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SHAPE AND TOPOLOGICAL SENSITIVITY ANALYSIS IN DOMAINS WITH CRACKS*

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Abstract. The framework for shape and topology sensitivity analysis in geometrical domains with cracks is established for elastic bodies in two spatial dimensions. The equilibrium problem for the elastic body with cracks is considered. Inequality type boundary conditions are prescribed at the crack faces providing a non-penetration between the crack faces. Modelling of such problems in two spatial dimensions is presented with all necessary details for further applications in shape optimization in structural mechanics. In the paper, general results on the shape and topology sensitivity analysis of this problem are provided. The results are of interest of their own. In particular, the existence of the shape and topological derivatives of the energy functional is obtained. The results presented in the paper can be used for numerical solution of shape optimization and inverse problems in structural mechanics.

Keywords: crack with non-penetration, shape sensitivity, derivative of energy functional, topological derivative

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1. Introduction

Shape optimization requires a few mathematical results, in the framework of modelling and numerical solution, for any specific class of problems governed by partial

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differential equations of mathematical physics. Usually, we need to show the well posedness of the specific problem, and also to propose a numerical method for the effective solution procedure. Hence, in order to solve a shape optimization problem we have to have results on

- the existence and continuous dependence with respect to the shape of solutions to the model, which may result in the existence of optimal shapes;
- the differentiability of solutions with respect to the boundary variations, which implies the existence of shape gradients and leads to some necessary conditions for optimality of the first order and possibly of the second order, which leads to the Newton method of shape optimization;
- in addition, to perform the asymptotic analysis of the related boundary value problem in singularly perturbed geometrical domains and to derive the form of the topological derivative for the shape functional of interest, which allows for the topology changes in the process of numerical optimization, if necessary;
- and finally, to device a numerical method and show its efficiency in numerical examples, and its convergence from the mathematical point of view.

One of the most important applications of shape optimization with long tradition is structural mechanics. From practical point of view, it is useful for applications that the analysis of a specific shape optimization problem is performed taking into account the possible presence of cracks, in particular in order to avoid the damage of the structure under considerations. Unfortunately, the analysis of elastic bodies with cracks is quite complex, and a little is known about mathematical modelling of cracks; however, the presence of cracks is evident, so the subject is important and the research well developed in the experimental branch of mechanics. In the present paper we consider cracks in two spatial dimensions, and we also prescribe the so-called nonpenetration conditions on the crack faces in the framework of linear elasticity. For such models we provide results on the existence of solutions, variational formulations, shape sensitivity analysis, and asymptotic analysis with respect to the singular perturbations of geometrical domains. We select the results in such a way that we construct the theoretical background for possible application of the level set method of shape optimization for the related problems. Most of the results presented here have been obtained recently, and are results of the long term collaboration between Nancy and Novosibirsk.

The aim of this paper is twofold, the modelling of elastic bodies with cracks and the sensitivity analysis of the energy functionals with respect to the boundary variations and singular perturbations of geometrical domains. The results presented in our paper can be implemented in the framework of the so-called level set method of shape optimization; we refer the reader to [4] for the numerical results obtained for the Signorini problem.

Thus, we provide some of the new results obtained in the course of the last years and related to the crack theory in elasticity with possible contact between crack faces, which are then required for the sensitivity analysis. The energy functional of an elastic body in two spatial dimensions is a representative example of a possible shape functional which can be minimized or maximized over a class of admissible domains.

First, for the modelling issue, we discuss problem formulations, peculiarities of the problems and possible relations between the topics under investigation. It is well known that the classical crack theory in elasticity is characterized by linear boundary conditions, which leads to linear boundary value problems. This approach has a clear shortcoming from the mechanical standpoint since opposite crack faces can penetrate each other. We consider nonlinear boundary conditions on crack faces, the so-called non-penetration conditions, written in terms of inequalities. From the standpoint of applications these boundary conditions are preferable since they provide a mutual non-penetration between crack faces. As a result a free boundary problem is obtained, which means that a concrete boundary condition at a given point can be found provided we have a solution of the problem.

The main attention in this paper is paid to the dependence of solutions of the problem on domain perturbations, and in particular, on the crack shape. The technique of boundary variations [28] is used in Section 5 in order to obtain the shape gradient of the energy functional. On the other hand, asymptotic analysis in singularly perturbed domains [24] is performed in Section 6 in order to obtain the topological derivative [29] of the same energy functional. In this way we have all tools necessary in the framework of the level set method of shape optimization, which is the subject of the subsequent publication.

The outline of the paper can be described as follows. We start with the strong formulation (1)–(5) of the elliptic free boundary problem. We have some inequality type conditions in (4)–(5) which lead to the free boundary and to the unknown coincidence set, which should be determined for the solution of the problem. In Section 2 the existence of weak solutions for variational formulations of problem (1)–(5) is presented. In particular, the smooth domain formulation introduced by the authors is given. In Section 3 the fictitious method is described in detail for the crack problem. In Section 4, the case of a crack on the boundary of rigid inclusion is analysed; this topic is new in the field of mathematical crack modelling to the best of our knowledge. In Section 5 the shape sensitivity analysis is performed for the singular parts of the boundary, i.e. the perturbations of the crack tips. Since the singularities of the displacement field are not explicitly known for our model, the treatment of the tips requires an appropriate technique proposed by the authors. We present in detail the construction which allows us to derive the shape derivative of the energy functional for the perturbations of the crack tips. In particular, our results can be used in the Griffith criteria for the crack propagation. The results of Section 6 are new, and constitute the topological sensitivity analysis part of the paper. The form of the topological derivative of the energy functional is obtained here for the first time, and it is exactly of the same form as in the case of linear problem. However, the proof is not the same, we use the method proposed for the Signorini problem in [31]. In Section 7 the shape sensitivity analysis is applied to the modelling of the kinking crack. Finally, in Section 8 some open problems, important for a progress in the field, are formulated.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and let $\Gamma_c \subset \Omega$ be a smooth curve without self-intersections, $\Omega_c = \Omega \setminus \overline{\Gamma}_c$ (see Fig. 1).

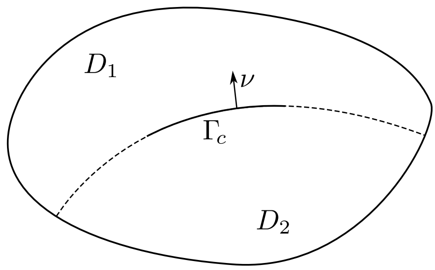

Figure 1. Domain Ω_c

It is assumed that Γ_c can be extended in such a way that this extension crosses Γ at two points, and Ω_c is divided into two subdomains D_1 and D_2 with Lipschitz boundaries ∂D_1, ∂D_2, $\operatorname{meas}(\Gamma \cap \partial D_i) > 0$, $i = 1, 2$. Denote by $\nu = (\nu_1, \nu_2)$ the unit normal vector to Γ_c. We assume that Γ_c does not contain its tip points, i.e. $\Gamma_c = \overline{\Gamma}_c \setminus \partial\Gamma_c$.

The equilibrium problem for a linear elastic body occupying Ω_c reads as follows. In the domain Ω_c we have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_c, \tag{1}$$

$$\sigma = A\varepsilon(u) \quad \text{in } \Omega_c, \tag{2}$$

$$u = 0 \quad \text{on } \Gamma, \tag{3}$$

$$[u]\nu \geqslant 0, \quad [\sigma_\nu] = 0, \quad \sigma_\nu \cdot [u]\nu = 0 \quad \text{on } \Gamma_c, \tag{4}$$

$$\sigma_\nu \leqslant 0, \quad \sigma_\tau = 0 \quad \text{on } \Gamma_c^\pm. \tag{5}$$

Here $[v] = v^+ - v^-$ is the jump of v on Γ_c, and the signs $\pm$ correspond to the positive and negative crack faces with respect to ν; $f = (f_1, f_2) \in \mathbf{L}^2(\Omega_c)$ is a given function,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu, \quad \sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2),$$
$$\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j),$$

the strain tensor components are denoted by $\varepsilon_{ij}(u)$,

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varepsilon(u) = \{\varepsilon_{ij}(u)\}, \quad i, j = 1, 2.$$

The elasticity tensor $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$, is given and satisfies the usual properties of symmetry and positive definiteness

$$a_{ijkl}\xi_{kl}\xi_{ij} \geqslant c_0|\xi|^2, \quad \forall\, \xi_{ij}, \;\; \xi_{ij} = \xi_{ji}, \quad c_0 = \text{const},$$

$a_{ijkl} = a_{klij} = a_{jikl}$, $a_{ijkl} \in L^\infty(\Omega)$.

Relations (1) are the equilibrium equations, and (2) is the generalized Hooke's law, $u_{i,j} = (\partial u_i/\partial u_j)$, $(x_1, x_2) \in \Omega_c$. All functions with two lower indices are symmetric in those indices, i.e. $\sigma_{ij} = \sigma_{ji}$ etc. Summation convention is assumed over repeated indices throughout the paper.

The first condition in (4) is called the non-penetration condition. It guarantees the mutual non-penetration between the crack faces $\Gamma_c^\pm$. The second condition of (5) guarantees zero friction on Γ_c. For simplicity we assume the clamping condition (3) at the external boundary Γ.

Note that a priori we do not know points on Γ_c where strict inequalities in (4), (5) are fulfilled. Due to this, the problem (1)–(5) is a free boundary value problem. If we have $\sigma_\nu = 0$ then, together with $\sigma_\tau = 0$, the classical boundary condition $\sigma\nu = 0$ follows which is used in the linear crack theory. On the other hand, due to (4), the condition $\sigma_\nu < 0$ implies $[u]\nu = 0$, i.e. we have a contact between the crack faces at a given point. The strict inequality $[u]\nu > 0$ at a given point means that we have no contact between the crack faces.

Hence, the first difficulty in studying the problem (1)–(5) is concerned with the boundary conditions (4)–(5). The second is related to the general crack problem difficulty—the presence of non-smooth boundaries.

2. The existence of weak solutions

We will show that the analysed problem is well posed. Therefore, there is a unique weak solution to the associated variational inequality. We introduce also the so-called smooth domain formulation [15] which has some implications in numerical analysis. For the related results in the case of a scalar problem of an elastic membrane with a cut we refer the reader to [1]. The smooth domain formulation allows to obtain variational solutions to the crack problem in the geometrical domain without any cut, the crack is *present* only in the subset of admissible functions for the variational solution, i.e., some inequality constraints are imposed on the admissible functions over the crack Γ_c.

First of all we note that problem (1)–(5) admits several equivalent formulations. In particular, it corresponds to minimization of the energy functional. To check this, we introduce the Sobolev space

$$\mathbf{H}^1_\Gamma(\Omega_c) = \{v = (v_1, v_2)\colon\ v_i \in H^1(\Omega_c),\ v_i = 0 \text{ on } \Gamma,\ i = 1, 2\}$$

and the closed convex set of admissible displacements

$$K = \{v \in \mathbf{H}^1_\Gamma(\Omega_c)\colon\ [v]\nu \geqslant 0 \text{ a.e. on } \Gamma_c\}. \tag{6}$$

In this case, due to the Weierstrass theorem, the problem

$$\min_{v\in K}\left\{\frac{1}{2}\int_{\Omega_c} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_c} f_i v_i\right\}$$

has (a unique) solution u satisfying the variational inequality

$$u \in K, \tag{7}$$

$$\int_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v-u) \geqslant \int_{\Omega_c} f_i(v_i - u_i), \quad \forall\, v \in K, \tag{8}$$

where $\sigma_{ij}(u) = \sigma_{ij}$ are defined by (2).

