]$. Set

$$(Ax)(t)=\int_0^1 G(t,s)x(s)\,ds,$$

Obviously, $A\colon \mathcal{C}([0,1],E)\to\mathcal{C}([0,1],E)$ is continuous. Let $y=-\ddot{x}$. Since $x(0)=x(1)=\theta$, we have $x(t)=(Ay)(t)$. Then Problem (1.1) becomes

$$\begin{cases} \ddot{y}(t)\in -F(t,(Ay)(t),-y(t)) \quad \text{a.e. in } [0,1];\\ y(0)=y(1)=\theta. \end{cases} \tag{2.1}$$

## 3 The Lipschitz case

In this section, our main purpose is to obtain the existence of a solution to (1.1), in the case when $F(.,.,.)$ is a closed multifunction, measurable with respect to the first argument and Lipschitz continuous with respect to the third argument. We use the fixed point theorem introduced by Covitz and Nadler for contraction multi-valued maps.

**Definitions 3.1** Let $G\colon E \to 2^E$ be a multifunction with closed values

1. G is a *k-Lipschitz* if
$$H\big(G(x), G(y)\big) \leq kd(x,y), \quad \text{for each } x, y \in E.$$
2. G is a *contraction* if it is $k$-Lipschitz with $k < 1$.
3. G has a *fixed point* if there exists $x \in E$ such that $x \in G(x)$.

Let us recall the following results that will be used in the sequel.

**Lemma 3.1** [4] *If $G\colon E \to 2^E$ is a contraction with nonempty closed values, then it has a fixed point.*

**Lemma 3.2** [12] *Assume that $F\colon [0,1] \times E \times E \to 2^E$ is a multifunction with nonempty closed values satisfying:*

- *For every $(x, y) \in E \times E$, $F(., x, y)$ is measurable on $[0, 1]$;*
- *For every $t \in [0, 1]$, $F(t, ., .)$ is (Hausdorff) continuous on $E \times E$.*

*Then, for any measurable functions $x(.)\colon [0,1] \to E$ and $y(.)\colon [0,1] \to E$, the multifunction $F(., x(.), y(.))$ is measurable on $[0, 1]$.*

**Definition 3.1** A measurable multi-valued function $F\colon [0,1] \to 2^E$ is said to be *integrably bounded* if there exists a function $h \in L^1([0,1], E)$ such that for all $v \in F(t)$, $\|v\| \leq h(t)$ for almost every $t \in [0, 1]$.

We shall prove the following theorem.

**Theorem 3.1** *Let $F\colon [0,1] \times E \times E \to 2^E$ be a set-valued map with nonempty closed values satisfying*

*(i) For all $x, y \in E$, $t \mapsto F(t, x, y)$ is measurable and integrably bounded;*

*(ii) There exists a function $m(.) \in L^1([0,1], \mathbb{R}^+)$ such that for all $t \in [0,1]$ and for all $(x_1, y_1), (x_2, y_2) \in E \times E$*
$$H\big(F(t, x_1, y_1), F(t, x_2, y_2)\big) \leq m(t)\|y_1 - y_2\|.$$

*Then, if*
$$\int_0^1 m(s)\, ds < 1$$
*the problem (1.1) has at least one solution on $[0, 1]$.*

**Proof** For $y(.) \in \mathcal{C}([0,1],E)$, set

$$S_{F,y(.)} := \big\{ f \in L^1([0,1],E) \colon f(t) \in F(t,(Ay)(t),-y(t)) \text{ for a.e. } t \in [0,1] \big\}.$$

By Lemma 3.2, for $y(.) \in \mathcal{C}([0,1],E)$, $F(.,(Ay)(.),-y(.)))$ is closed and measurable, then it has a measurable selection which, by hypothesis (i), belongs to $L^1([0,1],E)$. Thus $S_{F,y(.)}$ is nonempty. Let us transform the problem into a fixed point problem. Consider the multivalued map, $T \colon \mathcal{C}([0,1],E) \to 2^{\mathcal{C}([0,1],E)}$ defined as follows, for $y(.) \in \mathcal{C}([0,1],E)$,

$$T(y(.)) = \left\{ z(.) \in \mathcal{C}([0,1],E) \colon z(t) = \int_0^1 G(t,s) f(s)\, ds,\ \forall t \in [0,1],\ f \in S_{F,y(.)} \right\}.$$

We shall show that $T$ satisfies the assumptions of Lemma 3.1. The proof will be given in two steps:

**Step 1:** *$T$ has non-empty closed-values.* Indeed, let $(y_p(.))_{p \geq 0} \in T(y(.))$ converges to $\bar{y}(.)$ in $\mathcal{C}([0,1],E)$. Then $\bar{y}(.) \in \mathcal{C}([0,1],E)$ and for each $t \in [0,1]$

$$y_p(t) \in \int_0^1 G(t,s) F(s,(Ay)(s),-y(s))\, ds,$$

where

$$\int_0^1 G(t,s) F(s,(Ay)(s),-y(s))\, ds$$

is the Aumann integral of $G(t,.)F(.,(Ay)(.),-y(.))$, which is defined as

$$\int_0^1 G(t,s) F(s,(Ay)(s),-y(s))\, ds = \left\{ \int_0^1 G(t,s) f(s)\, ds,\ f \in S_{F,y(.)} \right\}.$$

Since the set

$$\int_0^1 G(t,s) F(s,(Ay)(s),-y(s))\, ds$$

is closed for all $t \in [0,1]$, we have

$$\bar{y}(t) \in \int_0^1 G(t,s) F(s,(Ay)(s),-y(s))\, ds.$$

Then, by the definition of the Aumann integral, there exists $f \in S_{F,y(.)}$ such that

$$\bar{y}(t) = \int_0^1 G(t,s) f(s)\, ds.$$

Hence $\bar{y}(.) \in T(y(.))$. So $T(y(.))$ is closed for each $y(.) \in \mathcal{C}([0,1],E)$.

**Step 2:** *$T$ is a contraction.* Indeed, let $y_1(.), y_2(.) \in \mathcal{C}([0,1], E)$ and consider $z_1(.) \in T(y_1(.))$. Then

$$z_1(t) = \int_0^1 G(t,s) f_1(s)\, ds,$$

where $f_1 \in S_{F,y_1(.)}$. Let $\varepsilon > 0$. Consider the valued map $U_\varepsilon \colon [0,1] \to 2^E$, defined by

$$U_\varepsilon(t) = \Big\{ x \in E \colon \|f_1(t) - x\| \le m(t)\|y_1(t) - y_2(t)\| + \varepsilon \Big\}.$$

For each $t \in [0,1]$, $U_\varepsilon(t)$ is nonempty. Indeed, let $t \in [0,1]$, we have

$$H\big(F(t, (Ay_1)(t), -y_1(t)), F(t, (Ay_2)(t), -y_2(t))\big) \le m(t)\|y_1(t) - y_2(t)\|.$$

Hence, there exists $x \in F(t, (Ay_2)(t), -y_2(t))$, such that

$$\|f_1(t) - x\| \le m(t)\|y_1(t) - y_2(t)\| + \varepsilon.$$

By Theorem III.40 in [3], the multifunction

$$V \colon t \to U_\varepsilon(t) \cap F(t, (Ay_2)(t), -y_2(t)) \text{ is measurable.} \tag{3.1}$$

Then there exists a measurable selection for $V$ denoted $f_2$ such that, for all $t \in [0,1]$,

$$f_2(t) \in F(t, (Ay_2)(t), -y_2(t))$$

and

$$\|f_1(t) - f_2(t)\| \le m(t)\|y_1(t) - y_2(t)\| + \varepsilon.$$

Now, set for all $t \in [0,1]$

$$z_2(t) = \int_0^1 G(t,s) f_2(s)\, ds.$$

Then

$$\begin{aligned}
\|z_1(t) - z_2(t)\| &\le \int_0^1 |G(t,s)| \|f_1(s) - f_2(s)\|\, ds \\
&\le \int_0^1 m(s)\|y_1(s) - y_2(s)\|\, ds + \varepsilon \\
&\le \|y_1(.) - y_2(.)\|_\infty \int_0^1 m(s)\, ds + \varepsilon.
\end{aligned}$$