Problem formulations (1)–(5) and (7)–(8) are equivalent. Any smooth solution of (1)–(5) satisfies (7)–(8) and conversely, (7)–(8) implies (1)–(5).

Below we provide two more equivalent formulations for the problem (1)–(5), the so-called mixed and smooth domain formulations. To this end, we first discuss in what sense boundary conditions (4)–(5) are fulfilled. Denote by Σ a closed curve without self-intersections of the class $C^{1,1}$ which is an extension of Γ_c such that $\Sigma \subset \Omega$ and the domain Ω is divided into two subdomains Ω_1 and Ω_2 (see Fig. 2). In this case Σ is the boundary of the domain Ω_1, and the boundary of Ω_2 is $\Sigma \cup \Gamma$.

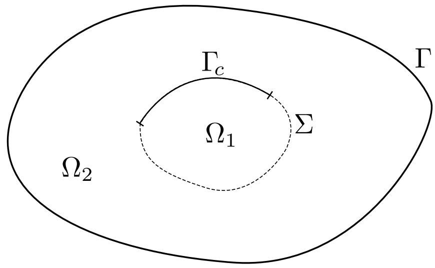

Figure 2. Extension of Γ_c to Σ.

We introduce the space $H^{1/2}(\Sigma)$ with the norm

$$\|v\|^2_{H^{1/2}(\Sigma)} = \|v\|^2_{L^2(\Sigma)} + \int_\Sigma\int_\Sigma \frac{|v(x)-v(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \tag{9}$$

and denote by $H^{-1/2}(\Sigma)$ a space dual of $H^{1/2}(\Sigma)$. Also, we denote by $H^{1/2}(\Gamma_c)$ the fractional Sobolev space defined by (9) with the set Σ replaced by Γ_c, and consider the space

$$H^{1/2}_{00}(\Gamma_c) = \Big\{v \in H^{1/2}(\Gamma_c)\colon\ \frac{v}{\sqrt{\varrho}} \in L^2(\Gamma_c)\Big\}$$

with the norm

$$\|v\|^2_{1/2,00} = \|v\|^2_{1/2} + \int_{\Gamma_c} \varrho^{-1}v^2,$$

where $\varrho(x) = \operatorname{dist}(x;\partial\Gamma_c)$ and $\|v\|_{1/2}$ is the norm in the space $H^{1/2}(\Gamma_c)$. It is known that functions from $H^{1/2}_{00}(\Gamma_c)$ can be extended to Σ by zero values, and moreover, this extension belongs to $H^{1/2}(\Sigma)$. More precisely, let v be defined on Γ_c, and let $\overline{v}$ be the extension of v by zero, i.e.

$$\overline{v}(x) = \begin{cases} v(x), & x \in \Gamma_c,\\ 0, & x \in \Sigma\setminus\Gamma_c.\end{cases}$$

Then (see [9])

$$v \in H^{1/2}_{00}(\Gamma_c) \quad \text{if and only if} \quad \overline{v} \in H^{1/2}(\Sigma).$$

With the above notation, it is possible to describe in what sense boundary conditions (4)–(5) are fulfilled. Namely, the condition $\sigma_\nu \leqslant 0$ in (5) means that

$$\langle \sigma_\nu, \varphi\rangle_{1/2,00} \leqslant 0, \quad \forall\,\varphi \in H^{1/2}_{00}(\Gamma_c), \quad \varphi \geqslant 0 \ \text{ a.e. on } \Gamma_c,$$

where $\langle\cdot,\cdot\rangle_{1/2,00}$ is a duality pairing between $H^{-1/2}_{00}(\Gamma_c)$ and $H^{1/2}_{00}(\Gamma_c)$. The condition $\sigma_\tau = 0$ in (5) means that

$$\langle \sigma^i_\tau, \varphi\rangle_{1/2,00} = 0, \quad \forall\,\varphi = (\varphi_1,\varphi_2) \in H^{1/2}_{00}(\Gamma_c), \quad i = 1,2.$$

The last condition of (4) holds in the following sense:

$$\langle \sigma_\nu, [u]\nu \rangle_{1/2,00} = 0.$$

2.1. Mixed formulation of the problem

Now we are interested in giving a mixed formulation of problem (1)–(5). Let us introduce the space of stresses

$$\mathbf{H}(\mathrm{div}) = \{\sigma = \{\sigma_{ij}\}\colon\ \sigma \in \mathbf{L}^2(\Omega_c),\ \mathrm{div}\,\sigma \in \mathbf{L}^2(\Omega_c)\}$$

with the norm

$$\|\sigma\|^2_{H(\mathrm{div})} = \|\sigma\|^2_{\mathbf{L}^2(\Omega_c)} + \|\mathrm{div}\,\sigma\|^2_{\mathbf{L}^2(\Omega_c)}$$

and the set of admissible stresses

$$\mathbf{H}(\mathrm{div};\Gamma_c) = \{\sigma \in \mathbf{H}(\mathrm{div})\colon\ [\sigma\nu] = 0 \text{ on } \Gamma_c;\ \sigma_\nu \leqslant 0,\ \sigma_\tau = 0 \text{ on } \Gamma_c^\pm\}.$$

We should note at this step that for $\sigma \in \mathbf{H}(\mathrm{div})$ the traces $(\sigma\nu)^\pm$ are correctly defined on $\Sigma^\pm$ as elements of $H^{-1/2}(\Sigma)$. The first condition in the definition of $\mathbf{H}(\mathrm{div};\Gamma_c)$ is fulfilled in the following sense:

$$(\sigma\nu)^+ = (\sigma\nu)^- \text{ on } \Sigma$$

for any curve Σ with the prescribed properties (see [9]). Relations $\sigma_\nu \leqslant 0$, $\sigma_\tau = 0$ on $\Gamma_c^\pm$ also make sense. The values σ_ν, σ_τ are defined as elements of the space $H_{00}^{-1/2}(\Gamma_c)$.

The mixed formulation of problem (1)–(5) is as follows. We have to find a displacement field $u = (u_1, u_2)$ and a stress tensor field $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in \mathbf{L}^2(\Omega_c), \quad \sigma \in \mathbf{H}(\mathrm{div};\Gamma_c), \tag{10}$$

$$-\mathrm{div}\,\sigma = f \quad \text{in } \Omega_c, \tag{11}$$

$$\int_{\Omega_c} C\sigma(\overline{\sigma} - \sigma) + \int_{\Omega_c} u(\mathrm{div}\,\overline{\sigma} - \mathrm{div}\,\sigma) \geqslant 0 \quad \forall \overline{\sigma} \in \mathbf{H}(\mathrm{div};\Gamma_c). \tag{12}$$

The tensor C is obtained by inverting Hooke's law (2), i.e.

$$C\sigma = \varepsilon(u).$$

It is possible to prove the existence of a solution to the problem (10)–(12) and check that (10)–(12) is formally equivalent to (1)–(5) (see [13]). The existence of a solution to (10)–(12) can be proved independently of (1)–(5). On the other hand, the solution exists due to the equivalence, and we already have the solution to problem (1)–(5).

2.2. Smooth domain formulation

Along with the mixed formulation (10)–(12) the so-called smooth domain formulation of problem (1)–(5) can be provided. In this case the solution of the problem is defined in the smooth domain Ω. To do this, we should notice that the solution of problem (1)–(5) satisfies (7)–(8), thus the condition

$$[\sigma\nu] = 0 \quad \text{on } \Gamma_c$$

holds, and therefore it can be proved that in the distributional sense

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega.$$

Hence, the equilibrium equations (1) hold in the smooth domain Ω.

Introduce the space of stresses defined in Ω,

$$\boldsymbol{\mathcal{H}}(\operatorname{div}) = \{\sigma = \{\sigma_{ij}\} \colon \ \sigma, \operatorname{div}\sigma \in \mathbf{L}^2(\Omega)\},$$

and the set of admissible stresses

$$\boldsymbol{\mathcal{H}}(\operatorname{div};\Gamma_c) = \{\sigma \in \mathcal{H}(\operatorname{div}) \colon \ \sigma_\tau = 0, \ \ \sigma_\nu \leqslant 0 \text{ on } \Gamma_c\}.$$

The norm in the space $\boldsymbol{\mathcal{H}}(\operatorname{div})$ is defined as

$$\|\sigma\|^2_{\boldsymbol{\mathcal{H}}(\operatorname{div})} = \|\sigma\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}\sigma\|^2_{\mathbf{L}^2(\Omega)}.$$

We see that for $\sigma \in \boldsymbol{\mathcal{H}}(\operatorname{div})$, the boundary conditions $\sigma_\tau = 0$, $\sigma_\nu \leqslant 0$ on Γ_c are correctly defined in the sense of $H^{-1/2}_{00}(\Gamma_c)$. Thus, we can provide the smooth domain formulation for problem (1)–(5). It is necessary to find a displacement field $u = (u_1, u_2)$ and a stress tensor field $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in \mathbf{L}^2(\Omega), \quad \sigma \in \boldsymbol{\mathcal{H}}(\operatorname{div};\Gamma_c), \tag{13}$$

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega, \tag{14}$$

$$\int_\Omega C\sigma(\overline{\sigma} - \sigma) + \int_\Omega u(\operatorname{div}\overline{\sigma} - \operatorname{div}\sigma) \geqslant 0 \quad \forall \overline{\sigma} \in \boldsymbol{\mathcal{H}}(\operatorname{div};\Gamma_c). \tag{15}$$

It is possible to prove the existence of a solution to problem (13)–(15) (see [15]). Moreover, any smooth solution of (1)–(5) satisfies (13)–(15) and conversely, (13)–(15) implies (1)–(5). The advantage of the formulation (13)–(15) is that it is given in the smooth domain. This formulation reminds contact problems with a thin obstacle when restrictions are imposed on sets of small dimensions (see [12]).

Numerical aspects for problems like (1)–(5) can be found, for example, in [1], [18]. In particular, in [1] the convergence of the finite element approximations is proved for a scalar problem, and some error estimates are derived.

3. Fictitious domain method

This type of modelling is also interesting from the numerical point of view, since theoretically allows for numerical computations in a fixed domain, its shape being defined by some additional constraints involving the Lagrangian multipliers. We discuss here in detail only one aspect of this technique which can be useful for numerical methods of shape optimization for frictionless contact problems.

In this section we establish the connection between problem (1)–(5) and the Signorini contact problem. It turnes out that the Signorini problem is a limit problem for a family of problems like (1)–(5). First we give a formulation of the Signorini problem. Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ_1, $\Gamma_1 = \Gamma_c \cup \Gamma_0$, $\Gamma_c \cap \Gamma_0 = \emptyset$, $\operatorname{meas} \Gamma_0 > 0$ (see Fig. 3).

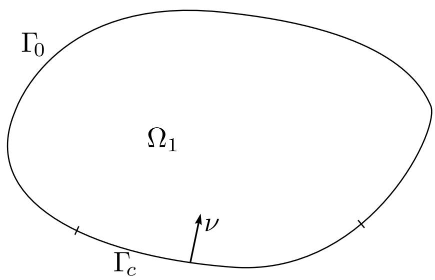

Figure 3. Signorini problem.

For simplicity, we assume that Γ_c is a smooth curve (without its tip points). Denote by $\nu = (\nu_1, \nu_2)$ the unit normal inward vector to Γ_c. We have to find a displacement field $u = (u_1, u_2)$ and a stress tensor field $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega_1, \tag{16}$$

$$\sigma = A\varepsilon(u) \quad \text{in } \Omega_1, \tag{17}$$

$$u = 0 \quad \text{on } \Gamma_0, \tag{18}$$

$$u\nu \geqslant 0, \quad \sigma_\nu \leqslant 0, \quad \sigma_\tau = 0, \quad u\nu \cdot \sigma_\nu = 0 \quad \text{on } \Gamma_c. \tag{19}$$

Here $f = (f_1, f_2) \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ is a given function, $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$ is a given elasticity tensor, $a_{ijkl} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$, with the usual properties of symmetry and positive definiteness.

It is well known that problem (16)–(19) has a variational formulation providing the existence of a solution. Namely, denote

$$\mathbf{H}^1_{\Gamma_0}(\Omega_1) = \{v = (v_1, v_2) \in \mathbf{H}^1(\Omega_1)\colon\ v_i = 0 \text{ on } \Gamma_0,\ i = 1, 2\}$$

and introduce the set of admissible displacements

$$K_c = \{v = (v_1, v_2) \in \mathbf{H}^1_{\Gamma_0}(\Omega_1)\colon\ v\nu \geqslant 0 \text{ a.e. on } \Gamma_c\}.$$

In this case problem (16)–(19) is equivalent to minimization of the functional

$$\frac{1}{2}\int_{\Omega_1} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_1} f_i v_i$$

over the set K_c and can be written in the form of variational inequality

$$u \in K_c, \tag{20}$$

$$\int_{\Omega_1} \sigma_{ij}(u)\varepsilon_{ij}(v - u) \geqslant \int_{\Omega_1} f_i(v_i - u_i) \quad \forall\, v \in K_c. \tag{21}$$

Here $\sigma_{ij}(u) = \sigma_{ij}$ are defined from the generalized Hooke's law (17). Variational inequality (20)–(21) is equivalent to (16)–(19), i.e., any smooth solution of (16)–(19) satisfies (20)–(21) and (20)–(21) implies (16)–(19). Along with the variational formulation (20)–(21) problem (16)–(19) admits a mixed formulation which is omitted here.

The aim of this section is to prove that problem (16)–(19) is a limit problem for a family of problems like (1)–(5). In what follows we will explain in this statement.

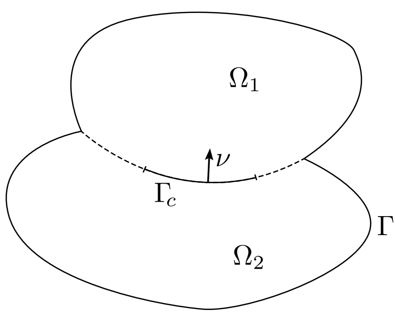

Figure 4. Extended domain Ω_c.

First of all we extend the domain Ω_1 by adding a domain Ω_2 with a piecewise smooth boundary Γ_2. Observe that the cut Γ_c in the extended domain is included in the intersection of the boundaries $\Gamma_1 \cap \Gamma_2$. The extended domain defined in this way is denoted by Ω_c, and it has a crack (cut) Γ_c. The boundary of Ω_c is $\Gamma \cup \Gamma_c^{\pm}$ (see Fig. 4). Denote $\Sigma_0 = \Gamma_1 \cap \Gamma_2$, $\Sigma = \Sigma_0 \setminus \Gamma$, hence Σ does not contain the tip points of Σ_0.

We introduce a family of elasticity tensors with a positive parameter λ,

$$a_{ijkl}^{\lambda} = \begin{cases} a_{ijkl} & \text{in } \Omega_1, \\ \lambda^{-1} a_{ijkl} & \text{in } \Omega_2. \end{cases}$$

Denote $A^{\lambda} = \{a_{ijkl}^{\lambda}\}$, and in the extended domain Ω_c consider the family of the crack problems. Find a displacement field $u^{\lambda} = (u_1^{\lambda}, u_2^{\lambda})$, and a stress tensor field $\sigma^{\lambda} = \{\sigma_{ij}^{\lambda}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div} \sigma^{\lambda} = f \quad \text{in } \Omega_c, \tag{22}$$

$$\sigma^{\lambda} = A^{\lambda} \varepsilon(u^{\lambda}) \quad \text{in } \Omega_c, \tag{23}$$

$$u^{\lambda} = 0 \quad \text{on } \Gamma, \tag{24}$$

$$[u^{\lambda}]\nu \geqslant 0, \quad [\sigma_{\nu}^{\lambda}] = 0, \quad \sigma_{\nu}^{\lambda} \cdot [u^{\lambda}]\nu = 0 \quad \text{on } \Gamma_c, \tag{25}$$

$$\sigma_{\nu}^{\lambda} \leqslant 0, \quad \sigma_{\tau}^{\lambda} = 0 \quad \text{on } \Gamma_c^{\pm}. \tag{26}$$

As before, $[v] = v^{+} - v^{-}$ is the jump of v through Γ_c, where $\pm$ fit the positive and negative crack faces $\Gamma_c^{\pm}$. All the remaining notation corresponds to that of Section 1. We see that for any fixed $\lambda > 0$ problem (22)–(26) describes an equilibrium state of the linear elastic body with the crack Γ_c where non-penetration conditions are prescribed. Hence, problem (22)–(26) is exactly the problem like (1)–(5), and we are interested in passing to the limit as $\lambda \to 0$. In particular, problem (22)–(26) admits a variational formulation. Boundary conditions (25)–(26) are fulfilled in the form as is explained in Section 2. It can be proved (see [7]) that the following convergence takes place as $\lambda \to 0$:

$$u^{\lambda} \to u^{0} \quad \text{strongly in } \mathbf{H}_{\Gamma}^{1}(\Omega_c), \tag{27}$$

$$\frac{u^{\lambda}}{\sqrt{\lambda}} \to 0 \quad \text{strongly in } \mathbf{H}^{1}(\Omega_2), \tag{28}$$

where $u^{0} = u$ on Ω_1, i.e. the restriction of the limit function from (27) to Ω_1 coincides with the unique solution of the Signorini problem (16)–(19). From (27)–(28) it is seen that the limit function u^{0} is zero in Ω_2. On the other hand, the convergence

$$\int_{\Omega_2} \sigma_{ij}^{\lambda} \varepsilon_{ij}(u^{\lambda}) = \frac{1}{\lambda} \int_{\Omega_2} a_{ijkl} \varepsilon_{ij}(u^{\lambda}) \varepsilon_{ij}(u^{\lambda}) \to 0$$

does not imply the existence of a limit for σ^λ in Ω_2 as $\lambda \to 0$. Thus, the domain Ω_2 can be understood as an undeformable body. This means that the Signorini problem is, in fact, a crack problem with the non-penetration condition between crack faces, where the crack Γ_c is located between the elastic body Ω_1 and the nondeformable (rigid) body Ω_2. It is worth noting that, in fact, we can write problem (22)–(26) in the equivalent form in the smooth domain $\Omega_c \cup \overline{\Gamma}_c$ by using the smooth domain formulation (Section 2.2). Some additional details of the fictitious domain method in the crack theory can be found in [7].

4. Crack on the boundary of rigid inclusion

The inclusions in elastic bodies are also important for applications, both in design procedures and in numerical solution of some inverse problems. We restrict ourselves to the limit case of a rigid inclusion, with a crack at the interface. This seems to be a new class of problems, both for the analysis and for the shape optimization. One can also attempt to find the shape derivative of the elastic energy with respect to the perturbations of the crack tip; some results in this direction are given with all details in Section 5.

We consider a rigid inclusion inside of the rigid body. This section is concerned with a crack situated on the boundary of the rigid inclusion.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ, and let $\omega \subset \Omega$ be a subdomain with a smooth boundary Σ, $\overline{\omega} \subset \Omega$. Assume that Σ is composed of two parts: $\Sigma = \Gamma_c \cup (\Sigma \setminus \Gamma_c)$, $\mathrm{meas}(\Sigma \setminus \Gamma_c) > 0$, see Fig. 5. Denote $\Omega_c = \Omega \setminus \overline{\Gamma}_c$. As

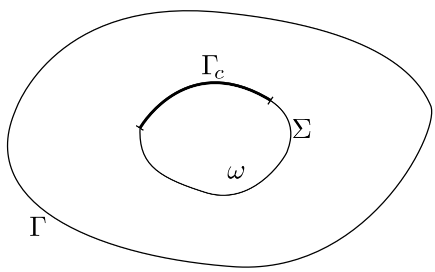

Figure 5. Rigid inclusion ω in elastic body.

before, by $A = \{a_{ijkl}\}$ we denote an elasticity tensor with the usual symmetry and positive definiteness properties, $a_{ijkl} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$. For a positive parameter $\lambda > 0$ we introduce the elasticity tensor

$$a^\lambda_{ijkl} = \begin{cases} a_{ijkl} & \text{in } \Omega \setminus \overline{\omega}, \\ \lambda^{-1} a_{ijkl} & \text{in } \omega, \end{cases} \qquad i, j, k, l = 1, 2,$$

and consider a boundary value problem for finding a displacement field $u^\lambda = (u_1^\lambda, u_2^\lambda)$ and a stress tensor field $\sigma^\lambda = \{\sigma_{ij}^\lambda\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma^\lambda = f \quad \text{in } \Omega_c, \tag{29}$$

$$\sigma^\lambda - A^\lambda \varepsilon(u^\lambda) = 0 \quad \text{in } \Omega_c, \tag{30}$$

$$u^\lambda = 0 \quad \text{on } \Gamma, \tag{31}$$

$$[u^\lambda]\nu \geqslant 0, \quad [\sigma_\nu^\lambda] = 0, \quad \sigma_\nu^\lambda \cdot [u^\lambda]\nu = 0 \quad \text{on } \Gamma_c, \tag{32}$$

$$\sigma_\tau^\lambda = 0, \quad \sigma_\nu^\lambda \leqslant 0 \quad \text{on } \Gamma_c^\pm. \tag{33}$$

Here $f = (f_1, f_2) \in \mathbf{L}^2(\Omega)$ is a given function. We see that for any $\lambda > 0$ the problem (29)–(33) is the problem like (1)–(5) describing an equilibrium state for the elastic body with the crack Γ_c. This problem has the variational formulation, the mixed formulation and the smooth domain formulation. Our aim is to consider the limit case as $\lambda \to 0$. This can be done by analyzing the variational inequality

$$u^\lambda \in K, \tag{34}$$

$$\int_{\Omega_c} \sigma_{ij}^\lambda(u^\lambda)\varepsilon_{ij}(v - u^\lambda) \geqslant \int_{\Omega_c} f_i(v_i - u_i^\lambda) \quad \forall\, v \in K. \tag{35}$$

Here $\sigma_{ij}^\lambda(u^\lambda) = \sigma_{ij}^\lambda$ are defined from (30), and the set K was introduced in (6).