So, we conclude that

$$\|z_1(.) - z_2(.)\|_\infty \le \|y_1(.) - y_2(.)\|_\infty \int_0^1 m(s)\, ds + \varepsilon.$$

By the analogous relation, obtained by interchanging the roles of $y_1(.)$ and $y_2(.)$, it follows that

$$H\big(T(y_1(.)),T(y_2(.))\big) \le \|y_1(.)-y_2(.)\|_\infty \int_0^1 m(s)\,ds+\varepsilon.$$

By letting $\varepsilon \to 0$, we get

$$H\big(T(y_1(.)),T(y_2(.))\big) \le \|y_1(.)-y_2(.)\|_\infty \int_0^1 m(s)\,ds.$$

Consequently, if $\int_0^1 m(s)\,ds < 1$, $T$ is a contraction. By Lemma 3.1, $T$ has a fixed point $y(.)$ which is a solution of (2.1). Then $x(.) = (Ay)(.)$ is a solution of (1.1). □

## 4 The lower semicontinuous case

In the sequel, we prove the existence of solutions of the problem (1.1), in the case where the set-valued map is lower semicontinuous. We use Schaefer's fixed point theorem combined with a selection theorem of Bressan and Colombo (see [2]), for lower semicontinuous and nonconvex multi-valued operators with decomposable values.

**Definition 4.1** A subset $B$ of $L^1([0,1],E)$ is *decomposable* if for all $u,v \in B$ and $I \subset [0,1]$ measurable, the function $u(.)\chi_I(.)+v(.)\chi_{[0,1]\setminus I}(.) \in B$, where $\chi(.)$ denotes the characteristic function.

**Definitions 4.1** Let $X$ a nonempty closed subset of $E$ and $G\colon X \to 2^E$ a multi-valued operator with nonempty closed values. We say that:

- $G$ is *lower semicontinuous* if the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is open for any open set $C$ in $E$.
- $G$ is *completely continuous* if $G(B)$ is relatively compact for every $B$ bounded set of $X$.

**Definition 4.2** Let $F\colon [0,1]\times E\times E \to 2^E$ be a set-valued map with nonempty compact values. Assign to $F$ the multi-valued operator

$$\mathcal{F}\colon \mathcal{C}([0,1],E)\times\mathcal{C}([0,1],E) \to 2^{L^1([0,1],E)},$$

defined by

$$\mathcal{F}(x(.),y(.)) = \big\{z(.) \in L^1([0,1],E) \colon z(t) \in F(t,x(t),y(t)) \text{ for a.e. } t \in [0,1]\big\}.$$

The operator $\mathcal{F}$ is called the *Niemytzki operator* associated with $F$. We say $F$ is *the lower semicontinuous type* if its associated Niemytzki operator $\mathcal{F}$ is lower semicontinuous, and has nonempty closed and decomposable values.

Let us recall the following result that will be used in the sequel.

**Lemma 4.1** [2] *Let $E$ be a separable metric space and let $G\colon E \to 2^{L^1([0,1],E)}$ be a multi-valued operator which is lower semicontinuous and has nonempty closed and decomposable values. Then $G$ has a continuous selection, i.e. there exists a continuous function $f\colon E \to L^1([0,1],E)$ such that $f(y) \in G(y)$ for every $y \in E$.*

We shall prove the following result.

**Theorem 4.1** *Assume that $dim(E) < \infty$. Let $F\colon [0,1] \times E \times E \to 2^E$ be a set-valued map with nonempty compact values satisfying*

*(i)* $(t,x,y) \mapsto F(t,x,y)$ *is $\mathcal{L} \otimes \mathcal{B}$-measurable;*

*(ii)* $(x,y) \mapsto F(t,x,y)$ *is lower semicontinuous for almost all $t \in [0,1]$;*

*(iii) There exist a function $m(.) \in L^1([0,1],\mathbb{R}^+)$ and positive constants $c$ and $d$ such that for almost all $t \in [0,1]$ and all $(x,y) \in E \times E$*

$$\big\|F(t,x,y)\big\| := \sup\Big\{\|z\| \colon z \in F(t,x,y)\Big\} \leq c\|x\| + d\|y\| + m(t).$$

*Then, if $1-c-d>0$ the problem (1.1) has at least one solution on $[0,1]$.*

**Proof** Remark that, by hypotheses, $F$ is the lower semicontinuous type (see [5]). Then, by Lemma 4.1, there exists a continuous function

$$f\colon \mathcal{C}([0,1],E) \times \mathcal{C}([0,1],E) \to L^1([0,1],E)$$

such that $f(x(.),y(.)) \in \mathcal{F}(x(.),y(.))$ for all $x(.),y(.) \in \mathcal{C}([0,1],E)$. Consider the problem:

$$\begin{cases} \ddot{y}(t) = -f((Ay)(.),-y(.))(t) & \text{a.e. in } [0,1]; \\ y(0) = y(1) = \theta. \end{cases} \tag{4.1}$$

Remark that, if $y(.) \in \mathcal{C}([0,1],E)$ is a solution of the problem (4.1), then $y(.)$ is a solution of the problem (2.1). Let us transform the problem (4.1) into a fixed point problem. Consider the operator, $T\colon \mathcal{C}([0,1],E) \to \mathcal{C}([0,1],E)$ defined as follows, for all $y(.) \in \mathcal{C}([0,1],E)$ and for all $t \in [0,1]$

$$T(y(.))(t) \;=\; \int_0^1 G(t,s)f((Ay)(.),-y(.))(s)\,ds.$$

We shall show that $T$ has a fixed point. The proof will be given in several steps:

**Step 1:** *$T$ is continuous.* Indeed, let $(y_p(.))_p$ converges to $y(.)$ in $\mathcal{C}([0,1],E)$. Then for each $t \in [0,1]$

$$\begin{aligned} &\big\|T(y_p(.))(t) - T(y(.))(t)\big\| \\ &\leq \int_0^1 |G(t,s)|\|f((Ay_p)(.),-y_p(.))(s) - f((Ay)(.),-y(.))(s)\|\,ds \\ &\leq \int_0^1 \|f((Ay_p)(.),-y_p(.))(s) - f((Ay)(.),-y(.))(s)\|\,ds. \end{aligned}$$

By the continuity of $f$, it is easy to deduce that $T$ is continuous.

**Step 2:** *$T$ is bounded on bounded sets of $\mathcal{C}([0,1],E)$.* Indeed, it is sufficient to show that $T(B_r)$ is bounded for all $r \geq 0$, where

$$B_r = \{y(.) \in \mathcal{C}([0,1],E)\colon \|y(.)\|_\infty \leq r\}.$$

Let $h \in T(B_r)$. For all $t \in [0,1]$ we have

$$\begin{aligned}\|h(t)\| &\leq \int_0^1 |G(t,s)|\|f((Ay)(.),-y(.))(s)\|\,ds\\ &\leq \int_0^1 \Big(c\|(Ay)(s)\| + d\|y(s)\| + m(s)\Big)\,ds \leq (c+d)r + \int_0^1 m(s)\,ds,\end{aligned}$$

because

$$\|(Ay)(t)\| \leq \int_0^1 |G(t,s)|\|y(s)\|\,ds \leq r.$$

Then

$$\|h\|_\infty \leq (c+d)r + \int_0^1 m(s)\,ds.$$

Hence $T(B_r) \subset B_\delta$, where $\delta$ is the right-hand side in the above inequality.

**Step 3:** *$T$ sends bounded sets of $\mathcal{C}([0,1],E)$ into equicontinuous sets.* Indeed, let $h \in T(B_r)$. Then $h = T(y(.))$ where $y(.) \in B_r$. Let $t, s \in [0,1]$ such that $t < s$. We have

$$\|h(s) - h(t)\| \leq \int_0^1 |G(s,\tau) - G(t,\tau)|\|f((Ay)(.),-y(.))(\tau)\|\,d\tau.$$

By the strong continuity of the function $G(.,.)$, the right-hand side of the above inequality tends to 0 as $s$ converges to $t$.