We can pass to the limit in (34)–(35) as $\lambda \to 0$. To this end, we introduce the space of infinitesimal rigid displacements

$$R(\omega) = \{\varrho = (\varrho_1, \varrho_2)\colon\ \varrho(x) = Bx + D,\ x \in \omega\},$$

where

$$B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad D = (d^1, d^2); \quad b, d^1, d^2 = \text{const}.$$

Consider next the space

$$\mathbf{H}_\Gamma^{1,\omega}(\Omega_c) = \{v \in \mathbf{H}_\Gamma^1(\Omega_c)\colon\ v = \varrho \text{ on } \omega,\ \ \varrho \in R(\omega)\}$$

and the set of admissible displacements

$$K_\omega = \{v \in \mathbf{H}_\Gamma^{1,\omega}(\Omega_c)\colon\ (v^+ - \varrho)\nu \geqslant 0 \ \text{ a.e. on } \Gamma_c\}.$$

Here v^+ corresponds to the crack face Γ_c^+. Now we substitute $v = 0$, $v = 2u^\lambda$ as test functions in (35). This yields the relation

$$\int_{\Omega_c} \sigma_{ij}^\lambda(u^\lambda)\varepsilon_{ij}(u^\lambda) = \int_{\Omega_c} f_i u_i^\lambda,$$

which implies two estimates

$$
\begin{aligned}
&(36) && \|u^\lambda\|_{\mathbf{H}^1_\Gamma(\Omega_c)} \leqslant c_1,\\
& && \frac{1}{\lambda}\int_\omega a_{ijkl}\varepsilon_{kl}(u^\lambda)\varepsilon_{ij}(u^\lambda) \leqslant c_2,
\end{aligned}
$$

being uniform in λ, $0<\lambda<\lambda_0$. Consequently, we can assume that as $\lambda \to 0$

$$
(37) \qquad u^\lambda \to u \quad \text{weakly in } \mathbf{H}^1_\Gamma(\Omega_c).
$$

Moreover, by (36)

$$
\int_\omega a_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(u) \leqslant \liminf \int_\omega a_{ijkl}\varepsilon_{kl}(u^\lambda)\varepsilon_{ij}(u^\lambda) \leqslant 0.
$$

Hence,

$$
\varepsilon_{ij}(u) = 0 \quad \text{in } \omega, \quad i,j = 1,2.
$$

This means the existence of a function ϱ_0 such that

$$
u = \varrho_0 \quad \text{in } \omega, \quad \varrho_0 \in R(\omega).
$$

Since u^λ converge to u weakly in $\mathbf{H}^1_\Gamma(\Omega_c)$ and $u^\lambda \in K$, it follows that

$$
(u^+ - \varrho_0)\nu \geqslant 0 \quad \text{on } \Gamma_c.
$$

In particular, $u \in K_\omega$. Now we take an arbitrary function $v \in R(\omega)$. In this case, there exists $\varrho \in R(\omega)$, such that $v = \varrho$ on ω. It is clear that the extension of v can be substituted in (35) as a test function. Since $A^\lambda = A$ in $\Omega \setminus \overline{\omega}$ we can pass to the limit as $\lambda \to 0$ in (34), (35) which leads to the variational inequality

$$
(38) \qquad u \in K_\omega,
$$

$$
(39) \qquad \int_{\Omega\setminus\overline{\omega}} \sigma_{ij}(u)\varepsilon_{ij}(v-u) \geqslant \int_{\Omega_c} f_i(v_i - u_i) \quad \forall\, v \in K_\omega.
$$

In other words, the above variational inequality follows from the boundary value problem written below. In order to derive the variational inequality it is sufficient to multiply (40) by $v \in K_c$, integrate over Ω_c and take into account (41)–(45).

Problem (38)–(39) describes an equilibrium state of the body occupying the domain Ω_c which has the crack Γ_c and the rigid inclusion ω. This means that any possible displacement in ω has the form $\varrho(x)$, $x \in \omega$, where $\varrho \in R(\omega)$. Problem (38)–(39) can be written in the differential form as follows. In the domain Ω_c, we have to find a displacement field $u = (u_1, u_2)$; $u = \varrho_0$ in ω; $\varrho_0 \in R(\omega)$; and in the domain $\Omega \setminus \overline{\omega}$ we have to find the stress tensor field $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega \setminus \overline{\omega}, \tag{40}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega \setminus \overline{\omega}, \tag{41}$$

$$u = 0 \quad \text{on } \Gamma, \tag{42}$$

$$(u - \varrho_0)\nu \geqslant 0, \quad \sigma_\tau = 0, \quad \sigma_\nu \leqslant 0 \quad \text{on } \Gamma_c^+, \tag{43}$$

$$\sigma_\nu \cdot (u - \varrho_0)\nu = 0 \quad \text{on } \Gamma_c^+, \tag{44}$$

$$-\int_\Sigma \sigma\nu \cdot \varrho = \int_\omega f_i \varrho_i \quad \forall\, \varrho \in R(\omega). \tag{45}$$

Problem formulations (38)–(39) and (40)–(45) are equivalent. This means that any smooth solution of (40)–(45) satisfies (38)–(39) and conversely, (38)–(39) implies (40)–(45).

Like in the previous sections, it is possible to describe in what sense the boundary conditions (43)–(45) are fulfilled. In particular, the last two conditions of (43) are fulfilled in the sense of $H_{00}^{-1/2}(\Gamma_c)$. As for (44), it is fulfilled in the form

$$\langle \sigma_\nu^+, (u - \varrho_0)\nu \rangle_{1/2,00,\Gamma_c} = 0.$$

Condition (45) holds as follows:

$$-\langle \sigma\nu, \varrho \rangle_{1/2,\Sigma} = \int_\omega f_i \varrho_i \quad \forall\, \varrho \in R(\omega).$$

To conclude this section, we note that the variational inequality (38) –(39) is equivalent to the minimization of the functional

$$\frac{1}{2} \int_{\Omega \setminus \overline{\omega}} \sigma_{ij}(v) \varepsilon_{ij}(v) - \int_{\Omega_c} f_i v_i$$

over the set K_ω.

5. Shape derivatives of energy functionals

This section and Section 6 are the most important contributions from the point of view of the shape optimization. Here, the boundary variation technique is applied in order to derive the form of the shape derivative of the energy functional with respect to the perturbations of the crack tips. We refer to [3], [19] for some results in this direction which apply to the domains with cracks. Such results constitute a complement to the monograph [28], where the shape sensitivity of elliptic boundary value problems in domains with cracks is not dealt with. On the other hand, in [28] the material derivatives of the solutions to the frictionless contact problem of an elastic body with the rigid foundation are obtained in the framework of the conical differentiability of solutions to variational inequalities.

The difficulty associated with the specific problem analysed in this section is in particular the lack of any information on the form of singularities of the displacement field at the crack tips. Therefore, we provide the precise form of the shape derivative using the path independent integrals, which is the standard procedure in the linear fracture mechanics. The structure of shape derivatives for shape differentiable functionals in domains with cracks is given in [19].

In the crack theory, the Griffith criterion is widely used to predict a crack propagation. This criterion says that a rack propagates provided the derivative of the energy functional with respect to the crack length reaches a critical value. In this section we discuss this question for the model (1)–(5). We refer also to [16] for some developments in the framework of finite strain elasticity and the rate-independent model.

The general point of view is that we should consider a perturbed problem with respect to (1)–(5). In particular, the crack length may be perturbed. The perturbation will be characterized by a small parameter t, and $t = 0$ corresponds to the unperturbed problem, i.e. to problem (1)–(5). To describe properly a perturbation of the problem, we should have a perturbation of the domain Ω_c. It will be done via the so-called velocity method (see [28]). This means that we consider a given velocity field V defined in $\mathbb{R}^2$ and describe a perturbation of Ω_c by solving a Cauchy problem for a system of ODE. Namely, let $V \in W^{1,\infty}(\mathbb{R}^2)^2$ be a given field, $V = (V_1, V_2)$. Consider a Cauchy problem for finding a function $\Phi = (\Phi_1, \Phi_2)$,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}(t,\cdot) = V(\Phi(t,\cdot)) \quad \text{for } t \neq 0, \quad \Phi(0,x) = x. \tag{46}$$

There exists a unique solution Φ to (46) such that

$$\Phi = (\Phi_1, \Phi_2)(t,x) \in C^1([0,t_0]; W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)^2), \quad |t_0| > 0. \tag{47}$$

Simultaneously, we can find a solution $\Psi = (\Psi_1, \Psi_2)$ to the Cauchy problem

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t}(t,\cdot) = -V(\Psi(t,\cdot)) \quad \text{for } t \neq 0, \quad \Psi(0,y) = y \tag{48}$$

with the same regularity

$$\Psi = (\Psi_1, \Psi_2)(t,y) \in C^1([0,t_0]; W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)^2), \quad |t_0| > 0. \tag{49}$$

It can be proved that for any fixed t, the function $\Psi(t,\cdot)$ is inverse to $\Phi(t,\cdot)$, which means the following (see the proof in [10]):

$$y = \Phi(t, \Psi(t, y)), \quad x \in \Psi(t, \Phi(t, x)), \quad x, y \in \mathbb{R}^2.$$

Due to this, we have a one-to-one mapping between the domain Ω_c and a perturbed domain Ω_c^t, namely

$$\begin{aligned} y &= \Phi(t, x)\colon\ \Omega_c \to \Omega_c^t, \\ x &= \Psi(t, y)\colon\ \Omega_c^t \to \Omega_c. \end{aligned}$$

Moreover, by (47), (49) we have the asymptotic expansions (I denotes the identity operator)

$$\Phi(t, x) = x + tV(x) + r_1(t), \tag{50}$$

$$\Psi(t, y) = y - tV(y) + r_2(t), \tag{51}$$

$$\frac{\partial \Phi(t)}{\partial x} = I + t\frac{\partial V}{\partial x} + r_3(t), \tag{52}$$

$$\frac{\partial \Psi(t)}{\partial y} = I - t\frac{\partial V}{\partial y} + r_4(t), \tag{53}$$

$$\begin{aligned} \|r_i(t)\|_{W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)^2} &= o(t), \quad i = 1, 2, \\ \|r_i(t)\|_{L^{\infty}_{\mathrm{loc}}(\mathbb{R}^2)^{2\times 2}} &= o(t), \quad i = 3, 4. \end{aligned}$$

Hence, in the domain Ω_c^t it is possible to consider the following boundary value problem (perturbed with respect to(1)–(5)): Find a displacement field $u^t = (u_1^t, u_2^t)$, and a stress tensor field $\sigma^t = \{\sigma_{ij}^t\}$, $i, j = 1, 2$, such that

$$-\operatorname{div} \sigma^t = f \quad \text{in } \Omega_c^t, \tag{54}$$

$$\sigma^t = A\varepsilon(u^t) \quad \text{in } \Omega_c^t, \tag{55}$$

$$u^t = 0 \quad \text{on } \Gamma^t, \tag{56}$$

$$[u^t]\nu^t \geqslant 0, \quad [\sigma_{\nu^t}^t] = 0, \quad \sigma_{\nu^t}^t \cdot [u^t]\nu^t = 0 \quad \text{on } \Gamma_c^t, \tag{57}$$

$$\sigma_{\nu^t}^t \leqslant 0, \quad \sigma_{\tau^t}^t = 0 \quad \text{on } \Gamma_c^{t\pm}. \tag{58}$$

Here

$$y = \Phi(t, x)\colon\ \Gamma \to \Gamma^t, \quad \Gamma_c \to \Gamma_c^t,$$

and we assume in this section that $f = (f_1, f_2) \in C^1(\mathbb{R}^2)$ and $a_{ijkl} = \text{const}$, $i, j, k, l = 1, 2$. All the other notation in (54)–(58) follows that of (1)–(5), in particular, $\nu^t = (\nu_1^t, \nu_2^t)$ is the unit normal vector to Γ_c^t.

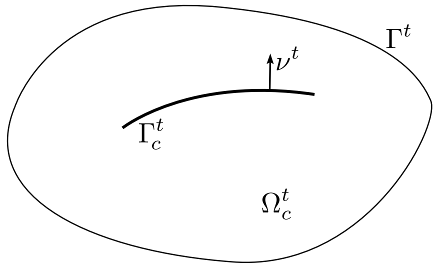

Figure 6. Perturbed domain Ω_c^t.

We can provide a variational formulation of problem (54)–(58). Indeed, let us introduce the Sobolev space

$$\mathbf{H}^1_{\Gamma^t}(\Omega_c^t) = \{v = (v_1, v_2) \colon \ v_i \in H^1(\Omega_c^t), \ v_i = 0 \text{ on } \Gamma^t, \ i = 1, 2\}$$

and the set of admissible displacements

$$K^t = \{v \in \mathbf{H}^1_{\Gamma^t}(\Omega_c^t) \colon \ [v]\nu^t \geqslant 0 \text{ a.e. on } \Gamma_c^t\}.$$

Consider the functional

$$\Pi(\Omega_c^t; v) = \frac{1}{2}\int_{\Omega_c^t} \sigma_{ij}^t(v)\varepsilon_{ij}(v) - \int_{\Omega_c^t} f_i v_i$$

and the minimization problem

$$\min_{v \in K^t} \Pi(\Omega_c^t; v). \tag{59}$$

Here $\sigma_{ij}^t(v)$ are defined from Hooke's law similar to (55). The solution of problem (59) exists and satisfies the variational inequality

$$u^t \in K^t, \tag{60}$$

$$\int_{\Omega_c^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(v - u^t) \geqslant \int_{\Omega_c^t} f_i(v_i - u_i^t) \quad \forall\, v \in K^t. \tag{61}$$

Having found a solution of problem (60)–(61), we can define the energy functional

$$\Pi(\Omega_c^t; u^t) = \frac{1}{2}\int_{\Omega_c^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(u^t) - \int_{\Omega_c^t} f_i u_i^t.$$

Note that for $t = 0$ we have $\Omega_c^0 = \Omega_c$ and $u^0 = u$, where u is the solution of the unperturbed problem (7), (8). The question is whether it is possible to differentiate the functional $\Pi(\Omega_c^t; u^t)$ with respect to t? We have in mind the existence of the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(\Omega_c^t; u^t)|_{t=0} = \lim_{t\to 0}\frac{\Pi(\Omega_c^t; u^t) - \Pi(\Omega_c; u)}{t}. \tag{62}$$

The answer is in affirmative in many practical situations. We consider two cases, where the derivative

$$I = \frac{\mathrm{d}}{\mathrm{d}t}\Pi(\Omega_c^t; u^t)\Big|_{t=0} \tag{63}$$

exists.

a) Assume that the normal vector ν to Γ_c keeps its value under the mapping $x \to \Phi(t, x)$, i.e. $\nu^t = \nu$. In this case, it has been proved that the formula for I can be obtained, namely (see [11], [17], [20]),

$$I = \frac{1}{2}\int_{\Omega_c}\{\operatorname{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u)\}\sigma_{ij}(u) - \int_{\Omega_c}\operatorname{div}(Vf_i)u_i, \tag{64}$$

where

$$E_{ij}(U; v) = \frac{1}{2}(v_{i,k}U_{k,j} + v_{j,k}U_{k,i}), \quad U = \{U_{i,j}\}, \quad i, j = 1, 2.$$

Note that the assumption concerning the normal vector ν takes place for rectilinear cracks Γ_c and vector fields V tangential to Γ_c (see Fig. 7). In this situation, (64) can provide a formula for the derivative of the energy functional with respect to the crack length which is practically needed for using the Griffith criterion. It will be the case when $V = (1, 0)$ in a vicinity of the right crack tip and $\operatorname{supp} V$ belongs to a small neighborhood of this tip (see Fig. 7).

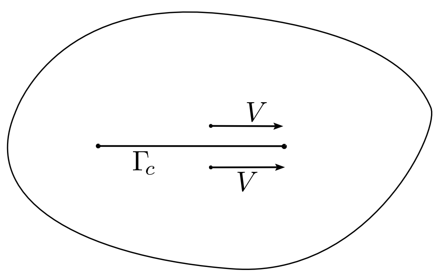

Figure 7. Rectilinear crack Γ_c and tangential field V.

b) A formula for the derivative (63) can be derived for curvilinear cracks when the above assumption on the normal vector ν is not fulfilled. We provide here the formula (63) when the crack Γ_c is described as the graph of a smooth function.

Let $\psi \in H^3(0, l_1)$ be a given function, $l_1 > 0$, and

$$\Sigma = \{(x_1, x_2)\colon\ x_2 = \psi(x_1),\ 0 < x_1 < l_1\}.$$

Consider a crack Γ_l, $\Gamma_l \subset \Sigma$, as the graph of a function ψ, see Fig. 8,

$$\Gamma_l = \{(x_1, x_2)\colon\ x_2 = \psi(x_1),\ 0 < x_1 < l\}, \quad 0 < l < l_1.$$

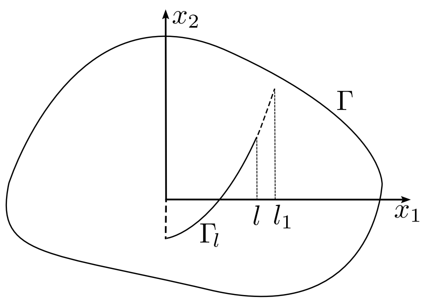

Figure 8. Domain Ω_l with a crack Γ_l.

Here l is a parameter that characterizes the length of the projection of the crack Γ_l onto the x_1 axis. Consider a smooth cut-off function θ with a support in a vicinity of the crack tip $(l, \psi(l))$. Moreover, we assume that $\theta = 1$ in a small neighborhood of $(l, \psi(l))$. We can consider a perturbation of the crack Γ_l along Σ via a small parameter t. Denote $\Omega_l = \Omega \setminus \overline{\Gamma}_l$. The perturbed crack Γ_l^t has the tip $(l+t, \psi(l+t))$, and we consider the perturbed domain $\Omega_l^t = \Omega \setminus \overline{\Gamma}_l^t$. It is possible to establish a one-to-one correspondence between Ω_l and Ω_l^t by the formulas

$$\begin{aligned} y_1 &= x_1 + t\theta(x), \\ y_2 &= x_2 + \psi(x_1 + t\theta(x)) - \psi(x_1), \end{aligned} \qquad (x_1, x_2) \in \Omega_l,\ (y_1, y_2) \in \Omega_l^t. \tag{65}$$

The transformation (65) is equivalent to (cf. (50))

$$y = x + tV(x) + r(t, x)$$

with the velocity field

$$V(x) = (\theta(x), \psi'(x_1)\theta(x)). \tag{66}$$

In the domain Ω_l^t we can consider a perturbed problem formulation. Namely, it is necessary to find a displacement field $u^t = (u_1^t, u_2^t)$ and the stress tensor field $\sigma^t = \{\sigma_{ij}^t\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma^t = f \quad \text{in } \Omega_l^t, \tag{67}$$

$$\sigma^t = A\varepsilon(u^t) \quad \text{in } \Omega_l^t, \tag{68}$$

$$u^t = 0 \quad \text{on } \Gamma, \tag{69}$$

$$[u^t]\nu^t \geqslant 0, \quad [\sigma_{\nu^t}^t] = 0, \quad \sigma_{\nu^t}^t \cdot [u^t]\nu^t = 0 \quad \text{on } \Gamma_l^t, \tag{70}$$

$$\sigma_{\nu^t}^t \leqslant 0, \quad \sigma_{\tau^t}^t = 0 \quad \text{on } \Gamma_l^{t\pm}. \tag{71}$$

Here $\nu^t = (\nu_1^t, \nu_2^t)$ is the unit normal vector to Γ_l^t. For a solution u^t of (67)–(71) it is possible to define the energy functional

$$\Pi(\Omega_l^t; u^t) = \frac{1}{2}\int_{\Omega_l^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(u^t) - \int_{\Omega_l^t} f_i u_i^t$$

and to find the derivative

$$\Pi'(l) = \frac{\mathrm{d}\Pi(\Omega_l^t; u^t)}{\mathrm{d}t}\bigg|_{t=0}$$

with the formula (see [25])

$$\begin{aligned}\Pi'(l) = {} & \frac{1}{2}\int_{\Omega_l} \{\operatorname{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u)\}\sigma_{ij}(u) \\ & - \int_{\Omega_l} \operatorname{div}(V f_i)u_i + \int_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(w) - \int_{\Omega_l} f_i w_i,\end{aligned} \tag{72}$$

where the vector field V is defined in (66) and $w = (0, \theta\psi'' u_1)$ is a given function. Note that the formula (72) contains the function θ, but in fact there is no dependence of the right-hand side of (72) on θ. In fact, the right-hand side in formula (62), and hence the right-hand side of (63) does not contain θ. Since the limit exists, hence there is no dependence on θ by the limit passage.

In particular, if $\psi'' = 0$, the formula (72) reduces to (64) with $\Omega_c = \Omega_l$. In this case we have a rectilinear crack and $\nu^t = \nu$. Formula (72) defines the derivative of the energy functional with respect to the length of the projection of the crack Γ_l onto the x_1 axis. Hence, the derivative of the energy functional with respect to the length of the curvilinear crack is

$$\Pi'(s) = \Pi'(l)(\psi'(l)^2 + 1)^{-1/2},$$

where

$$s = \int_0^l \sqrt{\psi'(t)^2 + 1}$$

is the length of the crack Γ_l.

To conclude this section we shortly discuss the existence of the so-called invariant integrals in the crack theory analyzed. It turnes out that the formula (64) for the derivative of the energy functional can be rewritten as an integral over an arbitrary closed curve surrounding the crack tip.

Consider the simplest case of a rectilinear crack $\Gamma_c = (0,1) \times \{0\}$ assuming that $\overline{\Gamma}_c \subset \Omega$, see Fig. 9. Let θ be a smooth cut-off function equal to 1 near the point

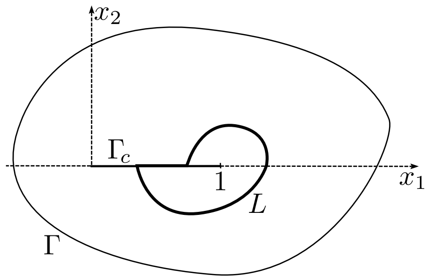

Figure 9. Curve L surrounding a crack tip.