**Step 4:** *The following set is bounded*

$$\Omega = \Big\{y(.) \in \mathcal{C}([0,1],E)\colon \lambda y(.) = T(y(.)),\ \text{for some } \lambda > 1\Big\}.$$

Indeed, let $y(.) \in \Omega$. Then

$$y(t) = \lambda^{-1}\int_0^1 G(t,s) f((Ay)(.),-y(.))(s)\,ds.$$

So, we conclude that

$$\begin{aligned}\|y(.)\|_\infty &\leq \lambda^{-1}(c\|(Ay)(.)\|_\infty + d\|y(.)\|_\infty) + \lambda^{-1}\int_0^1 m(s)\,ds\\ &\leq \lambda^{-1}(c\|y(.)\|_\infty + d\|y(.)\|_\infty) + \lambda^{-1}\int_0^1 m(s)\,ds.\end{aligned}$$

From which we get

$$(1-\lambda^{-1}(c+d))\|y(.)\|_\infty \leq \lambda^{-1}\int_0^1 m(s)\,ds.$$

Since $1-\lambda^{-1}(c+d) > 1-c-d$, we obtain

$$(1-c-d)\|y(.)\|_\infty \leq \lambda^{-1}\int_0^1 m(s)\,ds.$$

Hence, if $1-c-d>0$ we have

$$\|y(.)\|_\infty \leq \frac{\lambda^{-1}}{1-c-d}\int_0^1 m(s)\,ds.$$

This shows that $\Omega$ is bounded.

In conclusion, by the Steps 2 and 3 combined with the Ascoli's theorem, we can conclude that $T$ is completely continuous. Then by the step 1 and Schaefer's theorem (see [10] p. 29), we deduce that $T$ has a fixed point $y(.)$ which is a solution of (4.1). Then $x(.) = (Ay)(.)$ is a solution of (1.1). □

# 5 The Carathéodory case

In this section, we use the fixed point theorem for condensing maps due to Martelli [9], to prove the existence of solutions of the problem (1.1).

**Definition 5.1** A multi-valued map $F\colon [0,1]\times E\times E \to 2^E$ is said to be an $L^1$-*Carathéodory* if

(i) $t\mapsto F(t,x,y)$ is measurable for all $(x,y)\in E\times E$;

(ii) $(x,y)\mapsto F(t,x,y)$ is upper semicontinuous for almost all $t\in[0,1]$;

(iii) For each $k>0$, there exists $h_k\in L^1([0,1];\mathbb{R}^+)$ such that

$$\big\|F(t,x,y)\big\| := \sup\Big\{\|z\|\colon z\in F(t,x,y)\Big\} \leq h_k(t).$$

for all $\|(x,y)\| := \max(\|x\|,\|y\|)\leq k$ and for almost all $t\in[0,1]$.

**Definitions 5.1** Let $E$ be a separable Banach space, $X$ a nonempty subset of $E$ and $G\colon X\to 2^E$ a multi-valued map. We say that:

- $G$ is *upper semi-continuous* on $X$ if for each $x\in X$ the set $G(x)$ is a nonempty closed subset of $E$ and if for each open set $B$ of $E$ containing $G(x)$, there exists an open neighbourhood $V$ of $x$ such that $G(V)\subset B$.

- If $G$ is upper semi-continuous, it is said to be *condensing map* if for any subset $B\subset X$ with $\alpha(B)\neq 0$, we have $\alpha(G(B))<\alpha(B)$, where $\alpha$ denotes the Kuratowski measure of noncompactness. We remark that a completely continuous multivalued map is the easiest example of a condensing map.

It is known that if the multi-valued map $G$ is completely continuous with nonempty compact values, the $G$ is upper semi-continuous if and only if $G$ has a closed graph.

Let $S_F\colon \mathcal{C}([0,1],E) \to 2^{L^1([0,1],E)}$ be a set-valued map defined as follows, for $y(.) \in \mathcal{C}([0,1],E)$,

$$S_F(y(.)) := \big\{ f \in L^1([0,1],E) \colon f(t) \in F(t,(Ay)(t),-y(t)) \text{ for a.e. } t \in [0,1] \big\}.$$

**Remark 5.1** If $\dim(E) < \infty$ and $F\colon [0,1]\times E\times E \to 2^E$ is compact and convex then $S_F(y(.)) \neq \emptyset$ for all $y(.) \in \mathcal{C}([0,1],E)$ (see [7]).

In the sequel, we will use the following important Lemma.