$(1,0)$, and let $\operatorname{supp}\theta$ belong to a small neighborhood of the point $(1,0)$. Then we can take the vector field

$$V = (\theta, 0)$$

in (46), (48) which, according to (50), corresponds to the change of independent variables

$$\begin{aligned} y_1 &= x_1 + t\theta(x) + r_{11}(t), \\ y_2 &= x_2. \end{aligned}$$

In this case the formula (64) (or the formula (72) in the particular case $\psi = 0$) provides the derivative of the energy functional with respect to the crack length. This formula can be rewritten as an integral over a curve L surrounding the crack tip $(1,0)$ (see Fig. 9, solid line). Namely, the following formula is valid (see [14], [17]):

$$I = \int_L \Big\{ \frac{1}{2} \nu_1 \sigma_{ij}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u) u_{i,1} \nu_j \Big\} \tag{73}$$

provided f is equal to zero in a neighborhood of the point $(1,0)$. We should underline two important points. First, the formula (73) is independent of L, and second, the right-hand side of (73) is equal to the derivative of the energy functional with respect to the crack length.

In fact, invariant integrals like (73) can be obtained in more complex situations. For example, we can assume that the crack Γ_c is situated on the interface between two media which means that the elasticity tensor $A = \{a_{ijkl}\}$ is (see Fig. 9)

$$a_{ijkl} = \begin{cases} a^1_{ijkl} & \text{for } x_2 > 0, \\ a^2_{ijkl} & \text{for } x_2 < 0. \end{cases}$$

Here $a^1_{ijkl} = \text{const}$, $a^2_{ijkl} = \text{const}$, $i, j, k, l = 1, 2$, and $\{a^1_{ijkl}\}$, $\{a^2_{ijkl}\}$ possess the usual properties of symmetry and positive definiteness. In this case, formula (64) for the derivative of the energy functional holds true provided V is tangential to Γ_c. This formula provides the existence of an invariant integral of the form (73). We should remark at this point that when calculating the integral (73), the values $\sigma_{ij}(u)u_{i,1}\nu_j$ can be taken at Γ_c^+ or at Γ_c^-. It gives the same value of integral (73). This statement takes place due to the equality (see [8])

$$[\sigma_{ij}(u)u_{i,1}\nu_j] = 0 \quad \text{on } \Gamma_c.$$

On the other hand, we can analyze the case when the rigidity of the elastic body part $\Omega_c \cap \{x_2 < 0\}$ goes to infinity. Indeed, consider the elasticity tensor for a positive parameter $\lambda > 0$,

$$a^\lambda_{ijkl} = \begin{cases} a^1_{ijkl} & \text{for } x_2 > 0, \\ \lambda^{-1}a^2_{ijkl} & \text{for } x_2 < 0. \end{cases}$$

Then for any fixed $\lambda > 0$, the solution of the equilibrium problem like (1)–(5) exists, and we can pass to the limit as $\lambda \to 0$. As we already noted in Section 3, in the limit the following contact Signorini problem is obtained. Find a displacement field $u = (u_1, u_2)$ and a stress tensor field $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_c \cap \{x_2 > 0\}, \tag{74}$$

$$\sigma = A\varepsilon(u) \quad \text{in } \Omega_c \cap \{x_2 > 0\}, \tag{75}$$

$$u = 0 \quad \text{on } \partial(\Omega_c \cap \{x_2 > 0\}) \setminus \Gamma_c, \tag{76}$$

$$u\nu \geqslant 0, \quad \sigma_\nu \leqslant 0, \quad \sigma_\tau = 0, \quad \sigma_\nu \cdot u\nu = 0 \quad \text{on } \Gamma_c. \tag{77}$$

For problem (74)–(77) it is possible to differentiate the energy functional in the direction of the vector field $V = (\theta, 0)$, where the properties of θ are described above. The formula for the derivative has the form (cf. (64))

$$I = \frac{1}{2}\int_{\Omega_1} \{\operatorname{div} V \cdot \sigma_{ij}(u) - 2E_{ij}(V, u)\}\sigma_{ij}(u) - \int_{\Omega_1} \operatorname{div}(Vf_i)u_i. \tag{78}$$

Assume that $f = 0$ in a neighborhood of the point $(1, 0)$. In this case, formula (78) can be rewritten in the form of an invariant integral

$$I = \int_{L_1} \Big\{\frac{1}{2}\nu_1\sigma_{ij}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u)u_{i,1}\nu_j\Big\}, \tag{79}$$

where L_1 is a smooth curve "covering" the point $(1, 0)$ (see Fig. 10, solid line). Like for invariant integrals in the crack problems, formula (79) is independent of the choice of L_1.

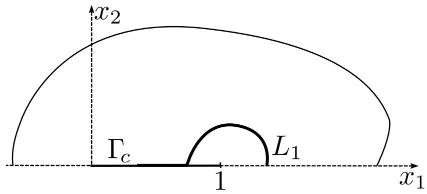

Figure 10. Curve L_1 "covering" a tip of the contact set.

6. Singular domain perturbations, topological derivatives

Boundary variations technique applied in Section 5 to prove the shape differentiability of the elastic energy functional in domains with a crack should be complemented by the asymptotic analysis of the functional [24] in singularly perturbed domains. Such an analysis is particularly related to the topology optimization, and the knowledge of the so-called *topological derivative* of the shape functional provides the information [29] whenever a small hole can be created in the process of numerical solution of some shape optimization problem, e.g., in the framework of the so-called *level set* method for variational inequalities [4].

In this section the topological derivative of the energy functional for elasticity boundary value problems in domains with cracks is obtained. To this end the domain decomposition technique is used in the same way as it is proposed in [31] for the Signorini problem, and used in [4] for the purposes of numerical methods of shape optimization. Therefore, the results given here can be applied in numerical solution of shape optimization in domains with cracks.

We briefly explain what we mean by the topological derivative of a shape functional. The notion of the topological derivative is new, the results are obtained in the framework of the asymptotic analysis of elliptic boundary value problems in singularly perturbed geometrical domains in the spirit of [22]. The full mathematical framework for linear elasticity boundary value problems can be found in [24].

First, let us make precise what is the meaning of a singularly perturbed geometrical domain for an elastic body with cracks. We introduce a small parameter $\varrho > 0$ which describes the singular perturbations of the elastic body under consideration. We divide the elastic body $\mathcal{D}$ into two parts denoted by Ω_0 and Ω_c, respectively, and denote by $B_\varrho(x)$ the hole which is located in Ω_0. The domain with the hole is denoted by $\Omega_\varrho = \Omega_0 \setminus \overline{B_\varrho(x)}$, the boundary Σ of Ω_0 is fixed and independent of the small parameter $\varrho > 0$, see Fig. 11. This means that for $\varrho > 0$ we consider the geometrical domain $\mathcal{D}_\varrho = \Omega_\varrho \cup \Sigma \cup \Omega_c$, the crack being located in Ω_c, and the hole $B_\varrho(x)$ being located in Ω_ϱ. The domain Ω_ϱ with the boundary $\Sigma \cup \partial B_\varrho(x)$ includes the hole $B_\varrho(x)$, see Fig. 11. For the purposes of the asymptotic analysis

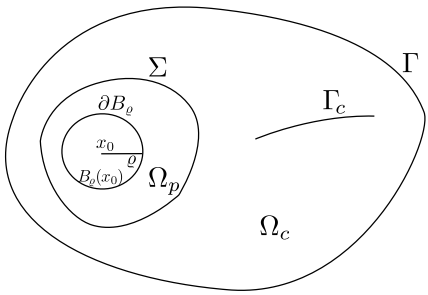

Figure 11.

with respect to the small parameter ϱ, we assume that the domain Ω_ϱ is located far from the outer boundary Γ, and far from the crack Γ_c. We assume also that in the domain Ω_ϱ the elastic body is isotropic and homogeneous, so we can perform the asymptotic analysis of the Steklov-Poincaré operator associated with the domain Ω_ϱ with respect to the small parameter $\varrho \to 0$ along the lines of [31]. We refer to [32] for all details of such analysis in the framework of exact solutions to elasticity boundary value problems by means of the elastic potentials [23]. For the convenience of the reader we recall here some facts on the topological derivatives of the shape functionals for linear elliptic boundary value problems; all proofs are given, e.g., in [29].

The topological derivative $\mathcal{T}_\Omega$ of a shape functional $\mathcal{J}(\Omega)$ is introduced in [29] in order to characterize the variation of $\mathcal{J}(\Omega)$ with respect to the infinitesimal variation of the topology of the domain Ω. In our context the notion of the *topological* derivative has the following meaning. Assume that $\Omega \subset \mathbb{R}^2$ is an open set and that we are given a shape functional

$$\mathcal{J}\colon\ \Omega \setminus D \to \mathbb{R}$$

for any compact subset $D \subset \overline{\Omega}$. We denote by $B_\varrho(x)$, $x \in \Omega$, the ball of radius $\varrho > 0$, $B_\varrho(x) = \{y \in \mathbb{R}^2\colon\ \|y - x\| < \varrho\}$, $\overline{B_\varrho(x)}$ is the closure of $B_\varrho(x)$, and assume that the limit

$$\mathfrak{T}(x) = \lim_{\varrho \downarrow 0} \frac{\mathcal{J}(\Omega \setminus \overline{B_\varrho(x)}) - \mathcal{J}(\Omega)}{|\overline{B_\varrho(x)}|}$$

exists. The function $\mathfrak{T}(x)$, $x \in \Omega$, is called the topological derivative of $\mathcal{J}(\Omega)$, and provides the information on the infinitesimal variation of the shape functional $\mathcal{J}$

if a small hole is made at $x \in \Omega$. This definition is suitable for the traction free boundary ∂B_ϱ of the hole $B_\varrho(x)$.

In several cases this characterization is constructive [5], [21], [6], [24], [30], [31], [32], i.e. the topological derivative can be evaluated for shape functionals depending on solutions of elliptic partial differential equations defined in the domain Ω.

6.1. Problem setting for elasticity systems

We introduce the elasticity system in the form convenient for the evaluation of topological derivatives. Let us consider the elasticity boundary value problem for an isotropic and homogeneous elastic body $\Omega_0 \subset \mathbb{R}^2$ with the boundary $\Gamma_D \cup \Gamma_N$,

$$\operatorname{div} \sigma(u) = 0 \quad \text{in } \Omega_0, \tag{80}$$

$$u = g \quad \text{on } \Gamma_D, \tag{81}$$

$$\sigma(u)n = T \quad \text{on } \Gamma_N, \tag{82}$$

and the same elasticity boundary value problem in the domain $\Omega_\varrho = \Omega_0 \setminus \overline{B_\varrho(x_0)}$ with a spherical cavity $B_\varrho(x_0) \subset \Omega_0$ centered at $x_0 \in \Omega_0$,

$$\operatorname{div} \sigma_\varrho(u_\varrho) = 0 \quad \text{in } \Omega_\varrho, \tag{83}$$

$$u_\varrho = g \quad \text{on } \Gamma_D, \tag{84}$$

$$\sigma_\varrho(u_\varrho)n = T \quad \text{on } \Gamma_N, \tag{85}$$

$$\sigma_\varrho(u_\varrho)n = 0 \quad \text{on } \partial B_\varrho(x_0), \tag{86}$$

where n is the unit outward normal vector to $\partial\Omega_\varrho = \partial\Omega_0 \cup \partial B_\varrho(x_0)$. In addition, g, T must be compatible with $u \in H^1(\Omega_0)$. Assuming that $0 \in \Omega_0$, we can consider the case $x_0 = 0$. Here u and u_ϱ denote the displacement vectors fields, g is a given displacement on the fixed part Γ_D of the boundary, T is a traction prescribed on the loaded part Γ_N of the boundary. In addition, σ is the Cauchy stress tensor given, for $\xi = u$ (80)–(82) or $\xi = u_\varrho$ (83)–(86), by

$$\sigma(\xi) = A\varepsilon(\xi), \tag{87}$$

where $\varepsilon(\xi)$ is the strain tensor $\varepsilon(\xi) = \{\varepsilon_{ij}(\xi)\}$, $i, j = 1, 2$, and A is the elasticity tensor

$$A = 2\mu\mathbb{I} + \lambda(I \otimes I), \tag{88}$$

with

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{\nu E}{1-\nu^2}, \tag{89}$$

E being Young's modulus, ν Poisson's ratio. In addition, I and $\mathbb{I}$ respectively are the second and the fourth order identity tensors. Thus, the inverse of A is

$$A^{-1} = \frac{1}{2\mu}\Big[\mathbb{I} - \frac{\lambda}{2\mu + 2\lambda}(I \otimes I)\Big].$$

The first shape functional under consideration depends on the displacement field. For our purposes it is sufficient to consider the linear form

$$J_1(\varrho) = \int_{\Omega_\varrho} F u_\varrho, \tag{90}$$

where F is a given function; in particular $F = f$, where f stands for the right-hand side in (1), is a possible choice. It is also useful to introduce the functional of the form

$$J_2(\varrho) = \int_{\Omega_\varrho} S\sigma(u_\varrho) \cdot \sigma(u_\varrho), \tag{91}$$

where S is an isotropic fourth-order tensor. Isotropy means here that S may be expressed as

$$S = 2m\mathbb{I} + l(I \otimes I),$$

where l, m are real constants. Their values may vary for specific cases, in particular $S = A^{-1}$ can be selected for our purposes. The following definition ensures that J_1, J_2 are well defined for solutions of the elasticity boundary value problems in Ω_0.

For simplicity the following notation is used for functional spaces

$$\begin{aligned}
\mathbf{H}^1_g(\Omega_\varrho) &= \{v \in [H^1(\Omega_\varrho)]^2 \colon\ v = g \text{ on } \Gamma_D\},\\
\mathbf{H}^1_{\Gamma_D}(\Omega_\varrho) &= \{v \in [H^1(\Omega_\varrho)]^2 \colon\ v = 0 \text{ on } \Gamma_D\},\\
\mathbf{H}^1_{\Gamma_D}(\Omega_0) &= \{v \in [H^1(\Omega_0)]^2 \colon\ v = 0 \text{ on } \Gamma_D\}.
\end{aligned}$$

The weak solutions to the elasticity systems are defined in the standard way. Find $u_\varrho \in \mathbf{H}^1_g(\Omega_\varrho)$ such that, for every $\varphi \in \mathbf{H}^1_{\Gamma_D}(\Omega_\varrho)$,

$$\int_{\Omega_\varrho} A\varepsilon(u_\varrho) \cdot \varepsilon(\varphi) = \int_{\Gamma_N} T \cdot \varphi. \tag{92}$$

The solution u_ϱ for $\varrho = 0$ is denoted by u.

We introduce the adjoint state equations in order to simplify the form of the shape derivatives of the functionals J_1, J_2. For the functional J_1 the equation takes on the form: Find $w_\varrho \in \mathbf{H}^1_{\Gamma_D}(\Omega_\varrho)$ such that, for every $\varphi \in \mathbf{H}^1_{\Gamma_D}(\Omega_\varrho)$,

$$\int_{\Omega_\varrho} A\varepsilon(w_\varrho) \cdot \varepsilon(\varphi) = -\int_{\Omega_\varrho} F \cdot \varphi, \tag{93}$$

whose Euler-Lagrange equation reads

$$\operatorname{div} \sigma_\varrho(w_\varrho) = F \quad \text{in } \Omega_\varrho, \tag{94}$$

$$w_\varrho = 0 \quad \text{on } \Gamma_D, \tag{95}$$

$$\sigma_\varrho(w_\varrho)n = 0 \quad \text{on } \Gamma_N, \tag{96}$$

$$\sigma_\varrho(w_\varrho)n = 0 \quad \text{on } \partial B_\varrho(x_0), \tag{97}$$

while $v_\varrho \in \mathbf{H}^1_{\Gamma_D}(\Omega_\varrho)$ is the adjoint state for J_2 that satisfies for all test functions $\varphi \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ the integral identity

$$\int_{\Omega_\varrho} A\varepsilon(v_\varrho) \cdot \varepsilon(\varphi) = -2 \int_{\Omega_\varrho} AS\sigma(u_\varrho) \cdot \varepsilon(\varphi), \tag{98}$$

whose associated Euler-Lagrange equation becomes

$$\operatorname{div} \sigma_\varrho(v_\varrho) = -2 \operatorname{div}(AS\sigma_\varrho(u_\varrho)) \quad \text{in } \Omega_\varrho, \tag{99}$$

$$v_\varrho = 0 \quad \text{on } \Gamma_D, \tag{100}$$

$$\sigma_\varrho(v_\varrho)n = -2AS\sigma_\varrho(u_\varrho)n \quad \text{on } \Gamma_N, \tag{101}$$

$$\sigma_\varrho(v_\varrho)n = -2AS\sigma_\varrho(u_\varrho)n \quad \text{on } \partial B_\varrho(x_0). \tag{102}$$

We denote the adjoint states for $\varrho = 0$ by $w = w_0$, $v = v_0$.

Remark 6.1. We observe that AS can be written as

$$AS = 4\mu m\mathbb{I} + \gamma(I \otimes I), \tag{103}$$

where

$$\gamma = 2\lambda l + 2(\lambda m + \mu l). \tag{104}$$

Thus, when $\gamma = 0$, the boundary condition on $\partial B_\varrho(x_0)$ in (99)–(102) becomes homogeneous and the tensor S must satisfy the constraint

$$\frac{m}{l} = -\Big(\frac{\mu}{\lambda} + 1\Big), \tag{105}$$

which is naturally satisfied for the energy shape functional, for instance. In fact, in this particular case, the tensor S is given by

$$S = \frac{1}{2}A^{-1} \Rightarrow \gamma = 0 \quad \text{and} \quad 2m + l = \frac{1}{2E}, \tag{106}$$

which implies that the adjoint solution associated with J_2 can be explicitly obtained.

Finally, we describe the construction of the Steklov-Poincaré operator

$$A_\varrho\colon\ H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$$

defined for the domain Ω_ϱ in the following way.

Given the solution z_ϱ to the boundary value problem

$$\operatorname{div} \sigma_\varrho(z_\varrho) = 0 \quad \text{in } \Omega_\varrho, \tag{107}$$

$$z_\varrho = g \quad \text{on } \Sigma, \tag{108}$$

$$\sigma_\varrho(z_\varrho)n = 0 \quad \text{on } \partial B_\varrho(x_0), \tag{109}$$

we define the traction on Σ as the value of the operator

$$A_\varrho(g) = \sigma_\varrho(z_\varrho)n.$$

6.2. Topological derivatives

The topological derivatives of shape functionals in elasticity in two spatial dimensions are obtained in [29]. In three spatial dimensions the results are less explicit, and can be found, e.g., in [24], [6]. The principal stresses associated with the displacement field u are denoted by $\sigma_I(u)$, $\sigma_{II}(u)$, the trace of the stress tensor $\sigma(u)$ is denoted by $\operatorname{tr}\sigma(u) = \sigma_I(u) + \sigma_{II}(u)$.

Theorem 6.1. *The expressions for the topological derivatives of the functionals J_1, J_2 have the form*

$$\begin{aligned} \mathcal{T} J_1(x_0) &= -\Big[F(u) + \frac{1}{E}(a_u a_w + 2b_u b_w \cos 2\delta)\Big]_{x=x_0} \\ &= -\Big[F(u) + \frac{1}{E}(4\sigma(u)\cdot\sigma(w) - \operatorname{tr}\sigma(u)\operatorname{tr}\sigma(w))\Big]_{x=x_0}, \end{aligned} \tag{110}$$

and

$$\begin{aligned} \mathcal{T} J_2(x_0) &= -\Big[(\alpha+\beta)a_u^2 + 2(\alpha-\beta)b_u^2 + \frac{1}{E}(a_u a_v + 2b_u b_v \cos 2\delta)\Big]_{x=x_0} \\ &= -\Big[4(\alpha-\beta)\sigma(u)\cdot\sigma(u) - (\alpha-3\beta)(\operatorname{tr}\sigma(u))^2 \\ &\quad + \frac{1}{E}(4\sigma(u)\cdot\sigma(v) - \operatorname{tr}\sigma(u)\operatorname{tr}\sigma(v))\Big]_{x=x_0}. \end{aligned} \tag{111}$$

Some of the terms in (110), (111) require explanation.

According to (104) the constants α and β are given by

$$\alpha = l + 2\Big(m + \gamma\frac{\nu}{E}\Big) \quad \text{and} \quad \beta = 2\frac{\gamma}{E}. \tag{112}$$

Furthermore, we denote

$$
\begin{aligned}
(113)\qquad a_u &= \sigma_I(u) + \sigma_{II}(u), & b_u &= \sigma_I(u) - \sigma_{II}(u),\\
a_w &= \sigma_I(w) + \sigma_{II}(w), & b_w &= \sigma_I(w) - \sigma_{II}(w),\\
a_v &= \sigma_I(v) + \sigma_{II}(v), & b_v &= \sigma_I(v) - \sigma_{II}(v).
\end{aligned}
$$

Finally, the symbol δ denotes the angle between the principal stress directions for the displacement fields u and w in (110), and for the displacement fields u and v in (111).

Remark 6.2. For the energy stored in a 2D elastic body, the tensor S is given by (106), $\gamma = 0$, $\alpha = 1/(2E)$ and $\beta = 0$. Thus, we obtain the well-known result

$$
(114)\qquad \mathcal{T}J_2(x_0) = \frac{1}{2E}[4\sigma(u)\cdot\sigma(u) - (\operatorname{tr}\sigma(u))^2]_{x=x_0}
$$

which we use below for derivation of the topological derivatives of the energy functional for domains with cracks.