**Lemma 5.1** [9] *Let $T\colon E \to 2^E$ be a convex compact condensing multi-valued maping. If the set*

$$\Omega := \big\{ y \in E \colon \lambda y \in T(y) \text{ for some } \lambda > 1 \big\}$$

*is bounded, then $T$ has a fixed point.*

We shall prove the following main result.

**Theorem 5.1** *Assume that dim$(E) < \infty$. Let $F\colon [0,1] \times E \times E \to 2^E$ be an $L^1$-Carathéodory set-valued map with nonempty compact convex values. Assume that there exist a function $m(.) \in L^1([0,1],\mathbb{R}^+)$ and positive constants $c$ and $d$ such that for almost all $t \in [0,1]$ and all $(x,y) \in E \times E$*

$$\big\|F(t,x,y)\big\| \leq c\|x\| + d\|y\| + m(t).$$

*Then, if $1-c-d > 0$ the problem (1.1) has at least one solution on $[0,1]$.*

**Proof** Let us transform the problem into a fixed point problem. Consider the multi-valued map, $T\colon \mathcal{C}([0,1],E) \to 2^{\mathcal{C}([0,1],E)}$ defined as follows, for each $y(.) \in \mathcal{C}([0,1],E)$,

$$T(y(.)) = \left\{ z(.) \in \mathcal{C}([0,1],E) \colon z(t) = \int_0^1 G(t,s)f(s)\,ds,\ \forall t \in [0,1],\ \forall f \in S_F(y(.)) \right\}.$$

We shall show that $T$ satisfies the assumptions of Lemma 5.1. By the same arguments as in the section 4, we can prove that

(a) $T$ is bounded on bounded sets of $\mathcal{C}([0,1],E)$.

(b) $T$ sends bounded sets of $\mathcal{C}([0,1],E)$ into equicontinuous sets.

(c) The following set is bounded

$$\Omega = \Big\{ y(.) \in \mathcal{C}([0,1],E) \colon \lambda y(.) \in T(y(.)),\ \text{for some } \lambda > 1 \Big\}.$$

**Claim 1** *$T(y(.))$ is convex for each $y(.) \in \mathcal{C}([0,1],E)$.*

**Proof** Let $h_1, h_2 \in T(y(.))$, then

$$h_i(t) = \int_0^1 G(t,s) f_i(s)\, ds,$$

where $f_i \in S_F(y(.))$ and $i = 1, 2$. Let $0 \le \alpha \le 1$. For all $t \in [0,1]$ we have

$$\Big(\alpha h_1 + (1-\alpha) h_2\Big)(t) = \int_0^1 G(t,s)(\alpha f_1(s) + (1-\alpha) f_2(s))\, ds.$$

The set $S_F(y(.))$ is convex because $F$ is convex. Hence

$$\Big(\alpha h_1 + (1-\alpha) h_2\Big) \in T(y(.)).$$

□

Now, by (a) and (b) combined with the Ascoli's theorem we can conclude that $T$ is completely continuous.

**Claim 2** *$T$ has a closed graph.*

**Proof** Let $(y_p(.))_p$ be a sequence which converges to $y(.)$ and consider the sequence $(h_p)_p$ such that $h_p \in T(y_p(.))$ and $(h_p)_p$ converges to $h$. We shall prove that $h \in T(y(.))$. We have

$$h_p(t) = \int_0^1 G(t,s) f_p(s)\, ds,$$

where $f_p \in S_F(y_p(.))$. Now, we consider the linear continuous operator

$$\Gamma\colon L^1([0,1], E) \to \mathcal{C}([0,1], E)$$

defined by

$$\Gamma(f)(t) = \int_0^1 G(t,s) f(s)\, ds.$$

One has $h_p \in \Gamma o S_F(y_p(.))$. Since $\Gamma o S_F$ has a closed graph (see [7]), we get $h \in \Gamma o S_F(y(.))$. So, there exists $f \in S_F(y(.))$ such that $h(t) = \Gamma(f)(t)$ witch implies that $h \in T(y(.))$. □

Consequently, $T$ is upper semi-continuous. Thus $T$ satisfies all the conditions of Lemma 5.1. So $T$ has a fixed point $y(.)$ which is a solution of (2.1). Then $x(.) = (Ay)(.)$ is a solution of (1.1). □

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