Now we consider the domain $\mathcal{D}_\varrho = \Omega_\varrho \cup \Sigma \cup \Omega_c$, see Fig. 11. The convex set K is defined by the same formula (6), with the only difference that in the present situation the boundary is $\partial\Omega_c = \Sigma \cup \Gamma \cup \Gamma_c^{\pm}$, and there is no condition prescribed on Σ, hence

$$
K = \{v \in \mathbf{H}^1_\Gamma(\Omega_c)\colon\ [v]\nu \geqslant 0 \text{ a.e. on } \Gamma_c\}.
$$

The energy in $\mathcal{D}_\varrho$ is given by the functional depending on the size of the cavity

$$
\begin{aligned}
j(\varrho) &= \min_{v\in K}\Big\{\frac{1}{2}\int_{\mathcal{D}_\varrho}\sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\mathcal{D}_\varrho} f_i v_i\Big\}\\
&= \min_{v\in K}\Big\{\frac{1}{2}\int_{\Omega_c}\sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_c} f_i v_i + \langle A_\varrho(v), v\rangle_{1/2,\Sigma}\Big\},
\end{aligned}
$$

where the expression for the energy in the domain Ω_ϱ with the hole $B_\varrho(x_0)$ uses the Steklov-Poincaré operator of the specific annulus domain Ω_ϱ. We refer the reader to [32] for the derivation of the asymptotics of arbitrary order for the operator. Thus, the argument of [31] applies, and in view of (114) we have

$$
j''(0^+) = -\frac{1}{4E}[4\sigma(u)\cdot\sigma(u) - (\operatorname{tr}\sigma(u))^2]_{x=x_0}
$$

which gives the expression for the topological derivative of the energy functional at the point x_0.

7. Evolution of a kinking crack

The problem of a kink is of special interest, because it represents a change of topology from a smooth crack to a non-smooth one. The topology change is the main difficulty in the mathematical analysis of cracks with a kink. In this section we apply the shape optimization approach to a two-parameter problem for a kinking crack. Namely, we fix a point of kink and find the unknown shape parameters of the kink angle and the crack length which minimize the total potential energy due to the Griffith approach. This nonlinear minimization problem describes the evolution of the kinking crack with respect to the time-like loading parameter. In the linear crack theory, the optimization Griffith approach was used in [2].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. Assuming that the origin $\mathcal{O} = (0,0)$ belongs to $\overline{\Omega}$, we consider a given crack $\Gamma_0 \subset \Omega$ with tips at Γ and at the origin, and the unknown part $C_{(r,\varphi)}$ of the crack whose tip is described in polar coordinates as

$$(r\cos\varphi, r\sin\varphi), \quad (r,\varphi) \in \overline{\omega},$$

where ω is the set of admissible parameters

$$\omega = \{(r,\varphi)\colon\ 0 < r < R(\varphi) \text{ for } \varphi \in (\varphi_0, \varphi_1)\}, \quad [\varphi_0, \varphi_1] \subset (-\pi, \pi),$$

with a given periodic function $R \in W^{2,\infty}(-\pi,\pi)$.

Admissible kinking cracks are defined as the union $\Gamma_{(r,\varphi)} = \Gamma_0 \cup C_{(r,\varphi)}$. Denote by $\Omega_{(r,\varphi)}$ the domain with a crack $\Gamma_{(r,\varphi)}$, i.e. $\Omega_{(r,\varphi)} = \Omega \setminus \overline{\Gamma}_{(r,\varphi)}$, see Fig. 12. In the

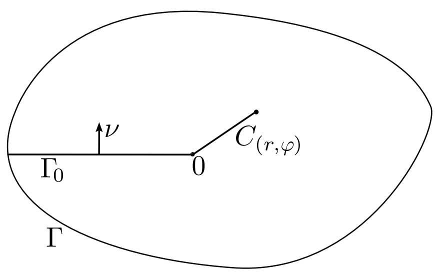

Figure 12. Kinking crack.

domain $\Omega_{(r,\varphi)}$ we can consider an equilibrium problem like (1)–(5). Namely, let ν be the normal vector to $\Gamma_{(r,\varphi)}$ and let $f = (f_1, f_2) \in C^1(\overline{\Omega})$ be a given function. The formulation of the problem reads as follows. In the domain $\Omega_{(r,\varphi)}$ we have to find a displacement vector $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_{(r,\varphi)}, \tag{115}$$

$$\sigma = A\varepsilon(u) \quad \text{in } \Omega_{(r,\varphi)}, \tag{116}$$

$$u = 0 \quad \text{on } \Gamma, \tag{117}$$

$$[u]\nu \geqslant 0, \quad [\sigma_\nu] = 0, \quad \sigma_\nu \cdot [u]\nu = 0 \quad \text{on } \Gamma_{(r,\varphi)}, \tag{118}$$

$$\sigma_\nu \leqslant 0, \quad \sigma_\tau = 0 \quad \text{on } \Gamma^{\pm}_{(r,\varphi)}. \tag{119}$$

For any given $(r,\varphi) \in \overline{\omega}$, a solution of problem (115)–(119) exists in the Sobolev space $\mathbf{H}^1_\Gamma(\Omega_{(r,\varphi)})$. Hence, for any $(r,\varphi) \in \overline{\omega}$ we can define a solution $u^{(r,\varphi)}$ and the energy functional

$$\Pi(\Omega_{(r,\varphi)}; u^{(r,\varphi)}) = \frac{1}{2}\int_{\Omega_{(r,\varphi)}} \sigma_{ij}(u^{(r,\varphi)})\varepsilon_{ij}(u^{(r,\varphi)}) - \int_{\Omega_{(r,\varphi)}} f_i u_i^{(r,\varphi)},$$

where $\sigma_{ij}(u^{(r,\varphi)}) = \sigma_{ij}$ are found from (116). Thus, differentiability of the energy functional with respect to (r,φ) can be analyzed. These results can be found in [10]. The main difficulty in the study of differentiability is the following one. Considering perturbations of problem (115)–(119), we have no one-to-one correspondence between sets of admissible displacements for perturbed and unperturbed problems. This requires additional considerations to prove differentiability of $\Pi(\Omega_{(r,\varphi)}; u^{(r,\varphi)})$ with respect to r, φ.

In what follows, we formulate an evolution problem for a kinking crack. Denote

$$P(r,\varphi) = \Pi(\Omega_{(r,\varphi)}; u^{(r,\varphi)}).$$

For a time-like loading parameter $t \geqslant 0$ we consider a family of forces tf in (115). Let the length of the crack Γ_0 be equal to $l_0 \geqslant 0$. Note that if the solution $u^{(r,\varphi)}$ corresponds to the force f in (115), we obtain a solution $tu^{(r,\varphi)}$ for the force tf due to the homogeneity property for problem (115)–(119). Let the initial crack (at $t = 0$) be given as Γ_0. For the loading tf, we look for a propagating crack $\Gamma_{(r(t),\varphi^*)} \subset \Omega$ with the kink at the origin $\mathcal{O}$ and unknown shape parameters of the crack length $l_0 + r(t)$ and the kink angle $\varphi^* \in [\varphi_0, \varphi_1]$. To this end, we use the shape optimization approach, which is based on the Griffith hypothesis. Following this hypothesis, we define a function of total potential energy

$$T(r,\varphi)(t) = 2\gamma(l_0 + r) + t^2 P(r,\varphi), \quad (r,\varphi) \in \overline{\omega}. \tag{120}$$

The first term in (120) represents the surface energy distributed uniformly at two crack faces with a constant density $\gamma > 0$ (the given material parameter). The second term in (120) represents the potential energy which is quadratic in t,

$$P(r,\varphi)(t) = \Pi(\Omega_{(r,\varphi)}; tu^{(r,\varphi)}) = t^2 P(r,\varphi).$$

Thus we arrive at the problem formulation of the evolution of the kinking crack:

$$r(0) = 0; \tag{121}$$

for $t > 0$, find parameters $(r(t), \varphi(t)) \in \overline{\omega}$ that

$$\text{minimize } T(r,\varphi)(t) \text{ over } (r,\varphi) \in \overline{\omega} \tag{122}$$

$$\text{subject to } \varphi \in \bigcap_{s<t} \{\varphi(s)\}. \tag{123}$$

The constraint (123) allows us to preserve the shape of the kinking crack during its evolution. This means that if the kinking angle φ^* is found, its value is preserved during the evolution. Problem (121)–(123) has a solution (see [10]). It turns out that the radius $r(t)$ during the evolution may be multi-valued, i.e. $r(t) \in [r^-(t), r^+(t)]$, which means a nonstable crack evolution.

8. 3D problems and open questions

Most problems discussed in the paper can be solved in the 3D case when a crack is presented as a 2D smooth surface. For example, the crack can be described as

$$x_i = x_i(y_1, y_2), \quad i = 1, 2, 3,$$

where $(y_1, y_2) \in D$, $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and the mapping $y \to x$ is non-degenerating.

All formulas and statements of Sections 1–5 hold true with suitable specifications of the situation. In particular, by discussing a fulfilment of the boundary conditions (4)–(5) we should introduce the Hilbert space $H^{1/2}(\Sigma)$, where Σ is the extension of Γ_c to a closed 2D smooth surface. The norm in $H^{1/2}(\Sigma)$ in this case is defined as (cf. (9))

$$\|v\|^2_{H^{1/2}(\Sigma)} = \|v\|^2_{L^2(\Sigma)} + \int_\Sigma \int_\Sigma \frac{|v(x) - v(y)|^2}{|x-y|^3} \, \mathrm{d}x \, \mathrm{d}y.$$

Mixed and smooth domain formulations as well as the fictitious domain method hold true in the 3D case.

Also, we can consider a crack located on the boundary of a rigid inclusion for a 3D elastic body and prove all statements of Section 5. Notice that in 3D case the space of infinitesimal rigid inclusions is defined as

$$R(\omega) = \{\varrho = (\varrho_1, \varrho_2, \varrho_3)\colon\ \varrho(x) = Bx + D,\ x \in \omega\},$$

where

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{21} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix}, \quad D = (d^1, d^2, d^3),$$
$$b_{ij}, d^i = \text{const},\ i, j = 1, 2, 3.$$

As for the differentiation of the energy functionals with respect to a perturbed parameter (Section 5), we have a big variety of perturbations in the 3D case. The simplest ones provide a perturbation of the crack front. For example, let Γ_c be chosen in the form

$$\Gamma_c = \{(x_1, x_2, 0)\colon\ 0 \leqslant x_1 \leqslant \varphi(x_2),\ x_2 \in [-1, 1],\ \varphi(x_2) > 0\}$$

with a given smooth function φ. In this case, the 3D vector field can be taken as

$$V(x) = (\theta(x), 0, 0),$$

where θ is a given smooth function with support in a vicinity of the crack front

$$\{(x_1, x_2, x_3)\colon\ x_1 = \varphi(x_2),\ x_3 = 0,\ x_2 \in [-1, 1]\}.$$

This allows us to differentiate the energy functional in the direction of the field V, which implies the formula (63) with $i, j = 1, 2, 3$; see [11], [17].

Like in Section 5, in the 3D case we can consider curvilinear cracks described as the graph of a function

$$x_3 = \psi(x_1, x_2), \quad (x_1, x_2) \in D,$$

where $D \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary. The necessary formulas for derivatives of the energy functional in this case can be found in [26].

As for invariant integrals, in the 3D case we should integrate over closed 2D surfaces surrounding the crack front, see [8], [14].

To conclude the paper, we formulate some open problems.

- For a crack Γ_c which crosses the external boundary Γ with a zero angle, there is no solvability of problem (1)–(5) in the general case, since Korn's inequality is non valid. Is it possible to overcome this difficulty?

- Is there uniqueness of solutions to problem (121)–(123)?
- Find the form of the shape derivative for the energy functional with respect to the perturbations of the crack tip in the case of the crack at the interface beween an elastic body and a rigid inclusion.

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